MATHEMATICS OF COMPUTATION
VOLUME 56, NUMBER 194
APRIL 1991, PAGES 463–476

SOME ESTIMATES FOR A WEIGHTED L^2 PROJECTION

JAMES H. BRAMBLE AND JINCHAO XU

ABSTRACT. This paper is devoted to the error estimates for some weighted L^2 projections. Nearly optimal estimates are obtained. These estimates can be applied to the analysis of the usual multigrid method, multilevel preconditioner and domain decomposition method for solving elliptic boundary problems whose coefficients have large jump discontinuities.

1. INTRODUCTION

This work was motivated by the study of the numerical solution of elliptic boundary value problems that have large discontinuity jumps in coefficients. If these jumps become larger, the corresponding discretized (by finite elements, for example) equation may be harder to solve. In some special cases, however, multigrid or domain decomposition methods can be properly designed so that the numerically observed convergence rate is actually independent of these jumps. We find that the theoretical justification of this phenomenon lies in certain approximation and stability properties of some weighted L^2 projections with weights provided by the discontinuous coefficients (cf. [10, 11, 4]). The point is that we want to get estimates which are uniform with respect to the weights.

A careful study of this type of weighted L^2 projection will be made in this paper. We shall establish estimates that are nearly optimal under some special circumstances. In a sequel of this paper, we shall present some negative results to demonstrate that the expected estimates are not always possible, in general, and the results in the paper are sharp in a certain sense.

Related to the topic of this paper is the usual L^2 projection. Some error and stability estimates for such a projection are also presented with complete proofs.

As is done in [10], we will use the following notation:

$$x \lesssim y, \quad f \gtrsim g, \quad \text{and} \quad u \asymp v$$

Received May 18, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 65M60, 65N15, 65N30.

Key words and phrases. Finite element space, weighted L^2 projections, multigrid, domain decomposition.

This work was supported in part under the National Science Foundation Grant Nos. DMS84-05352 and 8801346-02, also by the U.S. Army Research Office through the Mathematical Sciences Institute, Cornell University.

©1991 American Mathematical Society
0025-5718/91 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

which means that

$$x \le Cy, \quad f \ge cg, \quad \text{and} \quad cv \le u \le Cv,$$

where C and c are positive constants independent of the variables appearing in the inequalities and any other parameters related to meshes, spaces, etc.

The remainder of the paper is organized as follows. In §2, some preliminary material, such as the Sobolev spaces, finite element spaces, etc., will be presented. Section 3 is devoted to the analysis of the usual L^2 projection. The main estimates for weighted L^2 projections will be presented in §4.

2. Preliminaries

Let $\Omega \subset \mathbf{R}^d$ $(1 \le d \le 3)$ be a bounded domain. For simplicity, we assume that Ω is an interval for $d = 1$, a polygon for $d = 2$, and a polyhedron for $d = 3$. On Ω, $L^p(\Omega)$ denotes the usual Banach space consisting of p th power integrable functions. The Sobolev space of index (m, p) is defined by

$$W^{m,p}(\Omega) \stackrel{\text{def}}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \le m\},$$

with a norm

$$\|v\|_{W^{m,p}(\Omega)} \stackrel{\text{def}}{=} \left(\sum_{|\alpha| \le m} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-integer and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_d^{\alpha_d}}, \qquad |\alpha| = \sum_{i=1}^d \alpha_i.$$

For $p = 2$, by convention, we denote

$$H^m(\Omega) \stackrel{\text{def}}{=} W^{m,2}(\Omega).$$

We will have occasion to use the following seminorms:

$$|v|_{W^{m,p}(\Omega)} \stackrel{\text{def}}{=} \left(\sum_{|\alpha| = m} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p}.$$

For $m = 1$, $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ consisting of functions that vanish on $\partial\Omega$ in an appropriate sense. Similarly, for a measurable $\Gamma_0 \subset \partial\Omega$, $H_{\Gamma_0}^1(\Omega)$ is the space consisting of functions in H^1 that vanish on Γ_0.

We quote the following well-known Sobolev continuous imbeddings [1]:

$$H^1(\Omega) \hookrightarrow \begin{cases} L^\infty(\Omega), & \text{if } d = 1, \\ L^p(\Omega)\ (1 \le p < \infty), & \text{if } d = 2, \\ L^6(\Omega), & \text{if } d = 3. \end{cases} \tag{2.1}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Lemma 2.1. *We have*

$$\|u\|_{L^2(\partial\Omega)} \lesssim \varepsilon^{-1}\|u\|_{L^2(\Omega)} + \varepsilon\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \ \varepsilon \in (0, 1).$$

For a proof, we refer to [10].

The following result is a special case of Theorem 2.1 in [10]. (Cf. also [9].)

