THE STRUCTURE OF MULTIVARIATE SUPERSPLINE SPACES
OF HIGH DEGREE

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Abstract. We consider splines (of global smoothness \( r \), polynomial degree \( d \), in a general number \( k \) of independent variables, defined on a \( k \)-dimensional triangulation \( T \) of a suitable domain \( \Omega \) which are \( r2^{k-m-1} \)-times differentiable across every \( m \)-face \( m = 0, \ldots, k-1 \) of a simplex in \( T \). For the case \( d > r2^k \) we identify a structure that allows the construction of a minimally supported basis.

1. Introduction

A \( \kappa \)-simplex \( K \) \((0 \leq \kappa \leq k)\) is the convex hull of \( \kappa + 1 \) points in \( \mathbb{R}^k \) called the vertices of \( K \). \( K \) is nondegenerate if its \( \kappa \)-dimensional volume is nonzero, and degenerate otherwise. The dimension of a nondegenerate \( \kappa \)-simplex is \( \kappa \)

The convex hull of a subset of \( \mu + 1 \) vertices of \( K \) is a \( \mu \)-face of \( K \).

Let \( \mathcal{V} \subset \mathbb{R}^k \) be a given set of \( N \) distinct points.

A triangulation \( T \) of the set \( \mathcal{V} \) is a set of nondegenerate \( k \)-simplices satisfying the following requirements:

1. All vertices of each simplex in \( T \) are elements of \( \mathcal{V} \).
2. The interiors of the simplices in \( T \) are pairwise disjoint.
3. The set

\[
\Omega := \bigcup_{K \in T} K \subset \mathbb{R}^k
\]

is homeomorphic to \([0, 1]^k\).

4. Each \((k - 1)\)-face of a simplex in \( T \) is either on the boundary of \( \Omega \), or else is a common face of exactly two simplices in \( T \).

5. No simplex in \( T \) contains any points of \( \mathcal{V} \) other than its vertices.

Note that a \( \mu \)-face of a simplex in \( T \) is itself a \( \mu \)-dimensional simplex. On the triangulation \( T \) we define a spline space \( S_d^r(T) \) as usual by

\[
S_d^r(T) = \{ s \in C^r(\Omega) : s|_{\tau} \in \mathcal{P}_d^{k} \forall \tau \in T \},
\]

where \( \mathcal{P}_d^{k} \) is the \((k + d)\)-dimensional linear space of all \( k \)-variate polynomials of total degree less than or equal to \( d \).

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In this paper, we consider subspaces of $S'_d(\mathcal{T})$ obtained by increasing the smoothness requirements across faces of the underlying simplices. More precisely, denoting by $\mathcal{S}_\mu$ the set of all $\mu$-faces of the simplices in $\mathcal{T}$ ($\mu = 0, \ldots, k - 1$) and letting $\mathcal{S} = \bigcup_{\mu=0}^{k-1} \mathcal{S}_\mu$, we define the (superspline) space $S'_d(\mathcal{T})$ as a subspace of $S'_d(\mathcal{T})$ as follows:

$$S'_d(\mathcal{T}) = \{ s \in S'_d(\mathcal{T}) : s \text{ is } \rho\text{-times differentiable across } \sigma \text{ \forall } \sigma \in \mathcal{S} \},$$

where $\rho = r2^{k-\dim \sigma-1}$.

The concept of supersplines was introduced in Chui and Lai [8], [9]. The area in between finite elements and full spline spaces was further explored by Schumaker [16] and Ibrahim and Schumaker [11].

2. THE GENERALIZED BÉZIER-BERNSTEIN FORM

Crucial to analyzing the dimension of spline spaces is the Bézier-Bernstein form of a multivariate polynomial. In the case $k \leq 2$ this form is used widely and is well known. A review of the Bézier-Bernstein form for a general number of variables is in de Boor [6]. In this paper, we use a notation that is particularly suitable for our purposes. However, generalized barycentric coordinates and global control nets have also been proposed in Alfeld [2] and de Boor [6].

