THE STRUCTURE OF MULTIVARIATE SUPERSPLINE SPACES OF HIGH DEGREE

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Abstract. We consider splines (of global smoothness $r$, polynomial degree $d$, in a general number $k$ of independent variables, defined on a $k$-dimensional triangulation $\mathcal{T}$ of a suitable domain $\Omega$) which are $r^{2^k-m-1}$-times differentiable across every $m$-face ($m = 0, \ldots, k-1$) of a simplex in $\mathcal{T}$. For the case $d > r^{2^k}$ we identify a structure that allows the construction of a minimally supported basis.

1. Introduction

A $\kappa$-simplex $K$ ($0 \leq \kappa \leq k$) is the convex hull of $\kappa + 1$ points in $\mathbb{R}^k$ called the vertices of $K$. $K$ is nondegenerate if its $\kappa$-dimensional volume is nonzero, and degenerate otherwise. The dimension of a nondegenerate $\kappa$-simplex is $\kappa$. The convex hull of a subset of $\mu + 1$ vertices of $K$ is a $\mu$-face of $K$.

Let $\mathcal{V} \subset \mathbb{R}^k$ be a given set of $N$ distinct points.

A triangulation $\mathcal{T}$ of the set $\mathcal{V}$ is a set of nondegenerate $k$-simplices satisfying the following requirements:

1. All vertices of each simplex in $\mathcal{T}$ are elements of $\mathcal{V}$.
2. The interiors of the simplices in $\mathcal{T}$ are pairwise disjoint.
3. The set
   \[
   \Omega := \bigcup_{K \in \mathcal{T}} K \subset \mathbb{R}^k
   \]
   is homeomorphic to $[0,1]^k$.
4. Each $(k-1)$-face of a simplex in $\mathcal{T}$ is either on the boundary of $\Omega$, or else is a common face of exactly two simplices in $\mathcal{T}$.
5. No simplex in $\mathcal{T}$ contains any points of $\mathcal{V}$ other than its vertices.

Note that a $\mu$-face of a simplex in $\mathcal{T}$ is itself a $\mu$-dimensional simplex. On the triangulation $\mathcal{T}$ we define a spline space $S'_d(\mathcal{T})$ as usual by

\[
S'_d(\mathcal{T}) = \{ s \in C'(\Omega) : s|_\tau \in \mathscr{P}_d^{k+d} \forall \tau \in \mathcal{T} \},
\]

where $\mathscr{P}_d^{k+d}$ is the $(k+d)$-dimensional linear space of all $k$-variate polynomials of total degree less than or equal to $d$.
In this paper, we consider subspaces of \( S_\Delta^0(\mathcal{T}) \) obtained by increasing the smoothness requirements across faces of the underlying simplices. More precisely, denoting by \( \mathcal{S}_\mu \) the set of all \( \mu \)-faces of the simplices in \( \mathcal{T} \) (\( \mu = 0, \ldots, k - 1 \)) and letting \( \mathcal{S} = \bigcup_{\mu=0}^{k-1} \mathcal{S}_\mu \), we define the (superspline) space \( S_\Delta^p(\mathcal{T}) \) as a subspace of \( S_\Delta^0(\mathcal{T}) \) as follows:

\[
S_\Delta^p(\mathcal{T}) = \{ s \in S_\Delta^0(\mathcal{T}) : s \text{ is } \rho\text{-times differentiable across } \sigma \text{ } \forall \sigma \in \mathcal{S} \},
\]

where \( \rho = r2^{k-\dim \sigma - 1} \).

The concept of supersplines was introduced in Chui and Lai [8], [9]. The area in between finite elements and full spline spaces was further explored by Schumaker [16] and Ibrahim and Schumaker [11].

2. The Generalized Bézier-Bernstein Form

Crucial to analyzing the dimension of spline spaces is the Bézier-Bernstein form of a multivariate polynomial. In the case \( k \leq 2 \) this form is used widely and is well known. A review of the Bézier-Bernstein form for a general number of variables is in de Boor [6]. In this paper, we use a notation that is particularly suitable for our purposes. However, generalized barycentric coordinates and global control nets have also been proposed in Alfeld [2] and de Boor [6].