Lemma 2.2. *Assume D is a bounded domain in $\mathbf{R}^2$ with ∂D Lipschitz continuous. Then*

$$\|w\|_{L^\infty(D)} \lesssim |\log\varepsilon|^{1/2}\|w\|_{H^1(D)} + \varepsilon\|w\|_{W^{1,\infty}(D)} \quad \forall w \in W^{1,\infty}(D), \ \varepsilon \in (0, 1).$$

Next we introduce the finite element space. For $0 < h < 1$, let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$ with simplices K of diameter less than or equal to h. We assume the family $\{\mathcal{T}_h\}$ is quasiuniform, i.e., there are constants $c_0 > 0$ and $c_1 > 0$ such that

$$\max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \le c_0, \qquad \frac{\max_{K\in\mathcal{T}_h}}{\min_{K\in\mathcal{T}_h}} \le c_1 \quad \forall h,$$

where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K. Corresponding to each triangulation $\mathcal{T}_h$, we define a finite element subspace $S_h \subset H_0^1(\Omega)$ that consists of continuous piecewise (with respect to the elements in $\mathcal{T}_h$) linear polynomials vanishing on $\partial\Omega$.

For a given triangulation $\mathcal{T}_h$, we consider a finer quasiuniform mesh $\mathcal{T}_{\underline{h}}$ with $\underline{h} < h$ which is obtained by refining $\mathcal{T}_h$ in such a way that

$$S_h \subset S_{\underline{h}},$$

where $S_{\underline{h}} \subset H_0^1(\Omega)$ is the corresponding finite element space defined on $\mathcal{T}_{\underline{h}}$.

It is well known that, for any function $v \in S_h$

$$\|v\|^2_{L^2(\Omega)} \asymp h^d \sum_{x\in\mathcal{N}_h} v^2(x), \tag{2.2}$$

where $\mathcal{N}_h$ is the set of vertices of the triangulation $\mathcal{T}_h$, and

$$\|v\|_{L^\infty(\Omega)} \lesssim h^{-d/p}\|v\|_{L^p(\Omega)} \qquad (1 \le p \le \infty). \tag{2.3}$$

The right-hand side of (2.2) is often called the discrete L^2 norm. Inequality (2.3) is the well-known inverse property of the finite element spaces (cf. [5]).

Lemma 2.3. *For all $v \in S_h(\Omega)$,*

$$\|v\|_{L^\infty(\Omega)} \lesssim \begin{cases} \|v\|_{H^1(\Omega)}, & \text{if } d = 1, \\ |\log h|^{1/2}\|v\|_{H^1(\Omega)}, & \text{if } d = 2, \\ h^{-1/2}\|v\|_{H^1(\Omega)}, & \text{if } d = 3. \end{cases} \tag{2.4}$$

Proof. The first inequality (for $d = 1$) follows from the usual Sobolev imbedding (in (2.1)). The second is well known in the literature (cf. [3]) and can be

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

easily obtained by using Lemma 2.2 together with the inverse inequality (2.3) with $\varepsilon = h$.

The proof of the last case for $d = 3$ is also almost trivial and can be furnished by using the inverse inequality (2.3) and Sobolev imbedding (2.1):

$$\|v\|_{L^\infty(\Omega)} \lesssim h^{-1/2}\|v\|_{L^6(\Omega)} \lesssim h^{-1/2}\|v\|_{H^1(\Omega)}. \quad \square$$

The most interesting part in Lemma 2.3 is perhaps for $d = 2$. That is just the limiting case in which the Sobolev imbedding fails. The following is another such example.

Lemma 2.4. *Assume Ω is a polyhedral domain in $\mathbf{R}^3$. Then*

$$\|v\|_{L^2(\Gamma)} \lesssim |\log h|^{1/2}\|v\|_{H^1(\Omega)} \quad \forall v \in S^h(\Omega),$$

where Γ is any edge of Ω.

Proof. By breaking the domain Ω into (possibly overlapping) subdomains that have parallel faces in one direction, we may assume here, without loss of generality, that $\Omega = (0, 1)^3$ is the unit cube. Applying the inequality in Lemma 2.2 with the domain $D = (0, 1)^2$ and $w = v(x_1, x_2, x_3)$, we get

$$|v(0, 0, x_3)|^2 \lesssim |\log \varepsilon| \int_D \left(v^2 + \sum_{i=1}^{2}\left|\frac{\partial v}{\partial x_i}\right|^2\right) dx_1\, dx_2 + \varepsilon^2\|v\|^2_{W^{1,\infty}(\Omega)} \quad \forall \varepsilon \in (0, 1).$$

Integrating with respect to x_3, we get

$$\int_0^1 |v(0, 0, x_3)|^2\, dx_3 \lesssim |\log \varepsilon|\, \|w\|^2_{H^1(\Omega)} + \varepsilon^2\|v\|^2_{W^{1,\infty}(\Omega)} \quad \forall \varepsilon \in (0, 1).$$

Taking $\varepsilon = h^{3/2}$ and applying the inverse inequality yields

$$\|v\|_{L^2(\Gamma)} \lesssim |\log h|^{1/2}\|v\|_{H^1(\Omega)} \quad \forall v \in S^h(\Omega),$$

where $\Gamma = \{(0, 0, x_3): 0 \le x_3 \le 1\}$.