We use $\mathcal{Y}$ as an index set and denote by $\mathbb{N}$ the set of nonnegative integers. For vectors $I = [i_v]_{v \in \mathcal{Y}} \in \mathbb{N}^\mathcal{Y}$ and $a = [a_v]_{v \in \mathcal{Y}} \in \mathbb{R}^\mathcal{Y}$ we define

$$|I| = \sum_{v \in \mathcal{Y}} i_v,$$

$$a^I = |I|! \prod_{v \in \mathcal{Y}} a_v^{i_v} \left/ \prod_{v \in \mathcal{Y}} i_v! \right.,$$

where

$$0^0 := 1.$$

We also use the notation

$$\sigma(I) = \text{conv}\{v : i_v > 0\}, \quad \sigma(a) = \text{conv}\{v : a_v \neq 0\}$$

where $\text{conv} X$ denotes the convex hull of a point set $X$.

We now define generalized barycentric coordinates as cardinal piecewise linear functions $b_v \in S^0_1(\Omega)$ by the requirement

$$b_v(w) = \delta_{v,w} = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{else} \end{cases} \forall v, w \in \mathcal{Y}.$$

Clearly, in each $k$-simplex $K \in \mathcal{T}$ the functions $b_v$, where $v$ is a vertex of $K$, reduce to the ordinary barycentric coordinates. Globally, i.e., for all $x \in \Omega$, they satisfy

$$\sum_{v \in \mathcal{Y}} b_v = 1, \quad b_v \geq 0 \forall v \in \mathcal{Y}, \quad \text{and} \quad x = \sum_{v \in \mathcal{Y}} b_v(x)v.$$
For a given polynomial degree \( d \), we use the domain index set
\[
I_d = \{ \mathbf{I} \in \mathbb{N}^N : |\mathbf{I}| = d \text{ and } \sigma(\mathbf{I}) \in \mathcal{P} \cup \mathcal{I} \}.
\]

Letting
\[
\mathbf{b} = \mathbf{b}(x) = [b_{v}(x)]_{v \in \mathcal{V}},
\]
it is clear that every function \( s \in S_d^0(\mathcal{F}) \) can be written as
\[
s = \sum_{\mathbf{I} \in I_d} c_\mathbf{I} \mathbf{b}^{\mathbf{I}}.
\]

The coefficients \( c_\mathbf{I} \) are the Bézier ordinates of \( s \).

3. The de Casteljau algorithm

Let \( s \in S_d^0(\mathcal{F}) \), \( K \in \mathcal{F} \), and \( s|_K = p \in \mathbb{R}^k \). Without loss of generality we may relabel the vertices and assume that \( K \) is the \( k \)-simplex
\[
K = \{ \mathbf{a} = (a_1, \cdots, a_{k+1}) : a_1 + \cdots + a_{k+1} = d, \ a_j \geq 0, \ a_j \in \mathbb{R} \}.
\]
Then, with \( V_j \) denoting the vertices of \( K \), we have
\[
p = \sum_{\mathbf{I} \in K \cap I_d} c_\mathbf{I} \mathbf{b}^{\mathbf{I}} \quad \text{where} \quad x = \sum_{j=1}^{k+1} b_j V_j, \quad \sum_{j=1}^{k+1} b_j = 1.
\]

The Bernstein polynomials \( \mathbf{b}^{\mathbf{I}}(x) \) satisfy a recurrence relation, see Farin [10]
\[
\mathbf{b}^{\mathbf{I}}(x) = \sum_{j=1}^{k+1} b_j \mathbf{b}^{\mathbf{I}-\epsilon^j}(x), \ |\mathbf{I}| = d, \quad \text{and} \quad \epsilon^j = (0, \cdots, 1, \cdots, 0).
\]

This relation allows one to expand \( p \) in terms of Bernstein polynomials of lower degree with (polynomial) coefficients \( p_r^I(b) \).