We use \( \mathcal{Y} \) as an index set and denote by \( \mathbb{N} \) the set of nonnegative integers. For vectors \( \mathbf{I} = [i_v]_{v \in \mathcal{Y}} \in \mathbb{N}^\mathcal{Y} \) and \( \mathbf{a} = [a_v]_{v \in \mathcal{Y}} \in \mathbb{R}^\mathcal{Y} \) we define

\[
|\mathbf{I}| = \sum_{v \in \mathcal{Y}} i_v,
\]

\[
\mathbf{a}^\mathbf{I} = |\mathbf{I}|! \prod_{v \in \mathcal{Y}} a_v^{i_v} / \prod_{v \in \mathcal{Y}} i_v !,
\]

where

\[
0^0 := 1.
\]

We also use the notation

\[
\sigma(\mathbf{I}) = \text{conv}\{v : i_v > 0\}, \quad \sigma(\mathbf{a}) = \text{conv}\{v : a_v \neq 0\}
\]

where \( \text{conv} X \) denotes the convex hull of a point set \( X \).

We now define \textit{generalized barycentric coordinates} as cardinal piecewise linear functions \( b_v \in S_1^0(\Omega) \) by the requirement

\[
b_v(w) = \delta_{vw} = \begin{cases} 
1 & \text{if } v = w, \\
0 & \text{else, }
\end{cases} \quad \forall v, w \in \mathcal{Y}.
\]

Clearly, in each \( k \)-simplex \( K \in \mathcal{T} \) the functions \( b_v \), where \( v \) is a vertex of \( K \), reduce to the ordinary barycentric coordinates. Globally, i.e., for all \( x \in \Omega \), they satisfy

\[
\sum_{v \in \mathcal{Y}} b_v = 1, \quad b_v \geq 0 \quad \forall v \in \mathcal{Y}, \quad \text{and} \quad x = \sum_{v \in \mathcal{Y}} b_v(x)v.
\]
For a given polynomial degree $d$, we use the domain index set
\[ I_d = \{ I \in \mathbb{N}^N : |I| = d \text{ and } \sigma(I) \in \mathcal{S} \cup \mathcal{F} \}. \]

Letting
\[ b = b(x) = [b_\nu(x)]_{\nu \in \mathcal{N}}, \]
it is clear that every function $s \in S^0_d(\mathcal{F})$ can be written as
\[ s = \sum_{I \in I_d} c_I b^I. \]

The coefficients $c_I$ are the Bézier ordinates of $s$.

3. The de Casteljau algorithm

Let $s \in S^0_d(\mathcal{F})$, $K \in \mathcal{F}$, and $s|_K = p \in \mathbb{R}^k$. Without loss of generality we may relabel the vertices and assume that $K$ is the $k$-simplex
\[ K = \{ a = (a_1, \ldots, a_{k+1}) : a_1 + \cdots + a_{k+1} = d, a_j \geq 0, a_j \in \mathbb{R} \}. \]

Then, with $V_j$ denoting the vertices of $K$, we have
\[ p = \sum_{I \in K \cap I_d} c_I b^I \text{ where } x = \sum_{j=1}^{k+1} b_j V_j, \quad \sum_{j=1}^{k+1} b_j = 1. \]

The Bernstein polynomials $b^I(x)$ satisfy a recurrence relation, see Farin [10]
\[ b^I(x) = \sum_{j=1}^{k+1} b_j b^{I-\epsilon^j}(x), \quad |I| = d, \quad \text{and} \quad \epsilon^j = (0, \ldots, 1, \ldots, 0). \]

This relation allows one to expand $p$ in terms of Bernstein polynomials of lower degree with (polynomial) coefficients $p^r_1(b)$.

Theorem 1. We have
\[ p = \sum_{|I| = d-r} p^r_1(b) b^I, \quad 0 \leq r \leq d, \]
where
\[ p^0_1(b) = c_1, \]
\[ p^r_1(b) = \sum_{j=1}^{k+1} b_j p^{r-1}_{I+\epsilon^j}(b), \quad |I| = d - r, \quad 0 \leq r \leq d. \]

The intermediate coefficients $p^r_1(b)$ can also be written explicitly as
\[ p^r_1(b) = \sum_{|M| = r} c_{I+M} b^M, \quad |I| = d - r. \]
The formulas in Theorem 1 may be used to evaluate \( p \) at a given point, and are referred to as the de Casteljau Algorithm.