Similar arguments obviously apply to the other edges of Ω, and the proof is complete. $\square$

3. Ordinary L^2 projections

In this section, we shall consider the usual L^2 projection with respect to the ordinary L^2 inner product (namely without weights).

Associated with the finite element space S_h, the L^2 projection $Q_h: L^2(\Omega) \mapsto S_h$ is defined by

$$(Q_h u, v) = (u, v) \quad \forall u \in L^2(\Omega), \ v \in S_h.$$

The aim of this section is to establish some estimates for Q_h on H^1 in both the L^2 and H^1 norms, namely for all $u \in H^1_0(\Omega)$

$$\|u - Q_h u\|_{L^2(\Omega)} \lesssim h|u|_{H^1(\Omega)} \tag{3.1}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

and

$$|Q_h u|_{H^1(\Omega)} \lesssim |u|_{H^1(\Omega)}. \tag{3.2}$$

The above estimates are closely related to the so-called simultaneous approximation property:

$$\inf_{\chi\in S_h}(\|u-\chi\|_{L^2(\Omega)} + h\|u-\chi\|_{H^1(\Omega)}) \lesssim h\|u\|_{H^1(\Omega)} \quad \forall u\in H_0^1(\Omega). \tag{3.3}$$

More specifically, we have

Lemma 3.1. (3.1) *and* (3.2) *both hold if and only if* (3.3) *is true.*

The proof, which uses the triangle inequality and also the inverse property, is straightforward.

Inequality (3.3) has been assumed in some papers on finite elements, but it seems that little attention is paid to its proof. The stability of the L^2 projection in the H^1 norm was perhaps first established by Bank and Dupont in [2]. Their proof, however, requires the full elliptic regularity condition (which is unnecessary). A discussion of this problem in two dimensions may also be found in Crouzeix and Thomée [6]. Recently, Scott and Zhang [8] have constructed a kind of interpolation operator for nonsmooth functions that can also be used to give a proof of this result. As we pointed out earlier, it can be directly obtained by assuming the simultaneous approximation property (3.3), which is actually the approach that Mandel, McCormick, and Bank take in [7]. For avoiding a logical circle, the question remains as to how the simultaneous approximation property is justified. Our approach here is to establish the stability by a different argument and obtain the simultaneous approximation property as a consequence.

L^2 error estimates. As we have assumed that $d \le 3$, the Sobolev imbedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ holds. Therefore, the usual nodal value interpolant $I_h: C(\overline{\Omega}) \mapsto S_h$ is well defined in H^2. It is well known that (cf. [5])

$$\|u-Q_h u\|_{L^2(\Omega)} \le \|u-I_h u\|_{L^2(\Omega)} \lesssim h^2|u|_{H^2(\Omega)} \quad \forall u\in H^2(\Omega)\cap H_0^1(\Omega). \tag{3.4}$$

On the other hand,

$$\|u-Q_h u\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)} \quad \forall u\in L^2(\Omega). \tag{3.5}$$

An application of the standard interpolation technique to the above two estimates yields

Theorem 3.2. *For* $u\in H_0^1(\Omega)$,

$$\|u-Q_h u\|_{L^2(\Omega)} \lesssim h|u|_{H^1(\Omega)}. \tag{3.6}$$

H^1 stability. The main ingredient in our analysis is a local L^2 projection $Q_\tau: L^2(\tau) \mapsto \mathscr{P}_1(\tau)$, for any given $\tau\in\mathscr{T}_h$, defined by

$$(Q_\tau u, \phi)_{L^2(\tau)} = (u, \phi)_{L^2(\tau)} \quad \forall u\in L^2(\tau),\ \phi\in\mathscr{P}_1(\tau).$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Let $\hat\tau$ be the standard reference element, so that for any $\tau \in \mathscr{T}_h$ we have an affine diffeomorphism $F_\tau \colon \hat\tau \mapsto \tau$. For any function $v \in L^2(\tau)$, we adopt the following standard notation:

$$\hat v(\hat x) = v(F(\hat x)), \qquad \hat x \in \hat\tau.$$

If $Q_{\hat\tau}$ is defined similarly, it is then straightforward to verify that

$$\widehat{Q_\tau u} = Q_{\hat\tau}\hat u. \tag{3.7}$$

These locally defined operators have the desired stability and approximation properties, as shown by

Lemma 3.3. *For any* $\tau \in \mathscr{T}_h$,

$$|Q_\tau u|_{H^1(\tau)} \lesssim |u|_{H^1(\tau)} \quad \forall u \in H^1(\tau), \tag{3.8}$$

and

$$\|u - Q_\tau u\|_{L^2(\tau)} \lesssim h|u|_{H^1(\tau)} \quad \forall u \in H^1(\tau). \tag{3.9}$$

Proof. It follows from (3.7) that (3.8) is equivalent to

$$|Q_{\hat\tau}\hat u|_{H^1(\hat\tau)} \lesssim |\hat u|_{H^1(\hat\tau)} \quad \forall \hat u \in H^1(\hat\tau). \tag{3.10}$$