Theorem 1. We have
\[
p = \sum_{|\mathbf{I}| = d-r} p_r^I(b) \mathbf{b}^{\mathbf{I}}, \quad 0 \leq r \leq d,
\]
where
\[
p_0^I(b) = c_1,
\]
\[
p_r^I(b) = \sum_{j=1}^{k+1} b_j p_{r-1}^{\mathbf{I}+\epsilon^j}(b), \quad |\mathbf{I}| = d-r, \quad 0 \leq r \leq d.
\]

The intermediate coefficients \( p_r^I(b) \) can also be written explicitly as
\[
p_r^I(b) = \sum_{|\mathbf{M}| = r} c_{I+M} \mathbf{b}^{\mathbf{M}}, \quad |\mathbf{I}| = d-r.
\]
The formulas in Theorem 1 may be used to evaluate \( p \) at a given point, and are referred to as the de Casteljau Algorithm.

4. Smoothness across an interface

Given a vector \( e \in \mathbb{R}^k \), the directional derivative of \( p \) in the direction of \( \alpha = (\frac{\partial b_1}{\partial e}, \ldots, \frac{\partial b_n}{\partial e}) \), denoted \( D_\alpha \), is given by (Alfeld [2])

\[
D_\alpha p = d \sum_{|\mathbf{I}|=d-1} p_{\mathbf{I}}(\alpha) \mathbf{b}^\mathbf{I},
\]

where

\[
p_{\mathbf{I}}(\alpha) = \sum_{|\mathbf{M}|=1} c_{\mathbf{I}+\mathbf{M}} \mathbf{\alpha}^\mathbf{M}, \quad |\mathbf{I}| = d - 1.
\]

In general, the \( r \)th directional derivative of \( p \) in the direction of \( \alpha \) is given by (Farin [10])

\[
D_\alpha^r p = \frac{d!}{(d-r)!} \sum_{|\mathbf{I}|=d-r} p_{\mathbf{I}}(\alpha) \mathbf{b}^\mathbf{I},
\]

where

\[
p_{\mathbf{I}}(\alpha) = \sum_{|\mathbf{M}|=r} c_{\mathbf{I}+\mathbf{M}} \mathbf{\alpha}^\mathbf{M}, \quad |\mathbf{I}| = d - r, \quad \text{and} \quad p_{\mathbf{I}}^0(\alpha) = c_{\mathbf{I}}.
\]

The following theorem was proved by Farin [10] in the bivariate case; here we state without proof the result in any number of variables. Let \( \tau \) be an \( m \)-face of \( K \); without loss of generality, we will assume that

\[
\tau = \{ \mathbf{a} \in K: a_{m+2} = \cdots = a_{k+1} = 0 \}.
\]

**Theorem 2.** Let \( p, q \in \mathcal{P}^d \) be such that \( p_{|\tau} = q_{|\tau} \). Then \( D_\alpha^s p = D_\alpha^s q \) on \( \tau \) for all directions \( \alpha \), \( 0 \leq s \leq r \), if and only if \( p_{\mathbf{J}} = q_{\mathbf{J}} \) for all \( \mathbf{J} = (j_1, \ldots, j_{m+1}, 0, \ldots, 0), |\mathbf{J}| = d - r \).

5. Subsimplices and subpolynomials

Let \( K \in \mathcal{T} \) be a \( k \)-simplex and \( \tau \) be a face of \( K \) with \( \dim \tau = n \),

\[
K = \left\{ \mathbf{a} \in \mathbb{R}^N: \sum_{\mathbf{v} \in K} a_{\mathbf{v}} = d, a_{\mathbf{v}} \geq 0 \right\},
\]

\[
\tau = \left\{ \mathbf{a} \in K: \sum_{\mathbf{v} \in \tau} a_{\mathbf{v}} = d \right\}.
\]

For \( \mathbf{J} \in K \cap I_d \), so that \( \sum_{\mathbf{v} \in \tau} j_{\mathbf{v}} = d - \rho \), define

\[
\tau_{\mathbf{J}} = \{ \mathbf{a} = [a_{\mathbf{v}}]_{\mathbf{v} \in K}: j_{\mathbf{v}} \geq a_{\mathbf{v}}, \mathbf{v} \in \tau \}.
\]

