A HAMILTONIAN APPROXIMATION TO SIMULATE SOLITARY WAVES OF THE KORTEWEG-DE VRIES EQUATION

MINGYOU HUANG

Abstract. Given the Hamiltonian nature and conservation laws of the Korteweg-de Vries equation, the simulation of the solitary waves of this equation by numerical methods should be effected in such a way as to maintain the Hamiltonian nature of the problem. A semidiscrete finite element approximation of Petrov-Galerkin type, proposed by R. Winther, is analyzed here. It is shown that this approximation is a finite Hamiltonian system, and as a consequence, the energy integral

\[ I(u) = \int_0^1 \left( \frac{u_x^2}{2} + u^3 \right) \, dx \]

is exactly conserved by this method. In addition, there is a discussion of error estimates and superconvergence properties of the method, in which there is no perturbation term but instead a suitable choice of initial data. A single-step fully discrete scheme, and some numerical results, are presented.

1. The Hamiltonian nature and conservation laws

In this paper, we shall consider the following problem for the Korteweg-de Vries equation:

\[ u_t - 6u u_x + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \]

\[ u(x+1, t) = u(x, t), \]

\[ u(x, 0) = u_0(x) \] (a prescribed 1-periodic function).

To study the Hamiltonian nature of problem (P), we introduce the following function space with \( I = [0, 1], \)

\[ H^m_p = \{ v \in H^m(I); \; v^{(i)}(x+1) = v^{(i)}(x), \; i = 0, 1, \ldots, m - 1 \}, \]

and the functional

\[ H(u) = \int_0^1 \left( \frac{u_x^2}{2} + u^3 \right) \, dx, \]
where \( u^{(i)} = \partial^i u/\partial x^i \). Define

\[
\delta/\delta u := \sum_{k=0}^{\infty} (-1)^k (d/dx)^k \partial/\partial u^{(k)};
\]

then \( \delta H/\delta u = 3u^2 - u_{xx} \), and problem (P) is equivalent to finding a map \( u(t) \) from \( \mathbb{R}^+ \) to \( H_p^m \) such that

\[
(P') \quad u_t = J\delta H/\delta u, \quad J = \partial/\partial x.
\]

Since

\[
\int_0^1 \frac{\delta H}{\delta u} \frac{\partial}{\partial x} \left( \frac{\delta H}{\delta u} \right) dx = 0, \quad u \in H_p^m,
\]

then for any solution \( u = u(t) \) of (P') we have

\[
\frac{dH(u)}{dt} = \int_0^1 \frac{\delta H}{\delta u} \frac{\partial}{\partial t} \left( \frac{\delta H}{\delta u} \right) dx = \int_0^1 \frac{\delta H}{\delta u} \frac{\partial}{\partial x} \frac{\delta H}{\delta u} dx = 0,
\]

i.e., \( u = u(t) \) satisfies the energy conservation law: \( H(u(t)) = \text{const.} \).

For any functionals \( T \) and \( S: H_p^m \to \mathbb{R} \), define

\[
\{T, S\} := \int_0^1 \frac{\delta T}{\delta u} \frac{\partial}{\partial x} \frac{\delta S}{\delta u} dx \quad \text{(Poisson bracket),}
\]

which also is a functional defined on \( H_p^m \). It can be verified that the operation \( \{\ ,\ \} \) has the following properties:

(i) \( \{T, S\} = -\{S, T\}, \quad T, S: H_p^m \to \mathbb{R} \);
(ii) \( \{H, aT + bS\} = a\{H, T\} + b\{H, S\}, \quad a, b \in \mathbb{R}, \ H, T, S: H_p^m \to \mathbb{R} \);
(iii) (Jacobi identity) \( \{\{T, S\}, H\} + \{\{S, H\}, T\} + \{\{H, T\}, S\} = 0, \quad H, T, S: H_p^m \to \mathbb{R} \).

Lemma 1. The functional \( T(u) \) is a first integral of problem (P') if and only if \( \{T, H\} = 0 \).

Proof. Since, for any solution \( u = u(t) \) of (P'),

\[
\frac{dT(u)}{dt} = \int_0^1 \frac{\delta T}{\delta u} \frac{\partial}{\partial t} \left( \frac{\delta H}{\delta u} \right) dx = \int_0^1 \frac{\delta T}{\delta u} \frac{\partial}{\partial x} \frac{\delta H}{\delta u} dx = \{T, H\},
\]

the lemma follows immediately from this identity. \( \square \)

For a given functional \( H: H_p^m \to \mathbb{R} \), a family of mappings \( G^t_H \) containing a parameter \( t \) can be determined through (P'):

\[
u(t) = G^t_H u_0, \quad u_0 \in H_p^m,
\]

which is called the phase flow corresponding to \( H \). By Lemma 1 and the Jacobi
identity, we have

**Theorem 1.** Suppose $T$ and $S$ are two first integrals of $(P')$. Then $(T, S)$ is also a first integral of $(P')$. Therefore, the set of functionals consisting of all first integrals of $(P')$, equipped with the operation $(\cdot, \cdot)$, forms a Lie algebra $RH$.