As all the norms on $\mathscr{P}_1(\hat\tau)$ are equivalent, we have

$$|Q_{\hat\tau}\hat u|_{H^1(\hat\tau)} \lesssim \|Q_{\hat\tau}\hat u\|_{L^2(\hat\tau)} \le \|\hat u\|_{L^2(\hat\tau)} \lesssim \|\hat u\|_{H^1(\hat\tau)},$$

which, since $Q_{\hat\tau}\hat c = \hat c$ for any $\hat c \in \mathbf{R}^2$, implies that

$$|Q_{\hat\tau}\hat u|_{H^1(\hat\tau)} \lesssim \inf_{\delta \in \mathbf{R}^1} \|\hat u + \hat c\|_{H^1(\hat\tau)} \lesssim |\hat u|_{H^1(\hat\tau)}.$$

This proves (3.10) and hence (3.8).

Now we turn to the proof of (3.9). By changing variables and using (3.7), we get

$$\begin{aligned}
\|u - Q_\tau u\|_{L^2(\tau)} &\lesssim h^{d/2}\|\hat u - \hat Q_{\hat\tau}\hat u\|_{L^2(\hat\tau)} \\
&\lesssim h^{d/2} \inf_{\hat c \in \mathbf{R}^1} \|\hat u + \hat c\|_{H^1(\hat\tau)} \lesssim h^{d/2}|\hat u|_{H^1(\hat\tau)} \\
&\lesssim h^{d/2}h^{1-d/2}|u|_{H^1(\tau)} \lesssim h|u|_{H^1(\tau)}.
\end{aligned}$$

This completes the proof. □

We are now in a position to state and prove our stability theorem.

Theorem 3.4. *For all* $u \in H_0^1(\Omega)$,

$$|Q_h u|_{H^1(\Omega)} \lesssim |u|_{H^1(\Omega)}. \tag{3.11}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Proof. It follows from the inverse inequality, Lemma 3.2, and Lemma 3.3, that

$$\begin{aligned}|Q_h u|^2_{H^1(\Omega)} &= \sum_{\tau\in\mathcal{T}_h} |Q_h u|^2_{H^1(\tau)} \le 2\sum_{\tau\in\mathcal{T}_h}\{|Q_h u - Q_\tau u|^2_{H^1(\tau)} + |Q_\tau u|^2_{H^1(\tau)}\}\\ &\lesssim \sum_{\tau\in\mathcal{T}_h}\{h^{-2}|Q_h u - Q_\tau u|^2_{L^2(\tau)} + |u|^2_{H^1(\tau)}\}\\ &\lesssim \sum_{\tau\in\mathcal{T}_h}\{h^{-2}\|u - Q_\tau u\|^2_{L^2(\tau)} + |u|^2_{H^1(\tau)}\} + h^{-2}\|u - Q_h u\|^2_{L^2(\Omega)}\\ &\lesssim |u|^2_{H^1(\Omega)}.\end{aligned}$$

The desired result then follows. □

Remark 3.1. Notice that our proof of (3.11), which uses Q_τ, is carried out element-by-element. Such a "local" argument is crucial for us to establish the corresponding stability for the weighted L^2 projection (with trivial modification).

Simultaneous approximation properties. From the estimates we derived for Q_h, property (3.3) then becomes clear by Lemma 3.1. In fact, this simultaneous approximation property holds for more general boundary conditions. For example, if $\Gamma_0 \subset \partial\Omega$ is measurable, then we have

Proposition 3.5. *For any* $u \in H^1_{\Gamma_0}(\Omega)$, *there exists* $v_h \in S_h \cap H^1_{\Gamma_0}(\Omega)$ *such that* (3.3) *holds.*

4. Weighted L^2 projection

This section, which is the core of the paper, is devoted to the analysis of the weighted L^2 projections. Both the L^2 error estimates and H^1 stability will be investigated.

Assume the domain Ω admits the following decomposition:

$$\overline{\Omega} = \bigcup_{i=1}^{J} \overline{\Omega}_i, \tag{4.1}$$

where the Ω_i are mutually disjoint. Let Γ denote the set of interfaces, i.e., $\Gamma = \bigcup_{i=1}^{J} \partial\Omega_i \backslash \partial\Omega$. For simplicity, we assume that Γ consists only of segments ($d = 2$) or plane polygons ($d = 3$). In other words, no part of any $\partial\Omega_i$ is curved.

Given a set of positive constants $\{\omega_i\}_{i=1}^{J}$, we introduce the following weighted inner products:

$$(u, v)_{L^2_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i (u, v)_{L^2(\Omega_i)}, \tag{4.2}$$

and

$$(u, v)_{H^1_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v \, dx, \tag{4.3}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

with the induced norms denoted by $\|\cdot\|_{L^2_\omega(\Omega)}$ and $|\cdot|_{H^1_\omega(\Omega)}$, respectively. Moreover, we define a full weighted H^1 norm by

$$\|\cdot\|^2_{H^1_\omega(\Omega)} = \|\cdot\|^2_{L^2_\omega(\Omega)} + |\cdot|^2_{H^1_\omega(\Omega)}.$$

We assume that Ω is triangulated by a family of quasiuniform meshes $\{\mathcal{T}_h, h < 1\}$, as described earlier. An additional assumption we make here is that these triangulations will be lined up with the subdomains Ω_i's. Namely, the restriction of each $\mathcal{T}_h$ on each Ω_i is also a triangulation of Ω_i itself.