4. Smoothness across an interface

Given a vector \( e \in \mathbb{R}^k \), the directional derivative of \( p \) in the direction of \( \alpha = (\frac{\partial b_1}{\partial e}, \ldots, \frac{\partial b_k}{\partial e}) \), denoted \( D_\alpha \), is given by (Alfeld [2])

\[
D_\alpha p = d \sum_{|\mathbf{I}|=d-1} p^I(\alpha) \mathbf{b}^I,
\]

where

\[
p^I(\alpha) = \sum_{|\mathbf{M}|=1} c_{1+\mathbf{M}} \mathbf{a}^\mathbf{M}, \quad |\mathbf{I}| = d - 1.
\]

In general, the \( r \)th directional derivative of \( p \) in the direction of \( \alpha \) is given by (Farin [10])

\[
D_\alpha^r p = \frac{d!}{(d-r)!} \sum_{|\mathbf{I}|=d-r} p^I(\alpha) \mathbf{b}^I,
\]

where

\[
p^I(\alpha) = \sum_{|\mathbf{M}|=r} c_{1+\mathbf{M}} \mathbf{a}^\mathbf{M}, \quad |\mathbf{I}| = d - r, \quad \text{and} \quad p^0(\alpha) = c_1.
\]

The following theorem was proved by Farin [10] in the bivariate case; here we state without proof the result in any number of variables. Let \( \tau \) be an \( m \)-face of \( K \); without loss of generality, we will assume that

\[
\tau = \{ a \in K: a_{m+2} = \ldots = a_{k+1} = 0 \}.
\]

**Theorem 2.** Let \( p, q \in \mathcal{P}_d \) be such that \( p|_\tau = q|_\tau \). Then \( D_\alpha^s p = D_\alpha^s q \) on \( \tau \) for all directions \( \alpha, 0 \leq s \leq r \), if and only if \( p^J_\alpha = q^J_\alpha \) for all \( J = (j_1, \ldots, j_{m+1}, 0, \ldots, 0), |J| = d - r \).

5. Subsimplices and subpolynomials

Let \( K \in \mathcal{F} \) be a \( k \)-simplex and \( \tau \) be a face of \( K \) with \( \dim \tau = n \),

\[
K = \left\{ a \in \mathbb{R}^N: \sum_{v \in K} a_v = d, a_v \geq 0 \right\},
\]

\[
\tau = \left\{ a \in K: \sum_{v \in \tau} a_v = d \right\}.
\]

For \( J \in K \cap I_d \), so that \( \sum_{v \in \tau} j_v = d - \rho \), define

\[
\tau_J = \{ a = [a_v]_{v \in K}: j_v \geq a_v, v \in \tau \}.
\]

Clearly, \( \tau_J \) is a subsimplex of \( K \) similar to \( K \).
Let $p \in \mathcal{P}_d^k$, and $(I, c_I)_{I \in K}$ be the control points of $p$ with respect to $K$. The control points $(I, c_I)_{I \in \tau_J}$ define a polynomial $p_J$ of degree $p$. We call $\tau_J$ the subsimplex of $K$ associated with $I$, and $p_J$ the subpolynomial of $p$ associated with $J$.

De Boor [6] introduces the concepts of subsimplices and subpolynomials and proves most of the results of this section. Since our definitions are slightly different, we restate two key facts involving subpolynomials. Theorem 2 can be restated as follows:

**Theorem 3.** Let $p, q \in \mathcal{P}_d^k$, and $\tau$ be a face of $K$ with $\dim \tau = m$ such that $p|_\tau = q|_\tau$. Then

\[ D_\alpha^s p = D_\alpha^s q \quad \text{on } \tau, \quad \text{for all directions } \alpha \quad \text{and} \quad 0 \leq s \leq p, \]

if and only if

\[ p_I = q_I \quad \forall I \in K \quad \text{with} \quad \text{dist}(I, \tau) = p. \]

The above theorem immediately yields the following:

**Theorem 4.** Let $K, K' \in \mathcal{T}$, $\tau \subset K \cap K'$, $p, q \in \mathcal{P}_d^k$, and let $s \in S_d^0(\Omega)$ with $s|_K = p, \ s|_{K'} = q$. Then $s \in C^p(\tau)$ if and only if $p_I = q_I \quad \forall I \in K'$ with $\text{dist}(I, \tau) = p$.

### 6. Determining sets

We now generalize the concept of a determining set known from the bivariate case (see Alfeld and Schumaker [4]).