Let $L_u = -\partial^2/\partial x^2 + u$ (Schrödinger's operator). P. D. Lax proved in [5] that every eigenvalue $\lambda = \lambda(u)$ of the Sturm-Liouville problem $L_u f = \lambda f$ is a first integral of $(P')$, i.e., $\lambda(u) \in RH$. In fact, $(P')$ has infinitely many first integrals, such as

$$I_0(u) = \int_0^1 u \, dx, \quad I_1(u) = \int_0^1 u^2 \, dx, \quad I_2(u) = \int_0^1 \left( \frac{u^4}{2} + u^3 \right) \, dx, \ldots$$

From the form $(P')$ and the properties indicated above we see that problem $(P)$ is of the same nature as a Hamiltonian system of ordinary differential equations (see [1, Chapter 8]), which can be viewed as an infinite-dimensional Hamiltonian system. For a given functional $H: H^m_p \to \mathbb{R}$, we call $\partial H/\partial u$ the velocity vector of the phase flow $G^t_H$ with Hamiltonian function $H$. For any $I_z \in RH$, the phase flow determined by the equation $u_t = \partial H/\partial u$ commutes with $G^t_H$, i.e., $G^t_H G^t_z = G^t_z G^t_H$.

**2. The Hamiltonian approximation of problem $(P)$**

In this paper we seek to develop a numerical method for simulating the solitary waves of the Korteweg-de Vries equation which maintains the Hamiltonian nature of this equation. We believe that such a method will be able to preserve as much as possible the global properties of the original problem, for example, the energy conservation property

$$\frac{dH(u)}{dt} = \frac{d}{dt} \int_0^1 \left( \frac{u^2}{2} + u^3 \right) \, dx = 0,$$

which we consider to be particularly important. As is known, the conventional finite difference method (see [7]) and the Galerkin finite element method (see [8]) do not preserve the energy. In this section, we shall show that the Petrov-Galerkin finite element discretization is an appropriate way to derive a numerical method for problem $(P)$ which faithfully preserves the Hamiltonian nature and the energy conservation property of the continuous problem.

Let $L_h^0: 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of the interval $I = [0, 1]$, $I_j = [x_{j-1}, x_j]$, and $h = \max_{1 \leq j \leq N} (x_j - x_{j-1})$. For a given integer $r \geq 2$, we introduce the spaces

$$V_h = \{ v \in H^1_p; \ v|_{I_j} \in P_r(I_j), \ j = 1, 2, \ldots, N \},$$

$$H_h = \{ w \in H^2_p; \ w|_{I_j} \in P_{r+1}(I_j), \ j = 1, 2, \ldots, N \},$$

where $P_r(I_j)$ represents the set of all polynomials on $I_j$ with degree $< r$. It is easy to see that $\dim V_h = \dim H_h = (r - 1)N$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Based on the chosen pair of spaces $V_h$ and $H_h$, the Petrov-Galerkin finite element approximation of problem (P) is defined as follows: find a map $u^h(t)$ from $\mathbb{R}^+$ to $V_h$ such that

$$\begin{align*}
(P_h) \quad & (u_t^h, w^h) + 3((u^h)^2, w^h_x) + (u^h, w_{xx}^h) = 0 \quad \forall w^h \in H_h.
\end{align*}$$

Here and hereafter, $(\ , \ )$ and $\| \cdot \|$ stand for the inner product and the norm in $L_2(I)$, respectively.

For the purpose of the subsequent analysis, we introduce a linear integration operator $G: H_p^m \to H_p^{m+1}$ uniquely determined by

$$\begin{align*}
(2.1) \quad & (Gf)_x = f - f^0, \quad (Gf)^0 = f^0, \quad f \in H_p^m,
\end{align*}$$

where $f^0 = (f, 1)$ is the mean value of $f$ on the interval $I$. In fact, $Gf$ has the following explicit form:

$$(Gf)(x) = \int_0^x f(s) \, ds - f^0 x + \frac{3}{2} f^0 - \int_0^1 \int_0^x f(s) \, ds \, dx.$$ 

From the definition of $G$, we see that

$$\begin{align*}
(2.2) \quad & (Gf_1, f_2) = (Gf_1, (Gf_2)_x) + f_1^0 f_2^0, \\
(2.3) \quad & (Gf, f) = (f_0)^2.
\end{align*}$$