The weighted L^2 projection $Q_h^\omega: L^2(\Omega) \mapsto S_h$ is defined by

$$(Q_h^\omega u, v)_{L^2_\omega(\Omega)} = (u, v)_{L^2_\omega(\Omega)} \quad \forall u \in L^2(\Omega), \ v \in S_h. \tag{4.4}$$

We will derive error estimates for Q_h^ω of the following type:

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \le Ch|\log h|^\gamma |u|_{H^1_\omega(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

for some positive constant γ. The point here is that we require that the constant C appearing in the above estimate does not depend on the weights $\{\omega_i\}$. Again, we will use the notation "$\lesssim$" in place of "$\le C$", where C is in particular independent of the weights.

The derivation of such an estimate is not as simple as it might appear. For example, the argument used in the proof of Theorem 3.2 cannot be applied easily here, even though we can get the estimates analogous to (3.4) and (3.5) with proper weights. It is unclear if the interpolation between weighted H^2 and L^2 spaces would give rise to the right space. Nevertheless, the proof for $d = 1$ is almost trivial. To be more precise, we have the following

Proposition 4.1. *For $d = 1$, we have for all $u \in H_0^1(\Omega)$*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h|u|_{H^1_\omega(\Omega)}$$

and

$$|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim |u|_{H^1_\omega(\Omega)}.$$

Proof. Since $d = 1$, we have $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$. Hence the nodal value interpolant $I_h: C(\overline{\Omega}) \mapsto S_h$ is well defined in H^1. It is well known that (cf. [5]), for any $\tau \in \mathcal{T}_h$,

$$\|u - I_h u\|^2_{L^2(\tau)} \lesssim h^2 |u|^2_{H^1(\tau)} \quad \forall u \in H^1(\tau).$$

Summing up over all $\tau \in \mathcal{T}_h$ with proper weights, we then get

$$\|(I - I_h)u\|_{L^2_\omega(\Omega)} \le Ch|u|_{H^1_\omega(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

Our first inequality then follows, since $\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \le \|(I - I_h)u\|_{L^2_\omega(\Omega)}$. The proof of the second inequality is similar to Theorem 3.4 by Lemma 3.3. □

The above approach cannot, in general, be extended to higher dimensions because of the lack of the imbedding $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$, although a similar technique can be applied, as is done in §4.2.1 below in some special circumstances

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

when $d = 2$. The analysis for more general cases, especially for $d = 3$, is more complicated, and special techniques are needed.

4.1. The case of no internal cross point. By internal cross points we mean those points on Γ that belong to more than two $\overline{\Omega}_i$'s. If there is no such point on the interface, the analysis becomes very simple, and optimal estimates can be derived.

We shall first present a lemma that shows that the estimate we need can be reduced to the estimates on interfaces. To do this, let us introduce a weighted inner product on $L^2(\Gamma)$:

$$(u, v)_{L^2_\omega(\Gamma)} = \sum_{i=1}^{J} \int_{\partial\Omega_i \backslash \partial\Omega} \omega_i uv\, dx.$$

Denoting $S_h(\Gamma) \stackrel{\text{def}}{=} \{v|_\Gamma : v \in S_h\}$, let $P_\Gamma : L^2(\Gamma) \mapsto S_h(\Gamma)$ be the orthogonal projection with respect to $(\cdot, \cdot)_{L^2_\omega(\Gamma)}$.

Lemma 4.2. *For all* $u \in H_0^1(\Omega)$,

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h|u|_{H^1_\omega(\Omega)} + h^{1/2}\|u - P_\Gamma u\|_{L^2_\omega(\Gamma)}.$$

Proof. On each domain Ω_i, by Proposition 3.5, there exists a $w_i \in S_h(\Omega_i)$ such that

$$\|u - w_i\|^2_{L^2(\Omega_i)} + h^2\|u - w_i\|^2_{H^1(\Omega_i)} \lesssim h^2\|u\|^2_{H^1(\Omega_i)}. \tag{4.5}$$

Let $w \in S_h$ be such that

$$w = \begin{cases} w_i, & \text{at the nodes in } \Omega_i, \\ P_\Gamma u, & \text{on } \Gamma. \end{cases}$$