**Definition 5.** A set $D \subset I_d$ is a determining set of $S_d^r(\mathcal{T})$ if, for all $s \in S_d^r(\mathcal{T})$,

\[ c_I = 0 \quad \forall I \in D \implies s \equiv 0. \]

$D$ is a minimal determining set if there is no determining set which has fewer elements than $D$.

It is clear from elementary linear algebra that the number of elements in a determining set of $S_d^r(\mathcal{T})$ provides an upper bound on the dimension of $S_d^r(\mathcal{T})$, and that the number of elements in a minimal determining set is unique and equals the dimension of $S_d^r(\mathcal{T})$.

### 7. A minimal determining set of $S_d^r(\mathcal{T})$

For each simplex $\sigma \in \mathcal{T} \cup \mathcal{T}$, let $p = r2^{k - \dim \sigma - 1}$ as before; we define two sets of domain indices recursively by

\[ \overline{D}(\sigma) = \left\{ I \in I_d : \sum_{v \in \sigma} i_v \geq d - p \right\} \]

and

\[ D(\sigma) = \overline{D}(\sigma) \setminus \bigcup_{\tau < \sigma} D(\tau), \]

where $\tau$ is a proper face of $\sigma$. 

Definition 6. Let $\sigma \in \mathcal{S} \cup \mathcal{T}$. A set $\mathcal{A}(\sigma) \subset D(\sigma)$ is a determining set of $\overline{D}(\sigma)$ if, for all $s \in \mathcal{S}_d(\mathcal{T})$,

\begin{equation}
    c_t = 0 \quad \forall I \in \mathcal{A}(\sigma) \cup (\overline{D}(\sigma) \setminus D(\sigma)) \implies c_t = 0 \quad \forall I \in \overline{D}(\sigma).
\end{equation}

The set $\mathcal{A}$ is a minimal determining set of $\overline{D}(\sigma)$ if there is no determining set of $\overline{D}(\sigma)$ with fewer elements.

Lemma 7. Let $d > r_2^k$. Then, for all $I \in I_d$ there exists a unique $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that $I \in \mathcal{D}(\sigma)$.

Proof. To prove the lemma, we have to show first that for all $\sigma, \tau \in \mathcal{S} \cup \mathcal{T}$:

\begin{equation}
    \sigma \neq \tau \implies \mathcal{D}(\sigma) \cap \mathcal{D}(\tau) = \emptyset.
\end{equation}

To establish this, suppose there is a domain index $I \in \mathcal{D}(\sigma) \cap \mathcal{D}(\tau)$, for two simplices $\sigma, \tau \in \mathcal{S} \cup \mathcal{T}$, such that $\dim \sigma \geq \dim \tau$, and $\tau$ is not a face of $\sigma$. Thus,

\begin{equation}
    \sum_{v \in \sigma} i_v \geq d - r_2^{k-\dim \sigma - 1} \quad \text{and} \quad \sum_{v \in \tau} i_v \geq d - r_2^{k-\dim \tau - 1},
\end{equation}

which implies

\begin{equation}
    \sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v \geq 2d - r_2^{k-\dim \sigma - 1} - r_2^{k-\dim \tau - 1}.
\end{equation}

Moreover,

\begin{equation}
    \sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v = \sum_{v \in \Omega \tau} i_v + \sum_{v \in \Omega \sigma} i_v.
\end{equation}

Now, rearranging (37), substituting (36) into (37), and using $\sum_{v \in \sigma \cup \tau} i_v \leq d$, we obtain

\begin{equation}
    \sum_{v \in \Omega \tau} i_v \geq d - r_2^{k-\dim \tau}.
\end{equation}

So there exists a face $\tau$ of $\sigma$ such that $I \in \mathcal{D}(\tau)$, which is a contradiction.

Finally, we need to prove that $I_d = \bigcup_{\sigma \in \mathcal{S} \cup \mathcal{T}} \mathcal{D}(\sigma)$.

