CONVERGENCE OF AN ENERGY-PRESERVING SCHEME FOR THE ZAKHAROV EQUATIONS IN ONE SPACE DIMENSION

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Abstract. An energy-preserving, linearly implicit finite difference scheme is presented for approximating solutions to the periodic Cauchy problem for the one-dimensional Zakharov system of two nonlinear partial differential equations. First-order convergence estimates are obtained in a standard “energy” norm in terms of the initial errors and the usual discretization errors.

1. Introduction

In [11] Zakharov introduced a system of equations to model the propagation of Langmuir waves in a plasma. If we denote by $N(x, t)$ ($x \in \mathbb{R}$, $t > 0$) the deviation of the ion density from its equilibrium value, and by $E(x, t)$ the envelope of the high-frequency electric field, then the one-dimensional system takes the form

\begin{align*}
\text{(ZS.E)} & \quad iE_t + E_{xx} = NE, \\
\text{(ZS.N)} & \quad N_{tt} - N_{xx} = \frac{\partial^2}{\partial x^2}(|E|^2).
\end{align*}

We solve on $\{x \in \mathbb{R}, t > 0\}$ and supplement (ZS) by prescribing initial values for $E$, $N$, and $N_t$:

\begin{align*}
E(x, 0) &= E^0(x), \quad N(x, 0) = N^0(x), \quad N_t(x, 0) = N^1(x).
\end{align*}

Most of the interest to date in (ZS) stems from two particular features. Firstly, (ZS) admits solitary wave solutions [3]. Secondly, in three space dimensions, (ZS) was derived to model the collapse of caverns (cf. [11]). An intriguing and still unresolved question remains in three dimensions as to whether smooth data can generate a solution which becomes singular in finite time.

As is well known, (ZS) possesses the two formal invariants

\begin{align*}
\text{(2)} & \quad \int_{-\infty}^{\infty} |E(x, t)|^2 \, dx = \int_{-\infty}^{\infty} |E(x, 0)|^2 \, dx, \\
\text{(3)} & \quad \int_{-\infty}^{\infty} \left( |E_x|^2 + \frac{1}{2}(|v|^2 + N^2) + N|E|^2 \right) \, dx = \text{const},
\end{align*}

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where $v$ is given by

$$v = -u_x, \quad u_{xx} = N_t.$$  

We know that these are sufficient for global weak existence (cf. [9]). Also from [9] the same conclusion holds in three dimensions under an additional "smallness" condition. Moreover, higher-order estimates from [9] guarantee the existence of a smooth solution in one dimension provided smooth data are prescribed.

It is such a smooth solution of (ZS) with periodic boundary conditions which we approximate numerically in this paper. A spectral method is used in [5]; while practical results seem very good, the convergence issue is not rigorously addressed. Our algorithm uses an approximation of "Crank-Nicolson" type on the linear parts of (ZS). We approximate the solution over a fixed but arbitrary time interval $0 \leq t \leq T$.

The nonlinear terms in (ZS) are then approximated in such a way that:

(i) the discrete $L^2$-norm (over a period) of the approximation to $E$ is conserved; and

(ii) a discrete analogue of the total energy is conserved.

This discrete energy will be shown to be bounded below by a positive definite form. The scheme is linearly implicit and involves only two periodic tridiagonal solvers to advance one step in time. We obtain first-order convergence estimates in the natural "energy norm" in terms of initial errors and standard discretization errors.

In the references we list several papers where conservative schemes have been employed [2, 4, 6, 8]. Related results are to be found in [1, 10].

The standard summation by parts formula is

$$\sum_{j=1}^{J} v_j(u_{j+1} - 2u_j + u_{j-1}) = v_{j+1}(u_{j+1} - u_j) - v_1(u_1 - u_0)$$

$$- \sum_{j=1}^{J} (v_{j+1} - v_j)(u_{j+1} - u_j).$$

The "summed" terms cancel whenever $\{u_k\}, \{v_k\}$ are $J$-periodic mesh functions.

Although [9] treats the Cauchy problem on all of space, the methods given there (i.e., Galerkin) could be extended to deal with the periodic case studied here. Constants depending on $T$ and the Cauchy data are written $c_T$, while constants depending only on the data are generically written as $c$. These will change from line to line without explicit mention.

This scheme has been implemented; details will appear elsewhere.

### 2. The finite difference scheme

Let $T > 0$ be arbitrary; we will approximate the solution to the periodic Cauchy problem for (ZS) over the time interval $0 \leq t \leq T$. We first state hypotheses on the Cauchy data and the solution:

(H0) The Cauchy data

$$E(x, 0) = E^0(x), \quad N(x, 0) = N^0(x), \quad N_t(x, 0) = N^1(x)$$
are \( C^\infty \) and \( L \)-periodic. Moreover,

\[
\int_0^L N^1(x) \, dx = 0,
\]

\[
\sum_{j=1}^J N^1(jh) = 0 \quad \text{for any } h > 0 \text{ with } Jh = L.
\]

(HE) The periodic Cauchy problem possesses a unique smooth global solution.

In order to write the scheme, we define

\[
\delta u_k \equiv \Delta x^{-1}(u_{k+1} - u_k),
\]

\[
\delta^2 u_k \equiv \Delta x^{-2}(u_{k+1} - 2u_k + u_{k-1}),
\]

\[
\lambda = \frac{\Delta t}{\Delta x}, \quad \beta = \frac{\Delta t}{\Delta x^2}
\]

with \( \Delta t, \Delta x > 0 \). Now for \( J \) a positive integer we choose \( \Delta x = \frac{l}{J}, \Delta t > 0 \) such that

\[
n\Delta t \leq T
\]

and define \( t^l = l\Delta t, x_j = j\Delta x \quad (l = 0, \ldots, n; \quad j = 0, \ldots, J) \).

Our scheme is

\[
i \frac{E_k^{n+1} - E_k^n}{\Delta t} + \frac{1}{2} \delta^2 E_k^n + \frac{1}{2} \delta^2 E_k^{n+1} = \frac{1}{4}(N_k^n + N_k^{n+1})(E_k^n + E_k^{n+1}),
\]

\[
\frac{N_k^{n+1} - 2N_k^n + N_k^{n-1}}{\Delta t^2} - \frac{1}{2} \delta^2 N_k^n - \frac{1}{2} \delta^2 N_k^{n-1} = \delta^2(|E_k^n|^2).
\]

In both relations \( k = 1, \ldots, J, \quad n \geq 0 \) in the first and \( n \geq 1 \) in the second. Here we take \( E_k^n, N_k^n \) to be \( J \)-periodic mesh functions, i.e.,

\[
E_k^n = E_j^n, \quad N_k^n = N_j^n \quad \text{if } k \equiv j \pmod{J}.
\]

The scheme is supplemented with the initial values

\[
E_k^0 = E^0(x_k),
\]

\[
N_k^0 = N^0(x_k), \quad N_k^1 = N_k^0 + \Delta t N^1(x_k).
\]

We claim that the scheme is uniquely solvable: multiplying (8.N) by \( \Delta t^2 \), we see that the coefficient matrix for the unknown \( \{N_k^{n+1}\}_{k=1}^J \), of order \( J \times J \), is

\[
A_N = \begin{bmatrix}
1 + \lambda^2 & -\frac{\lambda^2}{2} & 0 & \cdots & -\frac{\lambda^2}{2} \\
-\frac{\lambda^2}{2} & 1 + \lambda^2 & -\frac{\lambda^2}{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{\lambda^2}{2} & 0 & \cdots & -\frac{\lambda^2}{2} & 1 + \lambda^2
\end{bmatrix},
\]

which is invertible by Gerschgorin for any \( \lambda > 0 \). The coefficient matrix for the unknown \( \{E_k^{n+1}\}_{k=1}^J \) has the form

\[
iI - A_E,
\]

where both matrices are square and of order \( J \times J \).
\( A_E \) is symmetric and has the form

\[
\begin{pmatrix}
(A_E)_{11} & -\frac{\beta}{2} & 0 & \ldots & -\frac{\beta}{2} \\
-\frac{\beta}{2} & (A_E)_{22} & -\frac{\beta}{2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\frac{\beta}{2} & 0 & \ldots & -\frac{\beta}{2} & (A_E)_{JJ}
\end{pmatrix}
\]