Moreover, with $\overset{\circ}{H}_p^m = \{ v \in H_p^m \mid v^0 = (v, 1) = 0 \}$ and $\overset{\circ}{V}_h = V_h \cap \overset{\circ}{H}^1_p$, we have

$$\begin{align*}
(2.4) \quad & (Gf_1, f_2) = (Gf_1, (Gf_2)_x) = -(f_1, Gf_2) \quad \text{for any } f_1, f_2 \in \overset{\circ}{H}_p^m,
\end{align*}$$

i.e., $G$ is a skewsymmetric operator on $\overset{\circ}{H}_p^m$. It can be verified that $G$ is a one-to-one map from $\overset{\circ}{H}_p^m$ onto $\overset{\circ}{H}_p^{m+1}$, and its inverse is precisely the differential operator $J = \partial / \partial x$.

**Theorem 2.** The solution $u^h = u^h(t)$ of the semidiscrete problem (P$_h$) satisfies the following conservation laws:

\[ (i) \quad I_0(u^h(t)) = \int_0^1 u^h \, dx = \text{const} \quad \text{for } t \geq 0, \]

\[ (ii) \quad I_2(u^h(t)) = \int_0^1 \left( \frac{(u_x^h)^2}{2} + (u^h)^3 \right) \, dx = \text{const} \quad \text{for } t \geq 0. \]

**Proof.** Since $1 \in H_h$, by choosing $w^h = 1$ in (P$_h$), we have

$$\frac{d}{dt} \int_0^1 u^h \, dx = \frac{d}{dt} (u^h, 1) = (u_t^h, 1) = 0,$$

so that (i) holds. To verify (ii), we choose $w^h = Gu_t^h \in H_h$; then

$$\begin{align*}
(u_t^h, Gu_t^h) + 3((u^h)^2, (Gu_t^h)_x) + (u^h, (Gu_t^h)_{xx}) = 0.
\end{align*}$$
Since \((u_t^h)^0 = (u_t^h, 1) = 0\), \((u_t^h, Gu_t^h) = 0\), and \((Gu_t^h)_x = u_t^h\), because of (2.1) and (2.4), we obtain from the above equation
\[
\frac{d}{dt} I_2(u_t^h(t)) = \frac{d}{dt} \left\{ ((u_t^h)^3, 1) + \frac{1}{2}(u_t^h, u_t^h_x) \right\} = 0,
\]
i.e., (ii) holds, and the theorem is proved. □

Theorem 2 tells us that the conservation laws \(I_0 = \text{const} \) and \(I_2 = \text{const} \) of problem (P) mentioned in §1 are faithfully preserved by the Petrov-Galerkin finite element approximation \((P_h)\), where \(I_2 = H\) represents the energy of the continuous system (P).

It is not difficult to see that the discrete problem \((P_h)\) is a system of ordinary differential equations. After some careful manipulations, we find that \((P_h)\) is precisely a finite Hamiltonian system. To show this, we introduce a kind of second-order discrete derivative \(d_{xx}^h u^h \in V_h\) for any given function \(u^h \in V_h\), which is uniquely determined by
\[
(d_{xx}^h u^h, v^h) = -(u_x^h, u_x^h), \quad \forall v^h \in V_h.
\]
By choosing \(v^h = 1\), we see that \((d_{xx}^h u^h, 1) = 0\), i.e., \(d_{xx}^h u^h \in \hat{V}_h = V_h \cap \hat{H}_p^1\).

Now let \(u^h = u^h(t)\) be a solution of problem \((P_h)\). Since \(d_{xx}^h u^h, u_t^h \in \hat{V}_h\), by using (2.1) and (2.2), equation \((P_h)\) can be rewritten in the form
\[
(Gu_t^h, v^h) - 3((u_t^h)^2, v^h) + (d_{xx}^h u^h, v^h) = 0, \quad v^h \in \hat{V}_h.
\]
In addition, let \(P_0\) be the \(L_2\) projector from \(L_2(I)\) into its subspace \(\hat{V}_h\), and let \(G_h := P_0G\); then for any \(f^h, g^h \in \hat{V}_h\),
\[
(G_h f^h, g^h) = (P_0 G f^h, g^h) = (G f^h, g^h) = -(f^h, G g^h) = -(f^h, G_h g^h),
\]
which shows that \(G_h\) is a skewsymmetric operator on \(\hat{V}_h\). In terms of these notations, we find that (2.5) is equivalent to
\[
G_h(u^h)_t = 3P_0(u_t^h)^2 - d_{xx}^h u^h.
\]
It can be verified by calculation that \(3P_0(u_t^h) - d_{xx}^h u^h = \delta H(u^h)/\delta u^h\). Therefore, the solution \(u^h(t)\) of \((P_h)\) satisfies
\[
G_h(u^h)_t = \delta H(u^h)/\delta u^h.
\]
Assume that \(P_0 H_h = \hat{V}_h\); then \(G_h\) restricted to \(\hat{V}_h\) is a one-to-one mapping, and the inverse \(G_h^{-1} = J_h\) exists, which also is a skewsymmetric operator on \(\hat{V}_h\). We thus obtain a new version of \((P_h)\),
\[
(u^h)_t = J_h \delta H(u^h)/\delta u^h.
\]
For any two functionals $T, S: V_h \rightarrow R$, a discrete analogue of the Poisson bracket, introduced in §1, can be defined by