To see this, we only need to show that for each domain index $I \in I_d$ there exists a simplex $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that $I \in \mathcal{D}(\sigma)$. However, this is trivial, since for each domain index $I$ there exists at least one $k$-simplex $K$ such that $\sum_{v \in K} i_v = d$. Thus, $I \in \overline{D}(K)$, and there must be a simplex $\sigma \prec K$ such that $I \in \mathcal{D}(\sigma)$. 

\begin{lemma}
    Let $d > r_2^k$, $r = r_2^{k-\dim \sigma - 1}$, and $\mathcal{A}(\sigma) := \overline{D}(\sigma) \cap K$, where $K \in \mathcal{T}$ and $\sigma \prec K$. Then $\mathcal{A}(\sigma)$ is a minimal determining set of $\overline{D}(\sigma)$.
\end{lemma}

Proof. First we establish that $\mathcal{A}(\sigma)$ is determining.

Let $s \in \mathcal{S}_d(\mathcal{T})$, and let $s|_K = p$, with $p \in \mathcal{R}_d^k$. If

\begin{equation}
    c_t = 0 \quad \forall I \in \mathcal{A}(\sigma) \cup (\overline{D}(\sigma) \setminus D(\sigma)),
\end{equation}

then

\begin{equation}
    p_t = 0 \quad \forall I \in \mathcal{A}(\sigma) \cup (\overline{D}(\sigma) \setminus D(\sigma)),
\end{equation}

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where $p_1$ is the subpolynomial of $p$ of degree $\leq \rho$ associated with $I$ (notice that $\text{dist}(I, \sigma) \leq \rho \forall I \in A(\sigma)$). Let $K'$ be any other $k$-simplex of $T$ such that $\sigma \prec K'$ and $K \neq K'$, and let $q = s|_{K'}$, $q \in P^k_d$. Now, $s \in C^\rho(\sigma)$; then $\forall I \in (D(\sigma) \cap K') \cup (\overline{D(\sigma)} \setminus D(\sigma))p_1 = q_1$, where $q_1$ is the subpolynomial of $q$ of degree $\leq \rho$ associated with $I$. But,

\begin{equation}
\begin{aligned}
p_1 &= 0 \quad \forall I \in (D(\sigma) \cap K) \cup (\overline{D(\sigma)} \setminus D(\sigma)) \\
\Rightarrow q_1 &= 0 \quad \forall I \in (D(\sigma) \cap K') \cup (\overline{D(\sigma)} \setminus D(\sigma)) \\
\Rightarrow c_1 &= 0 \quad \forall I \in (D(\sigma) \cap K') \cup (\overline{D(\sigma)} \setminus D(\sigma)) \\
\Rightarrow c_1 &= 0 \quad \forall I \in \left( \bigcup_{K \in T \atop \sigma \prec K} D(\sigma) \cap K \right) \cup (\overline{D(\sigma)} \setminus D(\sigma)) \\
\Rightarrow c_1 &= 0 \quad \forall I \in D(\sigma) \cup (\overline{D(\sigma)} \setminus D(\sigma)).
\end{aligned}
\end{equation}

Therefore,

\begin{equation}
\begin{aligned}
c_1 &= 0 \quad \forall I \in \overline{D(\sigma)}.
\end{aligned}
\end{equation}

To see that $A(\sigma)$ is indeed minimal, take $I \in A(\sigma)$ and consider the set $\tilde{A}(\sigma) = A(\sigma) \setminus \{I\}$, define a polynomial on $K$ whose Bézier coefficients are equal to zero except at the domain point $I$, where $c_1 = 1$, and extend this polynomial globally on the rest of the triangulation.

The coefficients $c_j$ are equal to zero on $\overline{D(\sigma)} \setminus D(\sigma)$, since the smoothness conditions there only involve domain indices in $D(\tau)$ for $\tau \prec \sigma$.

Hence, $c_j = 0 \quad \forall J \in \overline{A(\sigma)} \cup (\overline{D(\sigma)} \setminus D(\sigma))$, but this does not imply that $c_j = 0 \quad \forall J \in \overline{D(\sigma)}$ since in particular $I \in D(\sigma)$ and $c_1 = 1$.

Therefore, $\tilde{A}(\sigma)$ cannot be determining and $A(\sigma)$ is a minimal determining set of $\overline{D(\sigma)}$. $\square$

The following theorem is the central result of this paper.

**Theorem 9.** Let $r \geq 0$, $d > r2^k$ and let $A(\sigma) = D(\sigma) \cap K$, where $K \in T$ is a $k$-simplex so that $\sigma$ is a face of $K$. Then

\begin{equation}
\begin{aligned}
A &:= \bigcup_{\sigma \in T} A(\sigma)
\end{aligned}
\end{equation}

is a minimal determining set of $S_d^r(T)$.