(13)

where

\[
(A_E)_{kk} = \beta + \frac{\Delta t}{4}(N_k^n + N_{k+1}^{n+1}).
\]

(14)

Since \( A_E \) has only real eigenvalues, \( i I - A_E \) is invertible. Thus the scheme is uniquely solvable at each time step. Indeed, putting \( n = 0 \) in (8.E), we can solve for \( \{E_k^0\} \), since \( N_k^0, N_k^1, E_k^0 \) are known from the data. Putting \( n = 1 \) in (8.N), we can then solve for \( \{N_k^1\} \) and, using \( \{N_k^2\} \), we can put \( n = 1 \) in (8.E) and solve for \( \{E_k^2\} \), etc.

We summarize with

**Lemma 1.** Assume the data satisfy (H0). Then the scheme (8.E), (8.N) is uniquely solvable at each time step.

**Lemma 2.** Let the data satisfy (H0). Define \( \{u_k^n\} \) by

\[
\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} = \frac{N_{k+1}^{n+1} - N_k^n}{\Delta t}, \quad k = 1, \ldots, J - 1,
\]

\[
u_0 = u_J = 0.
\]

Extend \( \{u_k^n\} \) by defining

\[
u_k^n = u_j^n \quad \text{if} \quad k \equiv j \pmod{J}.
\]

Then

\[
u_k^n = -\Delta x \sum_{j=1}^{J-1} G(x_k, x_j) \frac{N_{k+1}^{n+1} - N_j^n}{\Delta t},
\]

where

\[
G(x, y) = \begin{cases} 
 x(1 - \frac{x}{L}), & 0 \leq x \leq y \leq L, \\
 y(1 - \frac{y}{L}), & 0 \leq y \leq x \leq L.
\end{cases}
\]

**Proof.** The proof that the given representation is indeed a solution is a straightforward computation and is omitted. The only issue is one of compatibility. Summing the definition of \( u_k^n \), we see that it is required that

\[
\sum_{k=1}^{J}(N_{k+1}^{n+1} - N_k^n) = 0.
\]

When \( n = 0 \), this is true by hypotheses (H0) and (10). Using (8.N), we can write

\[
N_{k+1}^{n+1} - N_k^n = N_k^n - N_k^{n-1} + \frac{\Delta t^2}{2} \delta^2(N_k^{n+1} + N_k^{n-1} + 2|E_k^n|^2).
\]

Using induction, we sum both sides over \( k \). The sum of the first two terms on the right vanishes by the induction hypothesis; the sum of the remaining terms vanishes by periodicity. \( \square \)
Theorem 1. Let the data satisfy (H0). Then the scheme (8) possesses the following two invariants:

(a) \[ \sum_k |E^n_k|^2 \Delta x = \text{const} \quad (n \Delta t < T). \]

(b) Define \( u^n_k \) as in Lemma 2, so that \( \delta^2 u^n_k = (N'^{n+1}_k - N^n_k) / \Delta t \). Then

\[ \mathcal{G}^{n+1}_d \equiv \Delta x \sum_k \left[ |\delta E^{n+1}_k|^2 + \frac{1}{2} (\delta u^n_k)^2 + \frac{1}{4} \left\{ (N^n_k)^2 + (N'^{n+1}_k)^2 \right\} \right. \]

\[ + \frac{1}{2} (N^n_k + N'^{n+1}_k)|E^{n+1}_k|^2 \] = \text{const} for \( n \Delta t < T \). The sums run over \( 1 \leq k \leq J \).

Thus the discrete \( L^2 \)-norm of \( E^n \) over a period is conserved, and the form of \( \mathcal{G}^n_d \) is similar to that for the exact solution in (2), (3).

We show that \( \mathcal{G}^n_d \) is bounded below by a positive definite form. For this purpose, we put

(15) \[ \|E^n\|_2^2 \equiv \sum_k |E^n_k|^2 \Delta x, \]

(16) \[ \|\delta E^n\|_2^2 \equiv \sum_k |\delta E^n_k|^2 \Delta x, \]

with similar quantities for \( N^n \). We make note of the discrete Sobolev inequality

(17) \[ \sup_k |u^n_k| \leq c \|u\|_2^{1/2} \|\delta u\|_2^{1/2} \]

valid for periodic mesh functions \( \{u_k\} \). Indeed, denoting the Fourier coefficients of the mesh function \( u \) by \( \{c_m\} \), we write

\[ |u_k| \leq c \left( \sum_{|m| \leq M} + \sum_{|m| > M} \right) |c_m| \]

\[ \leq c M^1/2 \left( \sum_m |c_m|^2 \right)^{1/2} + c M^{-(1/2)} \left( \sum_m |m|^2 |c_m|^2 \right)^{1/2} \]

and optimize on \( M \).

The last term \( \mathcal{L} \) in \( \mathcal{G}^n_d \) is estimable by

\[ |\mathcal{L}| \leq \frac{1}{2} \sum_k |N^n_k||E^{n+1}_k|^2 \Delta x + \frac{1}{2} \sum_k |N'^{n+1}_k||E^{n+1}_k|^2 \Delta x \]

\[ \leq \frac{\varepsilon}{4} \sum_k ((N^n_k)^2 + (N'^{n+1}_k)^2) \Delta x + \frac{1}{2 \varepsilon} \sum_k |E^{n+1}_k|^4 \Delta x \]

for any \( \varepsilon > 0 \). Choosing \( \varepsilon = 1/2 \), we get the bound

\[ |\mathcal{L}| \leq \frac{1}{8} \sum_k \Delta x ((N^n_k)^2 + (N'^{n+1}_k)^2) + \|E^{n+1}\|_4^2. \]
By the Sobolev inequality (17) and part (a) of the theorem,
\[ \|E^{n+1}\|^4 \leq c\|E^n\|_2^2\|E^{n+1}\|_\infty^2 \leq c\|E^n\|_\infty^2 \leq c\delta E^{n+1}\|_2 \]
\[ \leq \frac{1}{4}\|\delta E^{n+1}\|_2^2 + c. \]

This gives us

**Lemma 3.** There is a constant $c$, depending only on the data, such that the solution of the discrete scheme (8.8), (8.7) satisfies
\[ \sum_k \Delta x[|E^{n+1}_k|^2 + |\delta E^{n+1}_k|^2 + (\delta u^n_k)^2 + (N^n_k)^2 + (N^{n+1}_k)^2] \leq c, \]
and hence \( \sup_k |E^n_k| \leq c. \)

**Proof of Theorem 1.** As is well known, part (a) is obtained by multiplying (8.8) by $E^n_{k+1} - E^n_k$, summing over $k$, $k = 1, \ldots, J$, and taking the imaginary part.