$$\{T, S\} := \int_0^1 \frac{\delta T}{\delta u^h} J_h \frac{\delta S}{\delta u^h} \, dx,$$

and most of the analysis and conclusions in [5] can be carried over to the approximation problem $(P_h)$. Comparing the form (2.7) of problem $(P_h)$ with $(P')$, we see that the Hamiltonian nature of problem $(P)$ is maintained in the discrete approximation $(P_h)$. For this reason, we shall call $(P_h)$ a Hamiltonian approximation of problem $(P)$.

3. Error estimates and superconvergence of the approximate solution

The discrete approximation $(P_h)$ is identical to one of the methods proposed in [9], where $H^0$ and $H^1$ estimates for the error $e = u - u^h$ and its time derivative $e_t$ were derived. However, in the bound obtained for $e_t$ there exists an unknown term $\|Gw^h(0)\|_2$. In order to achieve superconvergence, D. N. Arnold and R. Winther in [2] altered the discrete equation by a perturbation term. In this section, we obtain superconvergence properties of the unperturbed equation $(P_h)$ by suitable choices of the initial data.

Since $G(H^1_p) = H^2_p$, and $G(V_h) = H_h$, problem $(P_h)$ can be formulated as follows: find a map $u^h(t): [0, T] \rightarrow V_h$ such that

$$-(Gu^h_t, v^h) + 3((u^h, v^h)^2) + a_0(u^h, v^h) = 0 \quad \forall v^h \in V_h,$$

where $a_0(u, v) = (u_x, v_x)$ and $u^h(0)$ assumes a prescribed value in $V_h$. In order to be sure that the problem has a unique solution, we assume $\mathcal{P}_0H_h = V_h$; then the coefficient matrix in front of the time derivative in (3.1) is nonsingular.

An elliptic projector $P^1: H^1_p \rightarrow V_h$ is defined by

$$a_0(P^1 \phi - \phi, v^h) = 0 \quad \text{for any } v^h \in V_h,$$

$$(P^1 \phi, 1) = (\phi, 1).$$

Let $u(t) = u(x, t)$ be the exact solution of $(P)$, which is assumed to be sufficiently smooth. From standard results for the Galerkin finite element method for elliptic equations, we know that

$$\|(P^1 u - u)(t)\|_s \leq C(u)h^{r-s} \quad -(r-2) \leq s \leq 1, \quad k \geq 0,$$

$$\|(P^1 u - u)(t)\|_{L_\infty(I)} \leq C(u)h^r,$$

where $\phi^{(k)}(t) = \left(\frac{d}{dt}\right)^k \phi(t)$. Moreover, the following superconvergence estimate at nodes holds (see [6]):

$$|(P^1 u - u)(x_i, t)| \leq C(u)h^{2r-2} \quad \text{when } r > 2.$$
Here and hereafter, $\| \cdot \|_s$ represents the norm in the Sobolev space $H^s(I)$, $s \geq 0$, and
\[
\| \cdot \|_{-s} = \sup_{0 \neq v \in H^s} \frac{\langle \cdot, v \rangle}{\|v\|_s}.
\]

In the subsequent analysis, we shall use the inverse properties of $\{V_h\}$, such as
\[
\|v^h\|_1 \leq C h^{-1} \|v^h\| \quad \forall v^h \in V_h.
\]

It is well known that such properties can be guaranteed by assuming the family $\{L_h, h > 0\}$ of partitions to be quasi-uniform, i.e., there is a constant $c > 0$ such that $h_j = x_j - x_{j-1} \geq ch$ for $1 \leq j \leq N$.

To begin with, we discuss the case $u^h(0) = P_1 u(0)$ and prove the following pointwise error estimates.

**Theorem 3.** Suppose that (P) has a unique solution $u(t)$ for $0 \leq t \leq T$, $u(t)$ is sufficiently smooth, and $\{L_h, h > 0\}$ is quasi-uniform. Assume $u^h(0) = P_1 u(0)$.