**Proof.** Let $s \in S_d^r(T)$ and assume that $c_1 = 0 \quad \forall I \in A$. Using induction on $\dim \sigma$, we first establish that $A$ is a determining set.

1. If $\dim \sigma = 0$ then $c_1 = 0 \quad \forall I \in A(\sigma)$ implies $c_1 = 0 \quad \forall I \in \overline{D(\sigma)}$, since $D(\sigma) = \overline{D(\sigma)}$.

2. We assume now that $c_1 = 0 \quad \forall I \in \overline{D(\sigma)}$ and $\forall \sigma$ with $\dim \sigma < n$. 

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(3) Let \( \dim \sigma = n \), \( c_1 = 0 \) \( \forall I \in \mathcal{A}(\sigma) \). Let \( \mathcal{B}(\sigma) = \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}(\sigma)} \setminus \mathcal{D}(\sigma)) \); notice that \( B(\sigma) \subset \mathcal{A}(\sigma) \cup (\bigcup_{\tau < \sigma} \overline{\mathcal{D}(\tau)}) \).

By the induction hypothesis, \( c_1 = 0 \) \( \forall I \in \bigcup_{\tau < \sigma} \overline{\mathcal{D}(\tau)} \) implies that \( c_1 = 0 \) \( \forall I \in \mathcal{B}(\sigma) \). But \( \mathcal{A}(\sigma) \) is a determining set of \( \mathcal{D}(\sigma) \); thus, \( c_1 = 0 \) \( \forall I \in \mathcal{D}(\sigma) \), hence

\[
(46) \quad c_1 = 0 \quad \forall I \in \bigcup_{\sigma \in \mathcal{P} \cup \mathcal{I}} \overline{\mathcal{D}(\sigma)},
\]

and by Lemma 7 we get that

\[
(47) \quad c_1 = 0 \quad \forall I \in \mathcal{I}_d.
\]

To show that \( \mathcal{A} \) is indeed minimal, we give arbitrary Bézier ordinates \( c_I \) \( \forall I \in \mathcal{A} \), and we construct the rest of the Bézier ordinates in such a way that the piecewise polynomial they determine is a superspline.

So, we assume \( c_I \) \( \forall I \in \mathcal{A} \) are given, and let \( J \in \mathcal{I}_d \). Then by Lemma 7 there is a unique \( \sigma \in \mathcal{P} \cup \mathcal{T} \) such that \( J \in \mathcal{D}(\sigma) \).

We define \( c_J \) inductively over \( \dim \sigma \).

1. If \( \sigma \) is a vertex, then there exists \( K \in \mathcal{T} \) so that \( \sigma \prec K \) and \( \mathcal{A}(\sigma) \subset K \).

Let \( \sigma_J \) be the subsimplex of \( K \) associated with \( J \), \( \sigma_J \subset \mathcal{A}(\sigma) \). Then the polynomial \( p_J \) is defined; so, for \( J \in \mathcal{D}(\sigma) \cap K' \) with \( K' \in \mathcal{T} \), \( K' \neq K \), let \( \tau_J \) be the subsimplex of \( K' \) associated with \( J \). Then there is a unique way to give Bézier ordinates to the domain points in \( \tau_J \) in such a way that they define a polynomial \( q_J \) which equals \( p_J \). Furthermore, in this manner, \( c_J \) can be defined for all \( J \in \mathcal{D}(\sigma) \setminus \mathcal{A}(\sigma) \).

2. Suppose \( c_J \) has been defined for all \( J \in \bigcup_{\dim \sigma < n} \mathcal{D}(\sigma) \).

3. Let \( J \in \mathcal{D}(\sigma) \) with \( \sigma \) an \( n \)-simplex and \( K \in \mathcal{T} \) such that \( \mathcal{A}(\sigma) \subset K \).

And let again \( \sigma_J \) be the subsimplex of \( K \) associated with \( J \). Note that

\[
(48) \quad \sigma_J \subset \overline{\mathcal{D}(\sigma)} \cap K \subset \mathcal{A}(\sigma) \cup \left( \bigcup_{\tau < \sigma} \mathcal{D}(\tau) \cap K \right).
\]

Then by the induction hypothesis and by the fact that the Bézier ordinates have been defined on \( \mathcal{A} \), we have that all of the Bézier ordinates on \( \sigma_J \) are defined; therefore \( p_J \) is defined. So as before, for \( J \in \mathcal{D}(\sigma) \cap K' \), \( K' \neq K \), we can define \( q_J \) in the same way we did when \( \sigma \) was a vertex. Thus, \( c_I \) has been defined for all \( I \in \mathcal{I}_d \).