In order to verify (b), we multiply (8.8) by $E^n_{k+1} - E^n_k$ and sum on $k$. Adding this to its conjugate, we obtain
\[ I_n + I_{n+1} = \frac{1}{4} \sum_k (N^{n+1}_k + N^n_k) \cdot 2 \text{Re}(E^{n+1}_k + E^n_k)(E^{n+1}_k - E^n_k), \]
where
\[ I_m = \frac{1}{\Delta x^2} \text{Re} \sum_k (E^{n+1}_k - E^n_k)(E^{m+1}_k - 2E^m_k + E^m_{k-1}) \quad (m = n, n + 1). \]

The right side of (18) equals
\[ \frac{1}{2} \sum_k (|E^{n+1}_k|^2 - |E^n_k|^2)(N^{n+1}_k + N^n_k). \]

Summing by parts, we get for the left side of (18)
\[ I_n + I_{n+1} = -\frac{1}{\Delta x^2} \sum_k |E^{n+1}_{k+1} - E^{n+1}_k|^2 + \frac{1}{\Delta x^2} \sum_k |E^n_{k+1} - E^n_k|^2. \]

Thus (19), (20) yield the identity
\[ -\sum_k |\delta E^{n+1}_k|^2 + \sum_k |\delta E^n_k|^2 = \frac{1}{2} \sum_k (|E^{n+1}_k|^2 - |E^n_k|^2)(N^{n+1}_k + N^n_k). \]

We obtain the contribution from \( \{N^n_k\} \) by recalling from Lemma 2 that
\[ \delta^2 u^n_k = \frac{u^n_{k+1} - 2u^n_k + u^n_{k-1}}{\Delta x^2} = \frac{N^{n+1}_k - N^n_k}{\Delta t} \]
and by multiplying (8.9) by $\frac{1}{2}(u^n_k + u^n_{k-1})$ and then summing on $k$. There results
\[ I - II = III, \]
where

\[
I = \frac{1}{2} \sum_k \frac{(N_k^{n+1} - 2N_k^n + N_k^{n-1})}{\Delta t^2} (u_k^n + u_k^{n-1}),
\]

\[
II = \frac{1}{4} \sum_k \frac{(u_k^n + u_k^{n-1})}{\Delta x^2} \left[ N_{k+1}^{n+1} - 2N_k^{n+1} + N_{k-1}^{n+1} + N_{k+1}^{n-1} - 2N_k^{n-1} + N_{k-1}^{n-1} \right],
\]

\[
III = \frac{1}{2} \sum_k \frac{(u_k^n + u_k^{n-1})}{\Delta x^2} \left[ |E_{k+1}^n|^2 - 2|E_k^n|^2 + |E_{k-1}^n|^2 \right].
\]

Term III is summed by parts:

\[
III = -\frac{1}{2\Delta x^2} \sum_k \left[ (u_{k+1}^n + u_{k+1}^{n-1}) - (u_k^n + u_k^{n-1}) \right] |E_k^n|^2
\]

\[
= -\frac{1}{2\Delta x^2} \sum_k \left[ u_k^n + u_k^{n-1} - u_{k-1}^n - u_{k-1}^{n-1} \right] |E_k^n|^2
\]

\[
+ \frac{1}{2\Delta x^2} \sum_k \left[ u_{k+1}^n + u_{k+1}^{n-1} - u_{k}^n - u_{k}^{n-1} \right] |E_k^n|^2,
\]

where we have shifted \( k \rightarrow k - 1 \) to obtain the first sum. Thus, by (22),

\[
III = \frac{1}{2\Delta x^2} \sum_k |E_k^n|^2 \left[ (u_{k+1}^n - 2u_k^n + u_{k-1}^n) + (u_{k+1}^{n-1} - 2u_k^{n-1} + u_{k-1}^{n-1}) \right]
\]

\[
= \frac{1}{2} \sum_k |E_k^n|^2 \left[ \frac{N_{k+1}^{n+1} - N_k^n}{\Delta t} + \frac{N_k^n - N_{k-1}^{n-1}}{\Delta t} \right]
\]

\[
= \frac{1}{2\Delta t} \sum_k |E_k^n|^2 (N_{k+1}^{n+1} - N_k^n).
\]

To evaluate I, we note that by (22)

\[
\delta^2 u_k^n - \delta^2 u_k^{n-1} = \frac{N_{k+1}^{n+1} - N_k^n}{\Delta t} - \left( \frac{N_k^n - N_{k-1}^{n-1}}{\Delta t} \right) = \frac{N_{k+1}^{n+1} - 2N_k^n + N_{k-1}^{n-1}}{\Delta t}.
\]

Thus,

\[
I = \frac{1}{2\Delta t} \sum_k (u_k^n + u_k^{n-1}) [\delta^2 u_k^n - \delta^2 u_k^{n-1}]
\]

and, summing this by parts, we get

\[
(26) \quad I = -\frac{1}{2\Delta t} \sum_k (\delta u_k^n)^2 + \frac{1}{2\Delta t} \sum_k (\delta u_k^{n-1})^2.
\]
Summing II now by parts, we find
\[
II = -\frac{1}{4\Delta x^2} \sum_k \left[ (u_{k+1}^n + u_{k+1}^{n-1}) - (u_k^n + u_{k-1}^{n-1}) \right] \cdot \left[ (N_{k+1}^{n+1} - N_k^{n+1}) + (N_{k+1}^{n-1} - N_k^{n-1}) \right]
\]
\[
= -\frac{1}{4\Delta x^2} \sum_k \left[ u_k^n + u_{k-1}^{n-1} - u_k^{n-1} - u_{k-1}^{n-1} \right] \left[ N_{k+1}^{n+1} + N_k^{n-1} \right]
\]
\[
+ \frac{1}{4\Delta x^2} \sum_k \left[ u_k^n + u_{k-1}^{n-1} + u_k^{n-1} - u_{k-1}^{n-1} \right] \left[ N_{k+1}^{n+1} + N_k^{n-1} \right],
\]
where we have again shifted \( k \to k - 1 \) to get the first sum. Thus, by (22),
\[
II = \frac{1}{4\Delta x^2} \sum_k \left( N_{k+1}^{n+1} + N_k^{n-1} \right) \left[ (u_k^n - 2u_k^n + u_{k-1}^n) + (u_{k+1}^{n-1} - 2u_k^{n-1} + u_{k-1}^{n-1}) \right]
\]
\[
= \frac{1}{4} \sum_k \left( N_{k+1}^{n+1} + N_k^{n-1} \right) \frac{N_{k+1}^{n+1} - N_k^n}{\Delta t} + \frac{N_k^n - N_{k-1}^{n-1}}{\Delta t}
\]
\[
= \frac{1}{4\Delta t} \sum_k \left[ (N_{k+1}^{n+1})^2 - (N_k^{n-1})^2 \right].
\]
Therefore, equation (23) yields
\[
-\frac{1}{2\Delta t} \sum_k (\delta u_k^n)^2 - \frac{1}{4\Delta t} \sum_k (N_{k+1}^{n+1})^2
\]
\[
= -\frac{1}{2\Delta t} \sum_k (\delta u_k^{n-1})^2 - \frac{1}{4\Delta t} \sum_k (N_k^{n-1})^2
\]
\[
+ \frac{1}{2\Delta t} \sum_k |E_k^n|^2 (N_{k+1}^{n+1} - N_k^{n-1}).
\]
Now multiply this by \( \Delta t \) and add the result to (21) to get
\[
-\frac{1}{2} \sum_k (\delta u_k^n)^2 - \frac{1}{4} \sum_k (N_{k+1}^{n+1})^2 - \sum_k |\delta E_k^{n+1}|^2
\]
\[
= -\frac{1}{2} \sum_k (\delta u_k^{n-1})^2 - \frac{1}{4} \sum_k (N_k^{n-1})^2 - \sum_k |\delta E_k^n|^2
\]
\[
+ \frac{1}{2} \sum_k [ |E_k^n|^2 (N_{k+1}^{n+1} - N_k^{n-1}) + (|E_k^{n+1}|^2 - |E_k^n|^2)(N_{k+1}^{n+1} + N_k^n) ].
\]
The last term here equals
\[
\frac{1}{2} \sum_k |E_k^{n+1}|^2 (N_{k+1}^{n+1} + N_k^n) - \frac{1}{2} \sum_k |E_k^n|^2 (N_{k+1}^{n+1} + N_k^{n-1}).
\]
Therefore, when we define \( \mathcal{E}_d^{n+1} \) as in part (b) of Theorem 1, (28) implies \( \mathcal{E}_d^{n+1} = \mathcal{E}_d^n \) and hence \( \mathcal{E}_d^n = \mathcal{E}_d^0 \) and energy is conserved. \( \square \)

In order to state the main theorem, we define the errors by
\[
e_k^n = E(x_k, t^n) - E_k^n,
\]
\[
\eta_k^n = N(x_k, t^n) - N_k^n.
\]
Here, $E^n_k$, $N^n_k$ are computed from the scheme \((8.E), (8.N)\) for $n\Delta t \leq T$, $1 \leq k \leq J$.