Then for small $h > 0$, the discrete problem $\text{(PA)}$ has a unique solution $u^h(t)$, $0 \leq t \leq T$, which satisfies
\[
(3.5) \quad \|u(t) - u^h(t)\|_{L^\infty(I)} \leq C(u) h^r,
\]
\[
(3.6) \quad |u(x_i, t) - u^h(x_i, t)| \leq C(u) h^{r+d}, \quad i = 1, 2, \ldots, N,
\]
where $d = 0$ for $r = 2$, and $d = 1$ for $r > 2$.

**Proof.** Set $z(t) = u(t) - P_1 u(t)$ and $w^h(t) = P_1 u(t) - u^h(t)$. Then $e(t) = u(t) - u^h(t) = z(t) + w^h(t)$, where $w^h(t) \in V_h$ satisfies
\[
(3.7) \quad -(Gw^h_t, v^h) + a_0(w^h, v^h) = (Gzt, v^h) + 3((u^h)^2 - u^2, v^h) \quad \forall v^h \in V_h.
\]

Since $(Gw^h_t, w^h) = 0$, choosing $v^h = w^h_i$ in (3.7) yields
\[
\frac{1}{2} \frac{d}{dt} \|w^h_x\|^2 = (Gz_t, w^h_i) + 3((u^h)^2 - u^2, w^h_i).
\]

Noting that $(u^h)^2 - u^2 = (w^h)^2 - 2(P_1 u)w^h - (P_1 u + u)z$, we have
\[
\frac{1}{2} \frac{d}{dt} \|w^h_x\|^2 = \frac{d}{dt} (Gz_t, w^h) - (Gz_{tt}, w^h)
\]
\[
+ \frac{d}{dt} [(w^h)^3, 1] - 3((P_1 u)w^h, w^h) - 3((u^h)^2 - u^2, w^h)]
\]
\[
+ 3((P_1 u)_t w^h, w^h) + 3((u^h)^2 - u^2)_t - (P_1 u + u)z_t + (P_1 u + u)_t z + (P_1 u + u)_t z_t.
\]

Without loss of generality we may assume
\[
\|w^h(t)\|_1 \leq 1 \quad \text{for} \ 0 \leq t \leq T.
\]

In fact, this assumption can be removed by the later estimates combined with the inverse inequalities in $V_h$ (see [8]). By the smoothness of $u(t)$ and estimate...
(3.2), \|P_1u\|_1$ and \|P_1u_t\|_1 are uniformly bounded for $0 < h < h_0$ in $0 \leq t \leq T$. Note that $w^h(0) = 0$ by the choice of $u^h(0)$. Integrating (3.8) from $0$ to $t$, we obtain in the usual way

$$
\|w^h(t)\|^2 \leq C \left\{ \|z(t)\|^2 + \|z^{(1)}(t)\|^2 + \|Gw^h(t)\|^2 + \int_0^t \|z(s)\|^2 + \|z^{(1)}(s)\|^2 + \|z^{(2)}(s)\|^2 ds \right\},
$$

(3.9)

where $C$ is a constant which does not depend on $h$, but depends on $u$ and its derivatives.

To derive an estimate for $Gw^h(t)$, we choose $v^h = P_0Gw^h$ in (3.7) and obtain

$$
\frac{1}{2} \frac{d}{dt} \|P_0Gw^h\|^2 = a_0(w^h, P_0Gw^h) - (Gz_t, P_0Gw^h) - 3((u^h)^2 - u^2, P_0Gw^h)
$$

\leq C(\|z\|^2 + \|z^{(1)}\|^2 + \|Gw^h\|^2).

Thus, by integration we have

$$
\|P_0Gw^h(t)\|^2 \leq C \int_0^t \|z(s)\|^2 + \|z^{(1)}(s)\|^2 + \|Gw(s)\|^2 ds
$$

and

$$
\|Gw^h(t)\|^2 \leq 2\|P_0Gw^h(t)\|^2 + 2\|(I - P_0)Gw^h(t)\|^2
$$

(3.10)

\leq C \left\{ h^4 \|Gw^h(t)\|^2 + \int_0^t \|z(s)\|^2 + \|z^{(1)}(s)\|^2 + \|Gw^h(s)\|^2 ds \right\}.