Clearly, \( \tau_{\mathbf{J}} \) is a subsimplex of \( K \) similar to \( K \).
Let \( p \in \mathcal{P}_d^k \), and \( (I, c_I)_{I \in \mathcal{K}} \) be the control points of \( p \) with respect to \( K \). The control points \( (I, c_I)_{I \in \tau_j} \) define a polynomial \( p_j \) of degree \( \rho \). We call \( \tau_j \) the subsimplex of \( K \) associated with \( J \), and \( p_j \) the subpolynomial of \( p \) associated with \( J \).

De Boor [6] introduces the concepts of subsimplices and subpolynomials and proves most of the results of this section. Since our definitions are slightly different, we restate two key facts involving subpolynomials. Theorem 2 can be restated as follows:

**Theorem 3.** Let \( p, q \in \mathcal{P}_d^k \), and \( \tau \) be a face of \( K \) with \( \dim \tau = m \) such that \( p|_{\tau} = q|_{\tau} \). Then
\[
D^s_\alpha p = D^s_\alpha q \quad \text{on} \quad \tau, \quad \text{for all directions} \quad \alpha \quad \text{and} \quad 0 \leq s \leq \rho,
\]
if and only if
\[
p_I = q_I \quad \forall I \in K \quad \text{with} \quad \text{dist}(I, \tau) = \rho.
\]

The above theorem immediately yields the following:

**Theorem 4.** Let \( K, K' \in \mathcal{T} \), \( \tau \subset K \cap K' \), \( p, q \in \mathcal{P}_d^k \), and let \( s \in \mathcal{S}_d^0(\Omega) \) with \( s|_K = p \), \( s|_{K'} = q \). Then \( s \in C^p(\tau) \) if and only if \( p_I = q_I \forall I \in K' \) with \( \text{dist}(I, \tau) = \rho \).

### 6. Determining sets

We now generalize the concept of a determining set known from the bivariate case (see Alfeld and Schumaker [4]).

**Definition 5.** A set \( D \subset I_d \) is a determining set of \( \mathcal{S}_d^r(\mathcal{T}) \) if, for all \( s \in \mathcal{S}_d^r(\mathcal{T}) \),
\[
c_I = 0 \quad \forall I \in D \implies s \equiv 0.
\]

\( D \) is a minimal determining set if there is no determining set which has fewer elements than \( D \).

It is clear from elementary linear algebra that the number of elements in a determining set of \( \mathcal{S}_d^r(\mathcal{T}) \) provides an upper bound on the dimension of \( \mathcal{S}_d^r(\mathcal{T}) \), and that the number of elements in a minimal determining set is unique and equals the dimension of \( \mathcal{S}_d^r(\mathcal{T}) \).

### 7. A minimal determining set of \( \mathcal{S}_d^r(\mathcal{T}) \)

For each simplex \( \sigma \in \mathcal{S} \cup \mathcal{T} \), let \( \rho = r2^{k - \dim \sigma - 1} \) as before; we define two sets of domain indices recursively by
\[
\overline{\mathcal{D}}(\sigma) = \left\{ I \in I_d : \sum_{\sigma \in \sigma} i_v \geq d - \rho \right\}
\]
and
\[
\mathcal{D}(\sigma) = \overline{\mathcal{D}}(\sigma) \setminus \bigcup_{\tau \subset \sigma} \mathcal{D}(\tau),
\]
where \( \tau \) is a proper face of \( \sigma \).
Definition 6. Let $\sigma \in \mathcal{S} \cup \mathcal{I}$. A set $\mathcal{A}(\sigma) \subset D(\sigma)$ is a determining set of $\overline{D}(\sigma)$ if, for all $s \in S_d^*(\mathcal{I})$,

$$c_t = 0 \quad \forall I \in \mathcal{A}(\sigma) \cup (\overline{D}(\sigma) \setminus D(\sigma)) \Rightarrow c_t = 0 \quad \forall I \in \overline{D}(\sigma).$$

The set $\mathcal{A}$ is a minimal determining set of $\overline{D}(\sigma)$ if there is no determining set of $\overline{D}(\sigma)$ with fewer elements.