**Lemma 4.** Let the data satisfy (H0). Define \(\{U^n_k\}\) by

\[
\frac{U^n_{k+1} - 2U^n_k + U^n_{k-1}}{\Delta x^2} = \frac{\eta^{n+1}_k - \eta^n_k}{\Delta t}, \quad k = 1, \ldots, J - 1,
\]

\[U_0 = U_J = 0.\]

Extend \(\{U^n_k\}\) by defining

\[U^n_k = U^n_j \quad \text{if} \quad k \equiv j \mod J.\]

Then

\[U^n_k = -\Delta x \sum_{j=1}^{J-1} G(x_k, x_j) \frac{\eta^{n+1}_j - \eta^n_j}{\Delta t},\]

where

\[G(x, y) = \begin{cases} x(1 - \frac{y}{L}), & 0 \leq x \leq y \leq L, \\ y(1 - \frac{x}{L}), & 0 \leq y \leq x \leq L. \end{cases}\]

**Proof.** The actual computation showing that the given representation is a solution is easy and is omitted. As in Lemma 2, there remains the compatibility question. Using the definition (30) of \(\eta^n_k\), we have

\[
\delta^2 U^n_k = \Delta t^{-1} \left[ N(x_k, t^{n+1}) - N^n_k + N(x_k, t^n) - N^n_k \right]
\]

\[= -\delta^2 u^n_k + \Delta t^{-1} [N(x_k, t^{n+1}) - N(x_k, t^n)].\]

Therefore, as in Lemma 2, we require that

\[S = \sum_{k=1}^J [N(x_k, t^{n+1}) - N(x_k, t^n)] = 0.\]

We expand $N(x, t)$ in a Fourier series with Fourier coefficients \(\{c_m\}\):

\[N(x, t) = \sum_m c_m(t) \exp \left( \frac{2im\pi x}{L} \right).\]

Thus, \(c_0(t)\) is proportional to \(\int_0^L N(x, t) \, dx\). Integrating (ZS.N) over a period, we see that this integral is a linear function of \(t\). In fact, \(c_0(t)\) is constant in time in view of (H0). Now we write

\[
\sum_{k=1}^J N(x_k, t) = \sum_m c_m(t) \sum_{k=1}^J \exp \left( \frac{2im\pi x_k}{L} \right)
\]

and evaluate the inner sum explicitly. Using \(x_k = k\Delta x = kL/J\), we see that this sum over \(k\) vanishes unless \(m = 0\), in which case

\[\sum_{k=1}^J N(x_k, t) = Jc_0(t).\]

Hence \(S = 0\) as desired. \(\Box\)

The norms are defined, e.g., as \(||e^n||^2_2 = \sum_{k=1}^J |e^n_k|^2 \Delta x\), etc.
Theorem 2. Let $T > 0$; assume (HE) and that the data satisfy (H0). Given any positive integer $J$, let $J \Delta x = L$ and choose $\Delta t = \Delta x$. Let $E^n_k$, $N^n_k$ be computed from the scheme (8.E), (8.N), (9), (10) for $n\Delta t \leq T$. Define

$$g^n_n = H_{fl}^{n+1} + H_{fl}^{n+1} + W_{fl}^{n+1} + W_{fl}^{n+1} + W_{fl}^{n+1}.$$  \(33\)

(Thus, $g^n_n$ is the (square of the) "energy norm" of the errors.)

Then there exists a constant $c_T$ depending only on the data and $T$, with the property that for $\Delta x$ sufficiently small, we have

$$g^n_n \leq c_T \left( g^0_n + \Delta x^2 \right).$$

Moreover, $g^0 = O(\Delta x^2)$, and hence

$$g^n_n \leq c_T \Delta x^2 \quad \text{as} \quad \Delta x \to 0.$$  

The proof of Theorem 2 will be given in the next section.

Remark. The choice $\Delta t = \Delta x$ allows us to easily combine several estimates. It is seen from the proof that the same estimates can be obtained provided $\Delta t$ is bounded both above and below by a constant times $\Delta x$.

### 3. Convergence estimates, proof of the main theorem

We begin by defining the standard discretization errors

$$\tau^n_k = \frac{i}{\Delta t} \left( E(x_k, t^{n+1}) - E(x_k, t^n) \right)$$

$$+ \frac{1}{2\Delta x^2} \left( E(x_{k+1}, t^n) - 2E(x_k, t^n) + E(x_{k-1}, t^n) \right)$$

$$+ \frac{1}{2\Delta x^2} \left( E(x_{k+1}, t^{n+1}) - 2E(x_k, t^{n+1}) + E(x_{k-1}, t^{n+1}) \right)$$

$$- \frac{1}{4} \left( N(x_k, t^n) + N(x_k, t^{n+1}) \right) \left( E(x_k, t^n) + E(x_k, t^{n+1}) \right).$$  \(34\)

and

$$\sigma^n_k = \frac{1}{\Delta t^2} \left( N(x_k, t^{n+1}) - 2N(x_k, t^n) + N(x_k, t^{n-1}) \right)$$

$$- \frac{1}{2\Delta x^2} \left( N(x_{k+1}, t^{n+1}) - 2N(x_k, t^{n+1}) + N(x_{k-1}, t^{n+1}) \right)$$

$$- \frac{1}{2\Delta x^2} \left( N(x_{k+1}, t^{n-1}) - 2N(x_k, t^{n-1}) + N(x_{k-1}, t^{n-1}) \right)$$

$$- \frac{1}{\Delta x^2} \left( |E(x_{k+1}, t^n)|^2 - 2|E(x_k, t^n)|^2 + |E(x_{k-1}, t^n)|^2 \right).$$  \(35\)

As usual, these measure the amount by which the exact solutions fail to satisfy the approximate equations.

Recall that $E, N$ are smooth solutions.