Since $\|w^h\|^2 \leq \|Gw^h\|^2 + \|w^h_x\|^2 + \|Gw^h\|^2$, combining (3.9) and (3.10) and applying Gronwall's lemma, we find for $h > 0$ small enough,

$$
\|Gw^h(t)\|^2 \leq C \left\{ \|z(t)\|^2 + \|z^{(1)}(t)\|^2 + \int_0^t \|z(s)\|^2 + \|z^{(1)}(s)\|^2 + \|z^{(2)}(s)\|^2 ds \right\},
$$

which shows by (3.2) that

$$
\|Gw^h(t)\|^2 \leq C(u)h^{r+d},
$$

(3.11)

where $d = 0$ for $r = 2$, and $d = 1$ for $r > 2$. In view of

$$
\|w^h(t)\|_{L_\infty(t)} \leq C\|Gw^h(t)\|_2,
$$

the desired estimates (3.5) and (3.6) can be derived from (3.11) combined with (3.3) and (3.4), respectively. \(\square\)
From estimate (3.6), we see that the approximate solution has a superconvergence property at the nodes, with one order higher when \( r > 2 \). Following a referee’s suggestion, we now improve this result. We shall use the technique of quasi-projection, introduced in [3] for linear second-order parabolic and hyperbolic equations. In [2], quasi-projection was used for the Korteweg-de Vries equation. Since we intend to conserve the energy integral and the Hamiltonian nature, we use this technique only for choosing a suitable initial data, unlike [2], where the discrete equation is altered.

Set \( V(t) = P_1 u(t) \), \( Z_0(t) = u(t) - V(t) \), and \( W^h_0(t) = V(t) - u^h(t) \). The quasi-projections \( Z_j(t) : [0, T] \to \hat{V}_h \), \( j = 1, 2, \ldots \), are defined inductively by

\[
a_0(Z_j, v^h) = \langle GZ_j, v^h \rangle - 6uZ_{j-1}, v^h \quad \forall v^h \in \hat{V}_h, \quad 0 \leq t \leq T.
\]

We shall use the sum \( Z_1(0) + Z_2(0) + \cdots + Z_m(0) \) to modify the previous initial data \( V(0) = P_1 u(0) \), i.e., we choose \( u^h(0) = V(0) - [Z_1(0) + Z_2(0) + \cdots + Z_m(0)] \), where \( m = \lceil (r - 1)/2 \rceil \).

The improved superconvergence result is then as follows:

**Theorem 4.** Assume (P) and \( \{L_h, h > 0\} \) to be as in Theorem 3 and \( u^h(0) = V(0) - [Z_1(0) + Z_2(0) + \cdots + Z_m(0)] \), \( m = \lceil (r - 1)/2 \rceil \). Then for \( h > 0 \) small enough, the approximate solution \( u^h(t) \) satisfies

\[
|u(x_i, t) - u^h(x_i, t)| \leq C(u) h^{2r-2}, \quad i = 1, 2, \ldots, N.
\]

To illustrate, let \( r = 4 \); then \( m = 1 \) and \( u^h(0) = V(0) - Z_1(0) \). The calculation of \( u^h(0) \) requires three projections \( V(0), (Z_0)_t(0), \) and \( Z_1(0) \), where \( (Z_0)_t(0) = u_t(0) - V_t(0) \) and \( V_t(0) \) is a solution of

\[
a_0(\dot{V}_t(0), v^h) = \langle Gu_t(0) - 6u(0)u_t(0), v^h \rangle, \quad v^h \in \hat{V}_h.
\]

The extra cost spent on calculating \( V_t(0) \) and \( Z_1(0) \) will be compensated by a convergence rate of order \( O(h^6) \).

Now we sketch the proof of Theorem 4.

Let \( Z(t) = \sum_{j=0}^m Z_j(t) \) and \( W^h(t) = W^h_0(t) - \sum_{j=1}^m Z_j(t) \). Then

\[
e(t) = u(t) - u^h(t) = Z_0(t) + W^h_0(t) = Z(t) + W^h(t),
\]

where \( W^h_0(t), W^h(t) \in \hat{V}_h \). It is not difficult to see that \( W^h_0(t) \) and the sum of \( Z_j(t) \), \( j = 1, 2, \ldots, m \), satisfy respectively the following two equations,

\[
-(G(W^h_0)^{(1)} - 6uW^h_0, v^h) + a_0(W^h_0, v^h) = (GZ_0^{(1)} - 6uZ_0 + 3e^2, v^h)
\]

and

\[
-(G\left( \sum_{j=1}^m Z_j \right)^{(1)} - 6u \sum_{j=1}^m Z_j, v^h) + a_0\left( \sum_{j=1}^m Z_j, v^h \right) = (GZ_0^{(1)} - 6uZ_0, v^h) - (GZ_m^{(1)} - 6uZ_m, v^h).
\]
Thus, by subtraction we derive an equation for $W^h(t)$,

$$-(G(W^h)^{(1)} - 6uW^h, v^h) + a_0(W^h, v^h) = (GZ_m^{(1)} - 6uZ_m + 3e^2, v^h).$$

By the assumption on $u^h(0)$, we have $W^h(0) = 0$.