Therefore, using (2.2),

$$\begin{aligned} \|u - w\|^2_{L^2_\omega(\Omega)} &\lesssim \sum_{i=1}^{J} \omega_i\|u - w_i\|^2_{L^2(\Omega_i)} + h^2\sum_{i=1}^{J} \omega_i \sum_{p\in\Gamma_i} |(w - P_\Gamma u)(p)|^2 \\ &\lesssim \sum_{i=1}^{J} \omega_i\|u - w_i\|^2_{L^2(\Omega_i)} + h\|w - P_\Gamma u\|^2_{L^2_\omega(\Gamma)} \\ &\lesssim \sum_{i=1}^{J} \omega_i h^2\|u\|^2_{H^1(\Omega_i)} + h\|u - P_\Gamma u\|^2_{L^2_\omega(\Gamma)} \\ &\lesssim h^2|u|^2_{H^1_\omega(\Omega)} + h\|u - P_\Gamma u\|^2_{L^2_\omega(\Gamma)}. \end{aligned}$$

The desired result then follows, since

$$\|u - Q_h^\omega u\|_{L^2_\omega(\Omega)} \le \|u - w\|_{L^2_\omega(\Omega)}. \quad \square$$

From the above proof, we see that the validity of Lemma 4.2 has nothing to do with cross points. Nevertheless, we only know its application to the case that the interface has no internal cross points.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Theorem 4.3. *Assume the decomposition* (4.1) *has no internal cross points. Then, for all* $u \in H_0^1(\Omega)$,

$$\|(I - Q_h^{\omega})u\|_{L^2_{\omega}(\Omega)} \lesssim h|u|_{H^1_{\omega}(\Omega)} \tag{4.6}$$

and

$$|Q_h^{\omega}u|_{H^1_{\omega}(\Omega)} \lesssim |u|_{H^1_{\omega}(\Omega)}. \tag{4.7}$$

Proof. Define a function $\phi \in S_h(\Gamma)$ by $\phi = P_{\Gamma_i}u$, on each Γ_i, where P_{Γ_i} is the orthogonal L^2 projection from $L^2(\Gamma_i)$ to the restriction of S_h to Γ_i. By the hypothesis that Γ has no cross point, ϕ is well defined. Note that on each Γ_i we have

$$\|u - \phi\|_{L^2(\Gamma_i)} \le \|u - w_i\|_{L^2(\Gamma_i)}.$$

By Lemma 2.1,

$$\|u - w_i\|^2_{L^2(\Gamma_i)} \lesssim h^{-1}\|u - w_i\|^2_{L^2(\Omega_i)} + h\|u - w_i\|^2_{H^1(\Omega_i)}.$$

Hence,

$$\begin{aligned} h\|u - w_i\|^2_{L^2(\Gamma_i)} &\lesssim \|u - w_i\|^2_{L^2(\Omega_i)} + h^2\|u - w_i\|^2_{H^1(\Omega_i)} \\ &\lesssim h^2|u|^2_{H^1(\Omega_i)}. \end{aligned}$$

Consequently,

$$\begin{aligned} h\|u - P_{\Gamma}u\|^2_{L^2_{\omega}(\Omega)} &\lesssim h\sum_{i=1}^{J}\omega_i\|u - \phi\|^2_{L^2(\Gamma_i)} \\ &\lesssim h^2\sum_{i=1}^{J}\omega_i|u|^2_{H^1(\Omega_i)} = h^2|u|^2_{H^1_{\omega}(\Omega)}. \end{aligned}$$

Applying Lemma 4.2 gives (4.6).

The proof of (4.7) is identical to that of (3.11). This completes the proof. □

4.2. General case. When the interface has some internal cross points, the problem becomes somewhat more subtle. We will derive certain estimates under some special circumstances.

4.2.1. *Estimates for "finer" finite element functions.* For $d = 2$, the embedding $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ is not true in general, but it is "almost right" for the functions in finite element subspaces, as is indicated by the second inequality in Lemma 2.3. This observation is the main motivation for the result in this subsection, and the argument is similar to that used in the proof of Proposition 4.1.

Lemma 4.4. *For any* $u \in S_{\underline{h}}$ *and* $\tau \in \mathcal{T}_h$,

$$\|(I - I_h)u\|_{L^2(\tau)} \lesssim \begin{cases} h\left(\log\frac{h}{\underline{h}}\right) 1/2|u|_{H^1(\tau)} & \text{if } d = 2, \\ h\left(\frac{h}{\underline{h}}\right)^{1/2}|u|_{H^1(\tau)} & \text{if } d = 3, \end{cases}$$

where $I_h\colon S_{\underline{h}} \mapsto S_h$ *is the interpolation operator.*

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Proof. We first consider the case that $d = 2$. Let $\hat{\tau}$ be the standard reference element; then

$$\|(I - I_h)u\|_{L^2(\tau)} \lesssim h\|(\hat{I} - \hat{I}_h)\hat{u}\|_{L^2(\hat{\tau})}.$$

It follows from the discrete Sobolev inequality in Lemma 2.3 for $d = 2$ that

$$\|(\hat{I} - \hat{I}_h)\hat{u}\|_{L^2(\hat{\tau})} \le 2\|\hat{u}\|_{L^\infty(\hat{\tau})} \lesssim \left(\log \frac{h}{\underline{h}}\right)^{1/2} \|\hat{u}\|_{H^1(\hat{\tau})}.$$