**Lemma 5.** We have $|\tau^n_k| + |\sigma^n_k| = O(\Delta t^2 + \Delta x^2)$ as $\Delta x, \Delta t \to 0$.

**Proof.** By Taylor's theorem and (ZS.E) we can write the first three terms $\tau_3$
in $\tau^n_k$ as

$$
\tau_3 = i \left( E_t(x_k, t^n) + \frac{1}{2}\Delta t E_{tt}(x_k, \beta^n_k) \right) + \frac{1}{2} \left( E_{xx}(x_k, t^n) + O(\Delta x^2) \right) + \frac{1}{2} \left( E_{xx}(x_k, t^{n+1}) + O(\Delta x^2) \right) (t^n < \beta^n_k < t^{n+1})
$$

$$
= i E_t(x_k, t^n) + \frac{i\Delta t}{2} E_{tt}(x_k, \beta^n_k) + O(\Delta x^2)
$$

$$
+ \frac{1}{2} \left[ N(x_k, t^n)E(x_k, t^n) - iE_t(x_k, t^n) \right]
+ \frac{1}{2} \left[ N(x_k, t^{n+1})E(x_k, t^{n+1}) - iE_t(x_k, t^{n+1}) \right]
$$

$$
= \frac{N(x_k, t^n)E(x_k, t^n) + N(x_k, t^{n+1})E(x_k, t^{n+1})}{2} + O(\Delta x^2)
$$

$$
+ \frac{i\Delta t}{2} E_{tt}(x_k, \beta^n_k) + \frac{i}{2} \left[ E_t(x_k, t^n) - E_t(x_k, t^{n+1}) \right]
$$

$$
= \frac{N(x_k, t^n)E(x_k, t^n) + N(x_k, t^{n+1})E(x_k, t^{n+1})}{2} + O(\Delta t^2 + \Delta x^2).
$$

Now the result for $\tau^n_k$ will follow if

$$
\frac{1}{2} \left( N(x_k, t^n)E(x_k, t^n) + N(x_k, t^{n+1})E(x_k, t^{n+1}) \right)
- \frac{1}{4} \left( N(x_k, t^n) + N(x_k, t^{n+1}) \right) \left( E(x_k, t^n) + E(x_k, t^{n+1}) \right)
$$

$$
= O(\Delta t^2 + \Delta x^2).
$$

Simple algebra shows that this expression equals

$$
\frac{1}{4} \left( E(x_k, t^{n+1}) - E(x_k, t^n) \right) \left( N(x_k, t^{n+1}) - N(x_k, t^n) \right),
$$

and hence is $O(\Delta t^2)$.

As for $\sigma^n_k$, we use Taylor's theorem again to write

$$
\sigma^n_k = \left( N_{tt}(x_k, t^n) + O(\Delta t^2) \right) - \frac{1}{2} \left( N_{xx}(x_k, t^{n+1}) + O(\Delta x^2) \right)
- \frac{1}{2} \left( N_{xx}(x_k, t^{n-1}) + O(\Delta x^2) \right) - \left( \frac{\partial^2}{\partial x^2} E(x_k, t^n) \right)^2 + O(\Delta x^2).
$$

The result follows from (ZS.N), since

$$
N_{xx}(x_k, t^n) - \frac{1}{2} \left( N_{xx}(x_k, t^{n+1}) + N_{xx}(x_k, t^{n-1}) \right) = O(\Delta t^2).
$$

Recall that the errors are defined by (29), (30). In order to obtain the error equations we subtract (8.E) from the definition (34) of $\tau^n_k$ to get

$$
i \left( \frac{e^{n+1}_k - e^n_k}{\Delta t} \right) + \frac{1}{2} \beta^2 e^n_k + \frac{1}{2} \beta^2 e^{n+1}_k
$$

$$
= \tau^n_k + \frac{1}{4} \left[ N(x_k, t^n) + N(x_k, t^{n+1}) \right] \left[ E(x_k, t^n) + E(x_k, t^{n+1}) \right]
$$

$$
- \frac{1}{4} \left[ N^n_k + N^{n+1}_k \right] \left[ E^n_k + E^{n+1}_k \right]
$$

$$
= \tau^n_k + \frac{1}{4} \left[ (e^n_k + e^{n+1}_k) \left( E(x_k, t^n) + E(x_k, t^{n+1}) \right)
+ (N^n_k + N^{n+1}_k) (e^n_k + e^{n+1}_k) \right].
$$

(36)
Subtracting (8.N) from (35), the definition of $\sigma^n_k$, we get similarly

$$\begin{align*}
\frac{\eta_{k+1}^n - 2\eta_k^n + \eta_{k-1}^n}{\Delta t^2} &= \frac{1}{2} \delta^2 \eta_{k+1}^n - \frac{1}{2} \delta^2 \eta_{k-1}^n \\
&= \sigma^n_k + \delta^2 (|E(x_k, t^n)|^2 - |E_k^n|^2).
\end{align*}$$

In a sequence of lemmas we will derive energy estimates on $e$ and $\eta$.

**Lemma 6 (L2-estimate of e).** There are constants $c, c_T$ such that for $\Delta x, \Delta t$ sufficiently small,

$$\|e^{n+1}\|_2^2 \leq (1 + c\Delta t)\|e^n\|_2^2 + c_T(\Delta t^2 + \Delta x^2)^2 \Delta t + c\Delta t (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).$$

**Proof.** As in Theorem 1(a), we multiply (36) by $e_{k+1}^n + e_k^n$, sum on $k$, and take the imaginary part to get

$$I + II = III + IV,$$

where

$$\begin{align*}
I &= \frac{1}{\Delta t} \Re \sum_k (e_{k+1}^n - e_k^n)(e_{k+1}^n + e_k^n) = \frac{1}{\Delta t} \sum_k (|e_{k+1}^n|^2 - |e_k^n|^2), \\
II &= \frac{1}{2} \Im \sum_k (e_{k+1}^n + e_k^n)(\delta^2 e_{k+1}^n + \delta^2 e_k^n), \\
III &= \Im \sum_k (e_{k+1}^n + e_k^n)\tau_k^n, \\
IV &= \frac{1}{4} \Im \sum_k (e_{k+1}^n + e_k^n)[(\eta_{k+1}^n + \eta_k^{n+1})(E(x_k, t^n) + E(x_k, t^{n+1}))],
\end{align*}$$

the last simplifying since $N$ is real. All sums are taken over indices $k$ with $1 \leq k \leq J$.

Term I is as desired. For III, we have from Lemma 5

$$|III| \leq c \sum_k (|e_{k+1}^n|^2 + |e_k^n|^2) + c \sum_k |\tau_k^n|^2$$

$$\leq c\Delta x^{-1}(\|e_{n+1}\|_2^2 + \|e^n\|_2^2) + c_T(\Delta t^2 + \Delta x^2) \cdot J,$$

and IV is easily estimable by

$$|IV| \leq c \sup_{x, t \leq T} |E(x, t)| \cdot \sum_k \frac{(|e_{k+1}^n| + |e_k^n|)\Delta x^{1/2} \cdot (|\eta_{k+1}^n| + |\eta_k^n|)\Delta x^{1/2}}{\Delta x}$$

$$\leq c\Delta x^{-1}[\|e_{n+1}\|_2^2 + \|e^n\|_2^2 + \|\eta_{n+1}\|_2^2 + \|\eta^n\|_2^2].$$

As before, term II vanishes upon summation by parts. Now we multiply (38) by $\Delta t\Delta x$ and use the bounds derived above to get

$$\|e^{n+1}\|_2^2 \leq \|e^n\|_2^2 + c\Delta t(\|e_{n+1}\|_2^2 + \|e^n\|_2^2) + c_T(\Delta t^2 + \Delta x^2)^2 \cdot J\Delta t \Delta x$$

$$+ c\Delta t(\|e_{n+1}\|_2^2 + \|e^n\|_2^2 + \|\eta_{n+1}\|_2^2 + \|\eta^n\|_2^2).$$

Thus, we have

$$\begin{align*}
(1 - c\Delta t)\|e^{n+1}\|_2^2 &\leq (1 + c\Delta t)\|e^n\|_2^2 + c_T(\Delta t^2 + \Delta x^2)^2 \Delta t \\
&\quad + c\Delta t(\|\eta_{n+1}\|_2^2 + \|\eta^n\|_2^2),
\end{align*}$$

and the result follows. □
When estimating the energy, we will need bounds on the discrete potentials $u^n_k$ from Lemma 2 and $U^n_k$ from Lemma 4.