The proof of (3.12) consists of estimating $Z(t)$ and $W^h(t)$.

Lemma 2. Let $s \geq -1$ and $k, j \geq 0$ be integers such that $2j + s \leq r - 2$. Then

$$|Z_j(x_i, t)| \leq C(u)h^{2r-2}, \quad j = 1, 2, \ldots, m; \quad i = 1, 2, \ldots, N.$$

These estimates may be proved by an argument as in [2] or [3], with some obvious changes.

The next step is to show

$$||W^h(t)||_1 \leq C(u)h^{2r-2}, \quad 0 \leq t \leq T.$$

Then the proof of (3.12) will be completed by (3.4), (3.15), and (3.16). We first choose $v^h = (W^h)_t$ in (3.13) to obtain

$$\frac{1}{2} \frac{d}{dt} ||W_x^h||^2 = -3 \frac{d}{dt} (uW^h, W^h) + 3(u, W^h, W^h)$$

$$+ \frac{d}{dt} (GZ_m^{(1)} - 6uZ_m, W^h)$$

$$- (GZ_m^{(2)} - 6uZ_m^{(1)} - 6uZ, W^h) + 3(e^2, (W^h)_t).$$

In addition to (3.17), by choosing $v^h = P_0GW^h$ in (3.13) and integrating this equation from 0 to $t$, we get

$$||P_0GW^h(t)||$$

$$\leq C \int_0^t \left( ||W^h(s)||_1^2 + \sum_{k=0}^1 ||Z_m^{(k)}(s)||_1^2 \right) ds.$$

For lack of available bounds for $(W^h)_t$ and $e^h$, we treat the nonlinear term $3(e^2, (W^h)_t)$ of (3.17) in the following way:

$$3(e^2, (W^h)_t) = 3(Z^2 + 2ZW^h + (W^h)^2, (W^h)_t)$$

$$= \frac{d}{dt} [3(Z^2, W^h) + 3(ZW^h, W^h) + ((W^h)^3, 1)]$$

$$- 6(ZZ_t, W^h) - 3(Z_tW^h, W^h).$$

As in the proof of Theorem 3, we may assume $||W^h(t)||_1 \leq 1, 0 \leq t \leq T$; then
Integrating (3.17), we obtain
\[
\|W_x(t)\|^2 \leq C\left\{\|W(t)\|^2 + \sum_{k=0}^{1} \|Z_m^{(k)}(t)\|_{L^1}^2 + \|Z(t)\|^2 \|Z(t)\|_1^2 \right.
\]
\[
+ \int_0^t \left[\|W(s)\|_{L^2}^2 + \sum_{k=0}^{2} \|Z_m^{(k)}(s)\|_{L^2}^2 \right.
\]
\[
+ \|Z(s)\|^2 \|Z^{(1)}(s)\|^2 ds\left.\right\},
\tag{3.19}
\]
where \(\|Z_m^{(k)} W_h, W_h\| \leq C\|W_h\|^2, \ k = 0, 1,\) are implicitly used. Lemma 2
tells us that \(\|Z_m^{(k)}(t)\|_{L^1} \leq C h^{2r-2}\) and \(\|Z^{(k)}(t)\|_s \leq C h^{s-r},\) for \(k = 0, 1, 2,\)
\(s = 0, 1,\) and \(0 \leq t \leq T.\) Thus, by (3.19),
\[
\|W_x(t)\|^2 \leq C\left\{h^{2(2r-2)} + \|W_h(t)\|^2 + \int_0^t \|W_h(s)\|_{L^2}^2 ds\right\}.
\tag{3.20}
\]
Since [9] \(\|e(t)\|_s \leq C h^{s-r},\) \(s = 0, 1,\) and \(\|(I - P_0)GW_h(t)\| \leq C h^2\|W_h(t)\|_1,\)
we have by (3.18)
\[
\|GW_h(t)\|^2 \leq C\left\{h^4\|W_h(t)\|^2_1 + h^{2(2r-2)} + \int_0^t \|W_h(s)\|_{L^2}^2 ds\right\}.
\tag{3.21}
\]
Similar to the proof of (3.11), when \(h > 0\) is small enough, the desired estimate
(3.16) can be derived from (3.20), (3.21), and Gronwall's lemma. Thus, the
proof of Theorem 4 is complete.