Replacing $\hat{u}$ by $\hat{u} + c$ for any constant c, we have

$$\begin{aligned}\|(\hat{I} - \hat{I}_h)\hat{u}\|_{L^2(\hat{\tau})} &\lesssim \left(\log \frac{h}{\underline{h}}\right)^{1/2} \inf_{c\in\mathbf{R}^1} \|\hat{u} + c\|_{H^1(\hat{\tau})} \\ &\lesssim \left(\log \frac{h}{\underline{h}}\right)^{1/2} |\hat{u}|_{H^1(\hat{\tau})} \lesssim \left(\log \frac{h}{\underline{h}}\right)^{1/2} |u|_{H^1(\tau)}.\end{aligned}$$

The desired result for $d = 2$ then follows. The proof for $d = 3$ only differs in the type of Sobolev inequality (in (2.1)) used, and the details obviously need not be repeated here. This completes the proof. □

As a direct consequence of Lemma 4.4, we have

Theorem 4.5. *For any* $u \in S_{\underline{h}}$,

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim \begin{cases} h\left(\log \frac{h}{\underline{h}}\right)^{1/2} |u|_{H^1_\omega(\Omega)} & \text{if } d = 2, \\ h\left(\frac{h}{\underline{h}}\right)^{1/2} |u|_{H^1_\omega(\Omega)} & \text{if } d = 3 \end{cases}$$

and

$$|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim \begin{cases} \left(\log \frac{h}{\underline{h}}\right)^{1/2} |u|_{H^1_\omega(\Omega)} & \text{if } d = 2, \\ \left(\frac{h}{\underline{h}}\right)^{1/2} |u|_{H^1_\omega(\Omega)} & \text{if } d = 3. \end{cases}$$

4.2.2. *Estimates for general* H^1 *functions.* In this subsection, we shall derive some estimates for functions in H^1.

The following lemma shows that nearly optimal estimates can be obtained in general if the full weighted H^1 norms are used.

Lemma 4.6. *For all* $u \in H_0^1(\Omega)$,

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h|\log h|^{1/2}\|u\|_{H^1_\omega(\Omega)}. \tag{4.8}$$

Proof. The proof will be carried out separately for different dimensions, even though the ideas in both cases are quite similar.

Case 1 : $d = 2$. Let w_i be as in the proof of Lemma 4.2 and define $w \in S_h$ by

$$w = \begin{cases} w_i, & \text{at the nodes inside } \Omega_i, \\ P_e u, & \text{at the nodes inside } e \subset \partial\Omega_i, \\ 0, & \text{at nodes elsewhere}, \end{cases}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

where $e \subset \partial\Omega_i$ is any edge of Ω_i and $P_e: L^2(e) \mapsto S_h(e)$ is the orthogonal $L^2(e)$ projection.

By (2.2),

$$\|w_i - w\|^2_{L^2(\Omega_i)} \lesssim h^2 \sum_{x\in\partial\Omega_i} (w_i - w)^2(x) \lesssim h^2 \sum_{e\subset\partial\Omega_i}\sum_{x\in e}(w_i - w)^2(x)$$
$$\lesssim h^2 \sum_{e\subset\partial\Omega_i}\left(\sum_{x\in e}(w_i - P_e u)^2(x) + \sum_{x\in\partial e} w_i^2(x)\right).$$

We need to bound the two terms appearing in the last expression above. The first term is easy:

$$\sum_{e\subset\partial\Omega_i} h\|w_i - P_e u\|^2_{L^2(e)} \lesssim h\|u - w_i\|^2_{L^2(\partial\Omega_i)}$$
$$\lesssim \|u - w_i\|^2_{L^2(\Omega_i)} + h^2\|u - w_i\|^2_{H^1(\partial\Omega_i)}$$
$$\lesssim h^2\|u\|^2_{H^1(\Omega_i)},$$

where we have used Lemma 2.1 and (4.5).

The second term can be bounded by the second discrete Sobolev inequality in Lemma 2.3:

$$\sum_{e\subset\partial\Omega_i} h^2 \sum_{x\in\partial e} w_i^2(x) \lesssim h^2|\log h|\,\|w_i\|^2_{H^1(\Omega_i)} \lesssim h^2|\log h|\,\|u\|^2_{H^1(\Omega_i)}.$$

Consequently,

$$\|w - w_i\|_{L^2(\Omega_i)} \lesssim h|\log h|^{1/2}\|u\|_{H^1(\Omega_i)}.$$

Applying the above estimate, with the triangle inequality and (4.5), we get

$$\|u - w\|_{L^2(\Omega_i)} \le \|u - w_i\|_{L^2(\Omega_i)} + \|w_i - w\|_{L^2(\Omega_i)} \lesssim h|\log h|^{1/2}\|u\|_{H^1(\Omega_i)}. \tag{4.9}$$

The desired result for $d = 2$ then follows.