**Lemma 7.** There is a constant $c$ depending only on the data such that

$$\sup_k |u^n_k| \leq c.$$  

**Proof.** We write, using the boundary condition $u^n_0 = 0$,

$$|u^n_k| = \left| \sum_{j=1}^k (u^n_j - u^n_{j-1}) \right| = \left| \Delta x \sum_{j=1}^k \delta u^n_{j-1} \right| \leq \|\delta u^n\|_2 (J \Delta x)^{1/2},$$

and this is bounded by Lemma 3 and the definition of $J$. □

**Lemma 8.** Let $U^n_k$ be defined as in Lemma 4. There is a constant $c$ such that

$$\sup_k |U^n_k| \leq c (\mathcal{E}^n)^{1/2}.$$  

**Proof.** The proof is the same as that of Lemma 7, but in the last step we use the definition of $\mathcal{E}^n$ from Theorem 2. □

**Lemma 9 (Energy of $e$).** Let $h = \Delta t = \Delta x$, and define

$$\Pi^n = \frac{1}{2} \text{Re} \sum_k (E(x_k, t^n) + E(x_k, t^{n+1}))(\eta^n_k + \eta^n_{k+1})e^{n+1}_k,$$

$$\Pi^n = \frac{1}{2} \sum_k (N^n_k + N^n_{k+1})|e^{n+1}_k|^2.$$  

Then

$$\frac{1}{2} \|\delta e^n\|^2_2 + h(\Pi^n - \Pi^{n-1}) - (\frac{1}{2} \|\delta e^{n+1}\|^2_2 + h(\Pi^n + \Pi^{n+1}))$$

$$= O[h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3].$$

**Proof.** As in Theorem 1(b), we multiply (36) by $(e^{n+1} - e^n_k)$, sum over $k$, $k = 1, \ldots, J$, add the result to its conjugate, and take the real part. There results the identity

$$I_0 = I + II + III,$$

where

$$I_0 = \text{Re} \sum_k (e^{n+1}_k - e^n_k)(\delta^2 e^n_k + \delta^2 e^{n+1}_k),$$

$$|I| = \left| 2 \text{Re} \sum_k \tau^n_k (e^{n+1}_k - e^n_k) \right|$$

$$\leq c_T h^2 J \|\delta e^n\|^2_2 \leq c_T h^{1/2} (\mathcal{E}^n + \mathcal{E}^{n-1})^{1/2},$$

$$II = \frac{1}{2} \text{Re} \sum_k (\eta^n_k + \eta^n_{k+1})(E(x_k, t^n) + E(x_k, t^{n+1}))(e^{n+1}_k - e^n_k),$$

$$III = \frac{1}{2} \sum_k (N^n_k + N^n_{k+1})(|e^{n+1}_k|^2 - |e^n_k|^2).$$
We sum $I_0$ by parts to get
\begin{equation}
I_0 = \frac{1}{2} \sum_k |\delta e_k^n|^2 - \frac{1}{2} \sum_k |\delta e_k^{n+1}|^2.
\end{equation}

Next, we rewrite term $III$ as
\begin{equation}
III = \frac{1}{2} \sum_k [(N_k^{n+1} + N_k^n)|e_k^{n+1}|^2 - (N_k^n + N_k^{n-1})|e_k^n|^2 + (N_k^{n-1} - N_k^{n+1})|e_k^n|^2]
\end{equation}
\begin{equation}
\equiv III^n - III^{n-1} + \frac{1}{2} \sum_k (N_k^{n-1} - N_k^{n+1})|e_k^n|^2,
\end{equation}
where
\begin{equation}
III^n = \frac{1}{2} \sum_k (N_k^{n+1} + N_k^n)|e_k^{n+1}|^2.
\end{equation}

Recall from the definition (Lemma 2) of $u_k^n$ that
\begin{equation}
\delta^2 u_k^n = \frac{N_k^{n+1} - N_k^n}{h}.
\end{equation}
Thus,
\begin{equation}
\delta^2 (u_k^n + u_k^{n-1}) = \frac{N_k^{n+1} - N_k^{n-1}}{h},
\end{equation}
and therefore
\begin{equation}
III = III^n - III^{n-1} - \frac{1}{2} h \sum_k |e_k^n|^2 \delta^2 (u_k^n + u_k^{n-1}).
\end{equation}

We sum by parts to get for the last term the bound
\begin{equation}
O \left( h \sum_k |e_k^n| |\delta e_k^n| (|\delta u_k^n| + |\delta u_k^{n-1}|) \right) = O(||e^n||_*||\delta e^n||_2 (||\delta u^n||_2 + ||\delta u^{n-1}||_2))
\end{equation}
\begin{equation}
= O(||e^n||_2^{1/2} ||\delta e^n||_3^{3/2}),
\end{equation}
where we have used Lemma 3. Hence,
\begin{equation}
III = III^n - III^{n-1} + O(\varepsilon^{n-1}).
\end{equation}

Consider now term $II$. For brevity we set
\begin{equation}
w_k^n = E(x_k, t^n) + E(x_k, t^{n+1}),
\end{equation}
so that
\begin{equation}
w_k^n - w_k^{n-1} = E(x_k, t^{n+1}) - E(x_k, t^{n-1}) = O(h).
\end{equation}

We write term $II$ as
\begin{equation}
II = \frac{1}{2} \text{Re} \sum_k (\eta_k^{n+1} + \eta_k^n) w_k^n (e_k^{n+1} - e_k^n)
\end{equation}
\begin{equation}
= \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^{n+1} e_k^{n+1} - \frac{1}{2} \text{Re} \sum_k w_k^{n-1} \eta_k^n e_k^n - \frac{1}{2} \text{Re} \sum_k (w_k^n - w_k^{n-1}) \eta_k^n e_k^n
+ \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^n e_k^{n+1} - \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^{n+1} e_k^n.
\end{equation}
Now we add and subtract the expression
\[ \frac{1}{2} \text{Re} \sum_k w_k^{n-1} \eta_k^{n-1} \varepsilon_k^n \]
and define
\[ \text{II}^n = \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^n \varepsilon_k^{n+1} + \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^n \varepsilon_k^{n+1}. \]

Then, using Lemma 4, we can write II as
\[
\begin{align*}
\text{II} & = \text{II}^n - \text{II}^{n-1} + O(\varepsilon^{n-1}) \\
& \quad - \frac{1}{2} \text{Re} \sum_k \varepsilon_k^n [(w_k^n - w_k^{n-1}) \eta_k^{n+1} + w_k^n (\eta_k^{n+1} - \eta_k^{n-1})] \\
& = \text{II}^n - \text{II}^{n-1} + O(\varepsilon^{n-1}) + O((\varepsilon^{n-1})^{1/2} (\varepsilon^n)^{1/2}) \\
& \quad - \frac{1}{2} \text{Re} \sum_k h \varepsilon_k^n w_k^{n-1} \delta^2 (U_k^n + U_k^{n-1}).
\end{align*}
\]

We sum the last term here once by parts; it equals
\[
\begin{align*}
\frac{1}{2} \text{Re} \sum_k h \delta (U_k^n + U_k^{n-1}) (w_k^{n-1} \delta \varepsilon_k^n + \varepsilon_k^{n+1} \delta w_k^{n-1}) \\
& = O[(\|\delta U^n\|_2^2 + \|\delta U^{n-1}\|_2^2)(\|E(t^n)\|_\infty \|\delta \varepsilon^n\|_2 + \|E_x(t^{n-1})\|_\infty \|\varepsilon^n\|_2)] \\
& = O((\varepsilon^n + \varepsilon^{n-1})).
\end{align*}
\]

Using these estimates in (46), we have
\[ \text{II} = \text{II}^n - \text{II}^{n-1} + O(\varepsilon^n + \varepsilon^{n-1}). \]