Lemma 7. Let $d > r_2^k$. Then, for all $I \in I_d$ there exists a unique $\sigma \in \mathcal{S} \cup \mathcal{I}$ such that $I \in D(\sigma)$.

Proof. To prove the lemma, we have to show first that for all $\sigma, \tau \in \mathcal{S} \cup \mathcal{I}$:

$$\sigma \neq \tau \implies D(\sigma) \cap D(\tau) = \emptyset.$$

To establish this, suppose there is a domain index $I \in D(\sigma) \cap D(\tau)$, for two simplices $\sigma, \tau \in \mathcal{S} \cup \mathcal{I}$, such that $\dim \sigma \geq \dim \tau$, and $\tau$ is not a face of $\sigma$. Thus,

$$\sum_{v \in \sigma} i_v \geq d - r_2^{k-\dim \sigma-1} \quad \text{and} \quad \sum_{v \in \tau} i_v \geq d - r_2^{k-\dim \tau-1},$$

which implies

$$\sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v \geq 2d - r_2^{k-\dim \sigma-1} - r_2^{k-\dim \tau-1}.$$

Moreover,

$$\sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v = \sum_{v \in \sigma \cap \tau} i_v + \sum_{v \in \sigma \cup \tau} i_v.$$

Now, rearranging (37), substituting (36) into (37), and using $\sum_{v \in \sigma \cup \tau} i_v \leq d$, we obtain

$$\sum_{v \in \sigma \cap \tau} i_v \geq d - r_2^{k-\dim \tau}.$$

So there exists a face $\tilde{\tau}$ of $\tau$ such that $I \in \overline{D}(\tilde{\tau})$, which is a contradiction.

Finally, we need to prove that $I_d = \bigcup_{\sigma \in \mathcal{S} \cup \mathcal{I}} D(\sigma)$.

To see this, we only need to show that for each domain index $I \in I_d$ there exists a simplex $\sigma \in \mathcal{S} \cup \mathcal{I}$ such that $I \in D(\sigma)$. However, this is trivial, since for each domain index $I$ there exists at least one $k$-simplex $K$ such that $\sum_{v \in K} i_v = d$. Thus, $I \in \overline{D}(K)$, and there must be a simplex $\sigma \prec K$ such that $I \in D(\sigma)$. ☐

Lemma 8. Let $d > r_2^k$, $p = r_2^{k-\dim \sigma-1}$, and $\mathcal{A}(\sigma) := D(\sigma) \cap K$, where $K \in \mathcal{I}$ and $\sigma \prec K$. Then $\mathcal{A}(\sigma)$ is a minimal determining set of $\overline{D}(\sigma)$.

Proof. First we establish that $\mathcal{A}(\sigma)$ is determining.

Let $s \in S_d^*(\mathcal{I})$, and let $s|_K = p$, with $p \in \mathcal{R}_d^k$. If

$$c_t = 0 \quad \forall I \in \mathcal{A}(\sigma) \cup (\overline{D}(\sigma) \setminus D(\sigma)),\]$$

then

$$p_t = 0 \quad \forall I \in \mathcal{A}(\sigma) \cup (\overline{D}(\sigma) \setminus D(\sigma)),\]$$