Next, we need to show that the piecewise polynomial function defined by the \( c_I \)'s is well defined.

Suppose \( p_L = q_L \) for \( L \in K \) and \( \sum_{v \in \xi} l_v = d - \rho \).

Let \( J \in \mathcal{D}(\sigma) \cap K' \) and suppose that \( J \in \xi_L \) with \( J \neq L \), where

\[
(49) \quad \xi_L = \{ I \in K' : i_v \geq l_v \quad v \in \xi, \quad \xi \in \mathcal{P} \cup \mathcal{T}, \quad \text{and} \quad \sigma \prec \xi \}
\]

and

\[
(50) \quad \tau_J = \{ I \in K' : i_v \geq j_v \quad v \in \sigma \}.
\]
Then $J \in \xi_L$ implies $j_v \geq l_v$, which in turn implies $\tau_J \subset \xi_L$. Therefore, $q_J$ is a subpolynomial of $q_L$. Similarly, $p_J$ is a subpolynomial of $p_L$. Hence, $p_J = q_J$ since $q_L \cup p_L \in C^\infty(\xi)$. Therefore, the Bézier ordinates are well defined and by construction, the piecewise polynomial defined is a superspline. □

Corollary 10. We have $\dim S_d'(\mathcal{T}) = \sum_{\sigma \in \mathcal{T} \cup \mathcal{F}} |\mathcal{A}(\sigma)|$.

In view of the above corollary, to compute $\dim S_d'(\mathcal{T})$ we need only to know the cardinality of $\mathcal{A}(\sigma)$ for every $\sigma \in \mathcal{T}$. Clearly, $|\mathcal{A}(\sigma)|$ depends only on $m = \dim \sigma$. The following theorem is proved in Alfeld and Sirvent [5].

Theorem 11. We have $\dim S_d'(\mathcal{T}) = \sum_{m=0}^{k} \phi(m) f_m$, where $f_m$ is the number of $m$-simplices of $\mathcal{S} \cup \mathcal{F}$, and for $m = 0, \ldots, k$, $\phi(m) = \phi_m^k(0)$, where, with $\rho_m = r2^{k-m-1}$, the quantities $\phi_m^p(p)$ are defined recursively by

$$
\phi_q^p(p) = \sum_{j=0}^{\rho_q-p} \binom{j + m - q - 1}{j} \left[ \binom{d - p - j + q}{q} - \sum_{i=0}^{q-1} \binom{q + 1}{i + 1} \phi_i^q(p + j) \right]
$$

if $0 < q < m$, and

$$
\phi_0^p(p) = \sum_{j=0}^{\rho_0-p} \binom{j + m - 1}{j} = \binom{\rho_0 - p + m}{m},
$$

$$
\phi_k^k(0) = \binom{d + k}{k} - \sum_{m=0}^{k-1} \binom{k + 1}{m + 1} \phi_m^k(0).
$$

8. Minimally supported bases

Definition 12. The star of a simplex $\sigma \in \mathcal{S} \cup \mathcal{F}$, denoted $\text{star}(\sigma)$, is the set of all $k$-simplices $K \in \mathcal{F}$ such that $\sigma$ is a face of $K$.

Definition 13. A basis $\{l_\mu : \mu = 1, 2, \ldots, \dim S_d'(\mathcal{T})\}$ is said to be minimally supported if for each basis function $l_\mu$ there exists a simplex $\sigma \in \mathcal{S} \cup \mathcal{F}$ such that the support of $l_\mu$ is contained in $\text{star}(\sigma)$.

The basis functions constructed in the proof of Theorem 9 are minimally supported: using the same construction, we can define cardinal supersplines

$$
l_1 \in S_d'(\mathcal{T}): n(J) = \delta_{IJ} \quad \forall I, J \in \mathcal{A}
$$

so that, if $I \in \mathcal{A}(\sigma)$, then $c_J = 0 \quad \forall J \in I_d \setminus \text{star}(\sigma)$ and $c_I = 1$.

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