4. Numerical results of simulating 1-solitary waves

A numerical experiment is performed for the following solitary wave of (P)
with initial data:
\[
u_0(x) = -(3d^2)^{-1}[1 + q(x)],\quad 0 \leq x \leq 1,
\]
\[
q(x) = q_0 + a \text{sech}^2(a/6d^2)(x - 0.5),
\]
\[
q_0 = -2d(a/24d^2)^{1/2} \text{tanh}(a/24d^2)^{1/2},
\]
where \(a = 0.2\) and \(d = 10^{-2}.\) Here, \(u_0(x)\) is extended as a 1-periodic function
to the whole real axis, and we denote the corresponding solution of (P) by
\(u(x, t);\) then \(q(x, s) = -1 - 3d^2 u(x, \frac{1}{2} d^2 s)\) solves the following equation:
\[
q_s + (1 + q)q_x + \frac{1}{2} d^2 q_{xxx} = 0.
\]

The solitary wave \(u(x, t)\) is simulated by means of the method (P_h) with
\(r = 2\) and uniform mesh \(x_j = jh, \ h = 1/47,\) while the approximate solution
\(u_h(t)\) is a piecewise linear function. Let \(\{q_j(x); \ j = 1, 2, \ldots, 47\}\) be the basis
of the subspace \(V_h,\) and
\[
u_h(x, t) = \sum_{j=1}^{47} u_j(t)q_j(x).
\]
Then it can be seen that \( \{u_j(t); j = 1, 2, \ldots, 47\} \) is the solution of the system of ordinary differential equations

\[
\sum_{j=1}^{47} a_{ij} \frac{du_j}{dt} - \frac{1}{h^3} (u_{i-1} - 2u_i + u_{i+1}) + \frac{1}{4h} (u_{i-1}^2 + 6u_i^2 + u_{i+1}^2) + \frac{1}{2h} (u_{i-1} - u_{i+1}) - \sum_{j=1}^{47} (u_j^2 + u_j u_{j+1} + u_{j+1}^2) = 0,
\]

(4.1)

where \( a_{ij} = (q_j, Gq_j)/h^2 \), and by the periodicity, \( u_0 = u_{47}, u_1 = u_{48} \).

We choose the time step \( \Delta t = 3.125 \times 10^{-7} \) and discretize (4.1) in the time variable by the midpoint rule; then a fully discrete scheme for (P) is obtained, namely

\[
\sum_{j=1}^{47} a_{ij} \frac{u_j^{n+1} - u_j^n}{\Delta t} = F_i \left( \frac{u_j^{n+1} + u_j^n}{2} \right), \quad i = 1, 2, \ldots, 47;
\]

(4.2)

\[
F_i(v) = \frac{1}{h^3} (v_{i-1} - 2v_i + v_{i+1}) - \frac{1}{4h} (v_{i-1}^2 + 6v_i^2 + v_{i+1}^2) - \frac{1}{2h} (v_{i-1}v_i - v_{i+1}) + \sum_{j=1}^{47} (v_j^2 + v_j v_{j+1} + v_{j+1}^2).
\]

As pointed out by Feng Kang in [4], the midpoint rule (i.e., the centered implicit Euler scheme) is a symplectic scheme, which behaves very well as far as preserving conservation laws is concerned.

Table 1 indicates the ability of scheme (4.2) to preserve the conservation laws \( I_i = \text{const}, i = 0, 1, 2, \) when this scheme is used to simulate the solitary waves of (P).

Figures 1–3 exhibit the shapes of solitary waves \( q(x, t) \) calculated by scheme (4.2) at time steps \( n = 0, 30, 60, \) respectively.

**Table 1**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( I_0(u) )</th>
<th>( I_1(u) )</th>
<th>( I_2(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3333.33333</td>
<td>11137605.2</td>
<td>-373082079 \times 10^2</td>
</tr>
<tr>
<td>30</td>
<td>-3333.33333</td>
<td>11137605.0</td>
<td>-373082041 \times 10^2</td>
</tr>
<tr>
<td>60</td>
<td>-3333.33448</td>
<td>11137624.0</td>
<td>-373082365 \times 10^2</td>
</tr>
<tr>
<td>90</td>
<td>-3333.33206</td>
<td>11138221.0</td>
<td>-373081466 \times 10^2</td>
</tr>
<tr>
<td>140</td>
<td>-3333.33251</td>
<td>11138159.6</td>
<td>-373081527 \times 10^2</td>
</tr>
<tr>
<td>190</td>
<td>-3333.33141</td>
<td>11138299.3</td>
<td>-373081693 \times 10^2</td>
</tr>
</tbody>
</table>
The shape of solitary wave $q(x, s)$ at time step $n = 0$

The shape of solitary wave $q(x, s)$ at time step $n = 30$

The shape of solitary wave $q(x, s)$ at time step $n = 60$

Bibliography


Department of Mathematics, Jilin University, Changchun 130023, People’s Republic of China