Case 2 : $d = 3$. Again, let ω_i be as in the proof of Lemma 4.2 and define $w \in S_h$ by

$$w = \begin{cases} w_i, & \text{at the nodes inside } \Omega_i, \\ P_F u, & \text{at the nodes inside } F \subset \partial\Omega_i, \\ 0, & \text{at nodes elsewhere}, \end{cases}$$

where $F \subset \partial\Omega_i$ is any face of Ω_i and $P_F: L^2(F) \mapsto S_h(F)$ is the orthogonal $L^2(F)$ projection. Then

$$\|w_i - w\|^2_{L^2(\Omega_i)} \lesssim h^3 \sum_{x\in\partial\Omega_i} (w_i - w)^2(x) \lesssim h^3 \sum_{F\subset\partial\Omega_i}\sum_{x\in F}(w_i - w)^2(x)$$
$$\lesssim h^3 \sum_{F\subset\partial\Omega_i}\left(\sum_{x\in F}(w_i - P_F u)^2(x) + \sum_{x\in\partial F} w_i^2(x)\right)$$
$$\lesssim \sum_{F\subset\partial\Omega_i}(h\|w_i - P_F u\|^2_{L^2(F)} + h^2\|w_i\|^2_{L^2(\partial F)}).$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Again, we need to bound two terms appearing in the last expression above. For the first term, we have

$$\begin{aligned}\sum_{F\subset\partial\Omega_i} h\|w_i - P_F u\|^2_{L^2(F)} &\lesssim h\|u - w_i\|^2_{L^2(\partial\Omega_i)}\\ &\lesssim \|u - w_i\|^2_{L^2(\Omega_i)} + h^2\|u - w_i\|^2_{H^1(\partial\Omega_i)}\\ &\lesssim h^2\|u\|^2_{H^1(\Omega_i)},\end{aligned}$$

where we have used Lemma 2.1 and (4.5).

The second term can be bounded by the discrete Sobolev inequality in Lemma 2.4:

$$\sum_{F\subset\partial\Omega_i} h^2\|w_i\|^2_{L^2(\partial F)} \lesssim h^2|\log h|\,\|u\|^2_{H^1(\Omega_i)} \lesssim h^2|\log h|\,\|u\|^2_{H^1(\Omega_i)}.$$

Consequently,

$$\|w - w_i\|_{L^2(\Omega_i)} \lesssim h|\log h|^{1/2}\|u\|_{H^1(\Omega_i)}.$$

As in (4.9), the estimate for $d = 3$ follows. This completes the proof. □

As a direct consequence of the above lemma, we have

Theorem 4.7. *If for all i, the $(d-1)$-dimensional Lebesgue measure of $(\partial\Omega_i \cap \partial\Omega)$ is positive, then for all $u \in H_0^1(\Omega)$*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h|\log h|^{1/2}|u|_{H^1_\omega(\Omega)} \tag{4.10}$$

and

$$|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim |\log h|^{1/2}|u|_{H^1_\omega(\Omega)}. \tag{4.11}$$

Proof. By hypothesis, we have $\|u\|_{H^1_\omega(\Omega)} \lesssim |u|_{H^1_\omega(\Omega)}$ by the Poincaré inequality. The estimate (4.10) then follows from (4.8). The estimate (4.11) can be proved similarly as (3.11) by using (4.10). □

Remark 4.1. The assumption concerning the measure of $\mathrm{meas}_{d-1}(\partial\Omega_i \cap \partial\Omega)$ in Theorem 4.7 cannot be removed, in general, and the deterioration $\underline{h}^{-1/2}$ in the estimate of Theorem 4.5 is also best possible. All these issues will be discussed in a separate paper.

Bibliography

1. R. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
2. R. E. Bank and T. F. Dupont, *An optimal order process for solving elliptic finite element equations*, Math. Comp. **36** (1981), 967–975.
3. J. Bramble, *A second order finite difference analog of the first biharmonic boundary value problem*, Numer. Math. **9** (1966), 236–249.
4. J. Bramble, J. Pasciak, J. Wang, and J. Xu, *Convergence estimates for multigrid algorithms without regularity assumptions*, Math. Comp. **57** (1991) (to appear).
5. P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, New York, 1978.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

6. M. Crouzeix and V. Thomée, *The stability in* L_p *and* W_p^1 *of the* L_2*-projection onto finite element function spaces*, Math. Comp. **48** (1987), 521–532.

7. J. Mandel, S. F. McCormick, and R. Bank, *Variational multigrid theory*, Multigrid Methods, (S. McCormick, ed.), SIAM, Philadelphia, 1988, pp. 131–178.

8. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483–493.

9. J. Xu, *Error estimates and improved algorithms for infinite elements*, Proc. DD5 Internat. Conf., Beijing, China, 1984.

10. ——, *Theory of multilevel methods*, Ph.D. thesis, Cornell, 1989; AM report 48, Dept. of Math., Penn State Univ., July 1989.

11. ——, *Iterative methods by space decomposition and subspace correction: a unifying approach* (in preparation).

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853
E-mail address: bramble@mssun7.msi.cornell.edu

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802
E-mail address: xu@math.psu.edu

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use