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where \( p_I \) is the subpolynomial of \( p \) of degree \( \leq \rho \) associated with \( I \) (notice that \( \text{dist}(I, \sigma) \leq \rho \ \forall I \in \mathcal{A}(\sigma) \)). Let \( K' \) be any other \( k \)-simplex of \( \mathcal{T} \) such that \( \sigma \prec K' \) and \( K \neq K' \), and let \( q = s|_{K'} \), \( q \in \mathcal{P}_d^k \). Now, \( s \in C^0(\sigma) \); then \( \forall I \in (\mathcal{D}(\sigma) \cap K') \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)) p_I = q_I \), where \( q_I \) is the subpolynomial of \( q \) of degree \( \leq \rho \) associated with \( I \). But,

\[
\begin{align*}
(39) & \quad p_I = 0 \ \forall I \in (\mathcal{D}(\sigma) \cap K) \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)) \\
(40) & \quad \implies q_I = 0 \ \forall I \in (\mathcal{D}(\sigma) \cap K) \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)) \\
(41) & \quad \implies c_I = 0 \ \forall I \in (\mathcal{D}(\sigma) \cap K) \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)) \\
(42) & \quad \implies c_I = 0 \ \forall I \in \left( \bigcup_{K \in \mathcal{P}_d^k, \sigma \prec K} \mathcal{D}(\sigma) \cap K \right) \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)) \\
(43) & \quad \implies c_I = 0 \ \forall I \in \mathcal{D}(\sigma) \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)).
\end{align*}
\]

Therefore,

\[
(44) \quad c_I = 0 \ \forall I \in \mathcal{D}(\sigma).
\]

To see that \( \mathcal{A}(\sigma) \) is indeed minimal, take \( I \in \mathcal{A}(\sigma) \) and consider the set \( \mathcal{A}(\sigma) = \mathcal{A}(\sigma) \setminus \{I\} \), define a polynomial on \( K \) whose Bézier coefficients are equal to zero except at the domain point \( I \), where \( c_I = 1 \), and extend this polynomial globally on the rest of the triangulation.

The coefficients \( c_I \) are equal to zero on \( \mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma) \), since the smoothness conditions there only involve domain indices in \( \mathcal{D}(\tau) \) for \( \tau \prec \sigma \).

Hence, \( c_I = 0 \ \forall J \in \mathcal{A}(\sigma) \cup (\mathcal{D}(\sigma) \setminus \mathcal{D}(\sigma)) \), but this does not imply that \( c_I = 0 \ \forall J \in \mathcal{D}(\sigma) \) since in particular \( I \in \mathcal{D}(\sigma) \) and \( c_I = 1 \).

Therefore, \( \mathcal{A}(\sigma) \) cannot be determining and \( \mathcal{A}(\sigma) \) is a minimal determining set of \( \mathcal{D}(\sigma) \). \( \square \)

The following theorem is the central result of this paper.

**Theorem 9.** Let \( r \geq 0, \ d > r2^k \) and let \( \mathcal{A}(\sigma) = \mathcal{D}(\sigma) \cap K \), where \( K \in \mathcal{T} \) is a \( k \)-simplex so that \( \sigma \) is a face of \( K \). Then

\[
\mathcal{A} := \bigcup_{\sigma \in \mathcal{P}_d^k} \mathcal{A}(\sigma)
\]

is a minimal determining set of \( \mathcal{S}_{d}(\mathcal{T}) \).

**Proof.** Let \( s \in \mathcal{S}_{d}(\mathcal{T}) \) and assume that \( c_I = 0 \ \forall I \in \mathcal{A} \). Using induction on \( \dim \sigma \), we first establish that \( \mathcal{A} \) is a determining set.

1. If \( \dim \sigma = 0 \) then \( c_I = 0 \ \forall I \in \mathcal{A}(\sigma) \) implies \( c_I = 0 \ \forall I \in \mathcal{D}(\sigma) \), since \( \mathcal{D}(\sigma) = \mathcal{D}(\sigma) \).

2. We assume now that \( c_I = 0 \ \forall I \in \mathcal{D}(\sigma) \) and \( \forall \sigma \) with \( \dim \sigma < n \).
Let \( \dim \sigma = n \), \( c_1 = 0 \) \( \forall I \in \mathcal{A}(\sigma) \). Let \( \mathcal{B}(\sigma) = \mathcal{A}(\sigma) \cup (\mathcal{D}(\sigma) \setminus \mathcal{B}(\sigma)) \); notice that \( B(\sigma) \subset \mathcal{A}(\sigma) \cup (\bigcup_{\tau < \sigma} \mathcal{B}(\tau)) \).