Finally, we multiply the relation
\[ I_0 = I + II + III \]
by \( h \) and use the estimates for each of these terms derived above to get
\[ \frac{1}{2} \|\delta \varepsilon^n\|_2^2 + \frac{1}{2} \|\delta \varepsilon^{n+1}\|_2^2 = O(h^3) + O(h(\varepsilon^n + \varepsilon^{n-1})) \]
+ \( II^n h + III^n h - II^{n-1} h - III^{n-1} h \),
or
\[
\begin{align*}
\frac{1}{2} \|\delta \varepsilon^n\|_2^2 + h(II^{n-1} + III^{n-1}) - (\frac{1}{2} \|\delta \varepsilon^{n+1}\|_2^2 + h(II^n + III^n)) \\
& = O(h(\varepsilon^n + \varepsilon^{n-1}) + h^3),
\end{align*}
\]
and this is the statement of Lemma 9. \( \square \)

**Lemma 10 (\( \eta \)-energy).** Let \( h = \Delta t = \Delta x \). Then
\[
\begin{align*}
- \frac{1}{2} \|\delta U^n\|_2^2 - \frac{1}{4} (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \frac{1}{2} \|\delta U^{n-1}\|_2^2 + \frac{1}{4} (\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2) \\
& = O(h^5 + h(\varepsilon^n + \varepsilon^{n-1})).
\end{align*}
\]

**Proof.** Recall from Lemma 4 the relation
\[
\delta^2 U_k^n = \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{h^2} = \frac{\eta_{k+1}^n - \eta_k^n}{h}.
\]
We multiply the \( \eta \)-equation (37) by \( \frac{1}{2}(U^n_k + U^{n-1}_k) \) and sum over \( k \) to get the identity

\[
I_1 - I_2 - I_3 = I_4 + I_5 ,
\]

where

\[
I_1 = \frac{1}{2} \sum_k (U^n_k + U^{n-1}_k) \frac{(\eta^{n+1}_k - 2\eta^n_k + \eta^{n-1}_k)}{h^2} ,
\]

\[
I_2 = \frac{1}{4} \sum_k (U^n_k + U^{n-1}_k) \delta^2 \eta^{n-1}_k ,
\]

\[
I_3 = \frac{1}{4} \sum_k (U^n_k + U^{n-1}_k) \delta^2 \eta^{n+1}_k ,
\]

\[
I_4 = \frac{1}{2} \sum_k \sigma_k^n (U^n_k + U^{n-1}_k) = O(h^{n} + \mathcal{E}^{n-1})^{1/2} \quad \text{(by Lemma 8)} ,
\]

\[
I_5 = \frac{1}{2} \sum_k (U^n_k + U^{n-1}_k) \delta^2 \{|E(x_k, t^n)|^2 - |E_k^n|^2\} .
\]

We sum \( I_2 + I_3 \) by parts, with the result

\[
I_2 + I_3 = -\frac{1}{4} \sum_k \delta(\eta^{n+1}_k + \eta^{n-1}_k) \delta(U^n_k + U^{n-1}_k) .
\]

Expansion of this yields

\[
-\frac{1}{4h^2} \sum_k (\eta^{n+1}_k + \eta^{n-1}_k - \eta^{n+1}_k - \eta^{n-1}_k)(U^n_{k+1} + U^{n-1}_{k+1} - U^n_k - U^{n-1}_k)
\]

\[
= -\frac{1}{4h^2} \sum_k (\eta^{n+1}_k + \eta^{n-1}_k)(U^n_k + U^{n-1}_k - U^n_{k-1} - U^{n-1}_{k-1})
\]

\[
+ \frac{1}{4h^2} \sum_k (\eta^{n+1}_k + \eta^{n-1}_k)(U^n_{k+1} + U^{n-1}_{k+1} - U^n_k - U^{n-1}_k) ,
\]

where we put \( k \to k - 1 \) to get the first sum. Thus,

\[
I_2 + I_3 = \frac{1}{4h^2} \sum_k (\eta^{n+1}_k + \eta^{n-1}_k)[U^n_{k+1} - 2U^n_k + U^{n-1}_{k+1} + U^n_{k-1} - 2U^{n-1}_k + U^{n-1}_{k-1}]
\]

\[
= \frac{1}{4} \sum_k (\eta^{n+1}_k + \eta^{n-1}_k)[\delta^2 U^n_k + \delta^2 U^{n-1}_k]
\]

\[
= \frac{1}{4h} \sum_k (\eta^{n+1}_k + \eta^{n-1}_k)[(\eta^{n+1}_k - \eta^n_k) + (\eta^n_k - \eta^{n-1}_k)]
\]

\[
= \frac{1}{4h} \sum_k ((\eta^{n+1}_k)^2 - (\eta^{n-1}_k)^2)
\]

\[
= \frac{1}{4h} \sum_k ((\eta^{n+1}_k)^2 + (\eta^n_k)^2) - \frac{1}{4h} \sum_k ((\eta^n_k)^2 + (\eta^{n-1}_k)^2) .
\]
Term $I_5$ is summed once by parts, with the result

$$I_5 = -\frac{1}{2h^2} \sum_k \left( U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1} \right)$$

(52)

\begin{align*}
&\cdot \left( |E(x_{k+1}, t^n)|^2 - |E_{k+1}^n|^2 - |E(x_k, t^n)|^2 + |E_k^n|^2 \right),
\end{align*}

and further expansion yields

$$I_5 = -\frac{1}{2h} \Re \sum_k \left( \delta U_k^n + \delta U_k^{n-1} \right)$$

\begin{align*}
&\cdot \left[ (E(x_{k+1}, t^n) - E_{k+1}^n) \left( E(x_{k+1}, t^n) + \overline{E}_{k+1}^n \right) \\
&\quad - (E(x_k, t^n) - E_k^n) \left( E(x_k, t^n) + \overline{E}_k^n \right) \right] \\
&= -\frac{1}{2h} \Re \sum_k \left( \delta U_k^n + \delta U_k^{n-1} \right) \left[ (e_k^n - e_k^{n-1}) \left( E(x_{k+1}, t^n) + \overline{E}_{k+1}^n \right) \\
&\quad + e_k^n \left( E(x_{k+1}, t^n) - \overline{E}(x_k, t^n) + \overline{E}_k^n \right) \right] \\
&= O \left( \sum_k (|\delta U_k^n| + |\delta U_k^{n-1}|)(|\delta e_k^n| + |e_k^n|(c_T + |\delta E_k^n|)) \right) \\
&= O(h^{-1}(\mathcal{G}^n + \mathcal{G}^{n-1} + h^{-1}\|e^n\|_\infty(\mathcal{G}^n + \mathcal{G}^{n-1})^{1/2}\|\delta E^n\|_2) \\
&= O(h^{-1}(\mathcal{G}^n + \mathcal{G}^{n-1}))
\end{align*}

by the Sobolev inequality applied to $\|e^n\|_\infty$.

Lastly, for the term $I_1$, we note from (31) that

$$\delta^2 U_k^n - \delta^2 U_k^{n-1} = \frac{1}{h} \left( \eta_{k+1}^{n+1} - \eta_k^n - (\eta_k^n - \eta_{k-1}^{n-1}) \right) = \frac{\eta_{k+1}^{n+1} - 2\eta_k^n + \eta_{k-1}^{n-1}}{h},$$

and hence

$$I_1 = \frac{1}{2h} \sum_k (U_k^n + U_k^{n-1}) \delta^2 (U_k^n - U_k^{n-1}).$$

(54)

Summing by parts we get

$$I_1 = -\frac{h^{-2}}{2h} \sum_k \left[ U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1} \right]$$

\begin{align*}
&\cdot \left[ U_{k+1}^{n-1} - U_{k+1}^{n-1} - (U_k^n - U_k^{n-1}) \right].
\end{align*}

(55)

This can be rewritten as

$$I_1 = -\frac{1}{2h} \sum_k \left[ (\delta U_k^n)^2 - (\delta U_k^{n-1})^2 \right] = -\frac{1}{2h^2} \left[ \|\delta U^n\|_2^2 - \|\delta U^{n-1}\|_2^2 \right].$$

(56)

Returning now to (50), we multiply it by $h^2$ to get

$$-\frac{1}{2} \|\delta U^n\|_2^2 + \frac{1}{2} \|\delta U^{n-1}\|_2^2$$

\begin{align*}
&\quad - \frac{1}{4} (\|\eta_{n+1}^n\|_2^2 + \|\eta_n^n\|_2^2) + \frac{1}{4} (\|\eta_{n-1}^{n-1}\|_2^2 + \|\eta_{n-1}^{n-1}\|_2^2) \\
&= O(h(\mathcal{G}^n + \mathcal{G}^{n-1}) + h^2).
\end{align*}

(57)
This completes the proof. □

Proof of Theorem 2. Let us define $h = \Delta t = \Delta x$ and

$$(58) \quad H^{n-1} = \frac{1}{2} \| \delta e^n \|_2^2 + \frac{1}{2} \| \delta U^{n-1} \|_2^2 + \frac{1}{4} (\| \eta^n \|_2^2 + \| \eta^{n-1} \|_2^2).$$

Recall the definitions of the terms $\Pi^n$, $\Pi^n$ from Lemma 9. Adding the conclusions of Lemmas 9 and 10, we get

$$(59) \quad H^n + h(\Pi^n + \Pi^n) = H^{n-1} + h(\Pi^{n-1} + \Pi^{n-1})$$

$$+ O(h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3),$$

where, from (33),

$$(60) \quad \mathcal{E}^n = \frac{1}{2} \| e^{n+1} \|_2^2 + H^n.$$

Now, for a (large) positive constant $\gamma$ (to be chosen below) set

$$(61) \quad \hat{\mathcal{E}}^n = \gamma \| e^{n+1} \|_2^2 + H^n + h(\Pi^n + \Pi^n).$$

From (59) and Lemma 6 it follows that

$$(62) \quad \hat{\mathcal{E}}^n \leq \gamma (1 + ch) \| e^n \|_2^2 + \gamma cT h^5 + c \gamma h(\| \eta^{n+1} \|_2^2 + \| \eta^n \|_2^2)$$

$$+ H^{n-1} + h(\Pi^{n-1} + \Pi^{n-1}) + O(h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3).$$

Now we estimate $\Pi^n$, $\Pi^n$ easily by

$$(63) \quad h|\Pi^n| = \left| \frac{h}{2} \text{Re} \sum_k (E(x_k, t^n) + E(x_k, t^{n+1}))(\eta^{n+1}_k + \eta^n_k)e_k^{n+1} \right|$$

$$\leq c(\| E(t^n) \|_\infty + \| E(t^{n+1}) \|_\infty)\| \eta^{n+1} + \eta^n \|_2 \| e^{n+1} \|_2$$

$$\leq \frac{1}{16} (\| \eta^{n+1} \|_2^2 + \| \eta^n \|_2^2) + c \| e^{n+1} \|_2^2$$

(with a constant $c$ depending only on the data), and

$$(64) \quad h|\Pi^n| \leq \left| \frac{h}{2} \sum_k (N^{n+1}_k + N^n_k)|e_k^{n+1}|^2 \right|$$

$$\leq c \| e^{n+1} \|_\infty \| N^{n+1} + N^n \|_2 \| e^{n+1} \|_2$$

$$\leq c(\| N^{n+1} \|_2 + \| N^n \|_2) \| e^{n+1} \|_2^{1/2} \| \delta e^{n+1} \|_2^{1/2}$$

by the Sobolev inequality. Since the first factor is bounded by Lemma 3, we obtain

$$(65) \quad h|\Pi^n| \leq \frac{1}{8} \| \delta e^{n+1} \|_2^2 + c \| e^{n+1} \|_2^2$$

with $c$ depending only on the data. Adding (63) to (64), we obtain

$$h(|\Pi^n| + |\Pi^n|) \leq \frac{1}{8} \| \delta e^{n+1} \|_2^2 + \frac{1}{16} (\| \eta^{n+1} \|_2^2 + \| \eta^n \|_2^2) + c \| e^{n+1} \|_2^2$$

$$\leq \frac{1}{4} H^n + c \| e^{n+1} \|_2^2$$

by the definition (58) of $H^n$. It follows that $\hat{\mathcal{E}}^n$ is strictly positive for a sufficiently large choice of $\gamma$, depending only on the data.

In fact, we can choose $\gamma$ large enough so that $\gamma > 1$ and

$$(66) \quad \hat{\mathcal{E}}^n \geq \frac{c}{2} \| e^{n+1} \|_2^2 + \frac{3}{4} H^n$$
with a constant $c > 0$ depending only on the data and on $\gamma$.

Hence, from (62),

$$\mathcal{E}^n \leq \mathcal{E}^{n-1} + c_T \gamma h (\mathcal{E}^n + \mathcal{E}^{n-1}) + c_T \gamma h^3.$$  

Now from its definition, we have, since $\gamma > 1$,

$$\mathcal{E}^n = \frac{1}{2} \|e^{n+1}\|_2^2 + H^n < \gamma \|e^{n+1}\|_2^2 + H^n$$

$$= \mathcal{E}^n - h(I''^n + III^n) \leq \mathcal{E}^n + \frac{1}{4} H^n + c \|e^{n+1}\|_2^2,$$

where we have used (65). Since $H^n \leq \mathcal{E}^n$ by (60), we conclude that

$$\frac{3}{4} \mathcal{E}^n \leq \mathcal{E}^n + c \|e^{n+1}\|_2^2 \leq c_T \mathcal{E}^n$$

in view of (66). For any such (fixed) choice of $\gamma$, we obtain from (67)

$$(1 - c_T h) \mathcal{E}^n \leq (1 + c_T h) \mathcal{E}^{n-1} + c_T h^3.$$

It follows that for $h = \Delta t = \Delta x$ sufficiently small, depending only on $T$ and the data, we have

$$\mathcal{E}^n \leq c_T (\mathcal{E}^0 + h^2).$$

Since $(\mathcal{E}^n)^{1/2}$ is equivalent to $(\mathcal{E}^n)^{1/2}$, the first part of the proof is complete.

It remains to estimate $\mathcal{E}^0$. From (29), (30) and (9), (10) we have

$$e_0^0 = 0, \quad \eta_0^0 = 0, \quad \eta_1^0 = O(h^2).$$

Thus, $\|\eta_1^0\|_2^2 + \|\eta_0^0\|_2^2 = O(h^4)$. From Lemma 6 with $n = 0$, $\|e^1\|_2 = O(h^5)$, and hence

$$\|\delta e^1\|_2^2 = h^{-1} \sum_{k=1}^J \|e^1_{k+1} - e^1_k\|^2 \leq 4h^{-1} \sum_{k=1}^J \|e^1_k\|^2 = O(h^3).$$

Finally, we bound $\|\delta U^n_k\|_2$. We multiply the definition of $U^n_k$ by $U^n_k$, sum over $k$, and then sum by parts to get

$$\|\delta U^0\|_2^2 = - \sum_{k=1}^J U^0_k (\eta^0_k - \eta^0) = \sum_{k=1}^{J-1} \sum_{j=1}^{J-1} G(x_k, x_j) \eta^1_k \eta^1_j,$$

where we have used Lemma 4 again. Since $G$ is continuous, it follows from general considerations (or from explicit computation, using $\eta^1_k = O(h^2)$) that the last expression is $O(h^2)$, and this completes the proof. □

Bibliography


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