By the induction hypothesis, \( c_1 = 0 \) \( \forall I \in \bigcup_{\tau < \sigma} \mathcal{D}(\tau) \) implies that \( c_1 = 0 \) \( \forall I \in \mathcal{B}(\sigma) \). But \( \mathcal{A}(\sigma) \) is a determining set of \( \mathcal{D}(\sigma) \); thus, \( c_1 = 0 \) \( \forall I \in \mathcal{D}(\sigma) \), hence
\[
(46) \quad c_1 = 0 \quad \forall I \in \bigcup_{\sigma \in \mathcal{P} \cup \mathcal{T}} \mathcal{B}(\sigma),
\]
and by Lemma 7 we get that
\[
(47) \quad c_1 = 0 \quad \forall I \in I_d.
\]

To show that \( \mathcal{A} \) is indeed minimal, we give arbitrary Bézier ordinates \( c_1 \) \( \forall I \in \mathcal{A} \), and we construct the rest of the Bézier ordinates in such a way that the piecewise polynomial they determine is a superspline.

So, we assume \( c_1 \) \( \forall I \in \mathcal{A} \) are given, and let \( J \in I_d \). Then by Lemma 7 there is a unique \( \sigma \in \mathcal{P} \cup \mathcal{T} \) such that \( J \in \mathcal{D}(\sigma) \).

We define \( c_J \) inductively over \( \dim \sigma \).

1. If \( \sigma \) is a vertex, then there exists \( K \in \mathcal{F} \) so that \( \sigma \prec K \) and \( \mathcal{A}(\sigma) \subset K \). Let \( \sigma_J \) be the subsimplex of \( K \) associated with \( J \), \( \sigma_J \subset \mathcal{A}(\sigma) \). Then the polynomial \( p_J \) is defined; so, for \( J \in \mathcal{D}(\sigma) \cap K' \) with \( K' \in \mathcal{F} \), \( K' \neq K \), let \( \tau_J \) be the subsimplex of \( K' \) associated with \( J \). Then there is a unique way to give Bézier ordinates to the domain points in \( \tau_J \) in such a way that they define a polynomial \( q_J \) which equals \( p_J \). Furthermore, in this manner, \( c_J \) can be defined for all \( J \in \mathcal{D}(\sigma) \setminus \mathcal{A}(\sigma) \).

2. Suppose \( c_J \) has been defined for all \( J \in \bigcup_{\dim \sigma < n} \mathcal{D}(\sigma) \).

3. Let \( J \in \mathcal{D}(\sigma) \) with \( \sigma \) an \( n \)-simplex and \( K \in \mathcal{F} \) such that \( \mathcal{A}(\sigma) \subset K \). And let again \( \sigma_J \) be the subsimplex of \( K \) associated with \( J \). Note that
\[
(48) \quad \sigma_J \subset \mathcal{D}(\sigma) \cap K \subset \mathcal{A}(\sigma) \cup \left( \bigcup_{\tau < \sigma} \mathcal{D}(\tau) \cap K \right).
\]

Then by the induction hypothesis and by the fact that the Bézier ordinates have been defined on \( \mathcal{A} \), we have that all of the Bézier ordinates on \( \sigma_J \) are defined; therefore \( p_J \) is defined. So as before, for \( J \in \mathcal{D}(\sigma) \cap K' \), \( K' \neq K \), we can define \( q_J \) in the same way we did when \( \sigma \) was a vertex. Thus, \( c_1 \) has been defined for all \( I \in I_d \).

Next, we need to show that the piecewise polynomial function defined by the \( c_1 \)'s is well defined.

Suppose \( p_L = q_L \) for \( L \in K \) and \( \sum_{v \in \xi} l_v = d - \rho \).

Let \( J \in \mathcal{D}(\sigma) \cap K' \) and suppose that \( J \in \xi_L \) with \( J \neq L \), where
\[
(49) \quad \xi_L = \{ I \in K' : i_v \geq l_v \quad v \in \xi, \quad \xi \in \mathcal{P} \cup \mathcal{T}, \quad \text{and} \quad \sigma \prec \xi \}
\]
and
\[
(50) \quad \tau_J = \{ I \in K' : i_v \geq j_v \quad v \in \sigma \}.
\]
Then \( J \in \xi_L \) implies \( j_v \geq l_v \), which in turn implies \( \tau_j \subset \xi_L \). Therefore, \( q_j \) is a subpolynomial of \( q_L \). Similarly, \( p_j \) is a subpolynomial of \( p_L \). Hence, \( p_j = q_j \) since \( q_L \cup p_L \in C^\infty(\xi) \). Therefore, the Bézier ordinates are well defined and by construction, the piecewise polynomial defined is a superspline. \( \square \)

**Corollary 10.** We have \( \dim S_d^\prime(\mathcal{F}) = \sum_{\sigma \in \mathcal{F} \cup T} |\mathcal{A}(\sigma)| \).

In view of the above corollary, to compute \( \dim S_d^\prime(\mathcal{F}) \) we need only to know the cardinality of \( \mathcal{A}(\sigma) \) for every \( \sigma \in \mathcal{F} \). Clearly, \( |\mathcal{A}(\sigma)| \) depends only on \( m = \dim \sigma \). The following theorem is proved in Alfeld and Sirvent [5].

**Theorem 11.** We have \( \dim S_d^\prime(\mathcal{F}) = \sum_{m=0}^{k} \phi(m)f_m \), where \( f_m \) is the number of \( m \)-simplices of \( \mathcal{F} \cup T \), and for \( m = 0, \ldots, k \), \( \phi(m) = \phi_m^k(0) \), where, with \( \rho_m = r2^{k-m-1} \), the quantities \( \phi^m_q(p) \) are defined recursively by

\[
\phi^m_q(p) = \sum_{j=0}^{p-\rho} \left( \begin{array}{c} j + m - q - 1 \\ j \end{array} \right) \left[ \left( \begin{array}{c} d - p + j + q \\ q \end{array} \right) - \sum_{i=0}^{q-1} \left( \begin{array}{c} q + 1 \\ i + 1 \end{array} \right) \phi^q_i(p + j) \right] 
\]

if \( 0 < q < m \), and

\[
\phi^m_0(p) = \sum_{j=0}^{p-\rho} \left( \begin{array}{c} j + m - 1 \\ j \end{array} \right) = \left( \begin{array}{c} \rho_0 - p + m \\ m \end{array} \right),
\]

\[
\phi^k_k(0) = \left( \begin{array}{c} d + k \\ k \end{array} \right) - \sum_{m=0}^{k-1} \left( \begin{array}{c} k + 1 \\ m + 1 \end{array} \right) \phi^k_m(0).
\]

8. **Minimally supported bases**

**Definition 12.** The star of a simplex \( \sigma \in \mathcal{F} \cup T \), denoted \( \operatorname{star}(\sigma) \), is the set of all \( k \)-simplices \( K \in \mathcal{F} \) such that \( \sigma \) is a face of \( K \).

**Definition 13.** A basis \( \{l_\mu : \mu = 1, 2, \ldots, \dim S_d^\prime(\mathcal{F})\} \) is said to be minimally supported if for each basis function \( l_\mu \), there exists a simplex \( \sigma \in \mathcal{F} \cup T \) such that the support of \( l_\mu \) is contained in \( \operatorname{star}(\sigma) \).

The basis functions constructed in the proof of Theorem 9 are minimally supported: using the same construction, we can define cardinal supersplines

\[
l_1 \in S_d^\prime(\mathcal{F}) : l_1(J) = \delta_{IJ} \quad \forall I, J \in \mathcal{A}
\]

so that, if \( I \in \mathcal{A}(\sigma) \), then \( c_J = 0 \) \( \forall J \in I_d \setminus \operatorname{star}(\sigma) \) and \( c_1 = 1 \).

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