MULTIPlicITIES OF DIHEDRAL DISCRIMINANTS

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Dedicated to the memory of Helmut Hasse

Abstract. Given the discriminant $d_k$ of a quadratic field $k$, the number of cyclic relative extensions $N|k$ of fixed odd prime degree $p$ with dihedral absolute Galois group of order $2p$, which share a common conductor $f$, is called the multiplicity of the dihedral discriminant $d_N = f^{2(p-1)}d_k^p$. In this paper, general formulas for multiplicities of dihedral discriminants are derived by analyzing the $p$-rank of the ring class group mod $f$ of $k$. For the special case $p = 3$, $d_k = -3$, an elementary proof is given additionally. The theory is illustrated by a discussion of all known discriminants of multiplicity $\geq 5$ of totally real and complex cubic fields.

Introduction

Let $p$ be an odd prime and $K|\mathbb{Q}$ a cyclic extension of degree $p$. Then it is well known [9, 15, 7] that the conductor of $K$ must have the form $f = p^e \cdot q_1 \cdots q_t$, where $e = 0$ or $e = 2$, $t \geq 0$, and the $q_i$ are pairwise distinct rational primes satisfying $q_i \equiv 1 \pmod{p}$ for $i = 1, \ldots, t$. The discriminant of $K$ is just a power of the conductor, $d_K = f^{p-1}$. If, for any positive integer $f$, the number of cyclic extensions $K|\mathbb{Q}$ of degree $p$ which share the same conductor $f$ is denoted by $m(f)$, then

$$\sum_{f|f'} m(f') = \frac{1}{p-1}(p^p - 1),$$

where $\rho = \dim_{\mathbb{F}_p}(\mathbb{Q}(f)^\times/Q_f^\times \cdot \mathbb{Q}(f^p)^\times) = \dim_{\mathbb{F}_p}(\text{Syl}_p U(\mathbb{Z}/f^\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p) = t + w$ with $Q_f^\times = \{ r \in \mathbb{Q}^\times \mid (r, f) = 1 \}$, $Q_f^\times = \{ r \in \mathbb{Q}^\times \mid r \equiv 1 \pmod{f} \}$, and $w = \frac{1}{2}e$. Moebius inversion yields an explicit formula for $m(f')$:

$$m(p^e \cdot q_1 \cdots q_t) = (p - 1)^{t+w-1}.$$

It is the aim of the present paper to establish similar formulas for multiplicities of discriminants in the case of non-Galois extensions $L|\mathbb{Q}$ of degree $p$ with dihedral normal closure $N$ of degree $2p$. For the sake of illustration, the formulas are applied to discriminants with multiplicities up to 16 of non-Galois

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cubic fields which occur in the most extensive recent numerical tables [10, 5, 11].

1. Cubic discriminants

Assume that $k$ is a fixed quadratic field with discriminant $d_k$. If $N|k$ is a cyclic cubic relative extension with conductor $f$ and with absolute Galois group $\text{Gal}(N|Q) \simeq S_3$, the symmetric group on three symbols, then [8, p. 578] $f$ must have the form

$$ f = 3^e \cdot q_1 \cdots q_t $$

with $0 \leq e \leq 2$, $t \geq 0$, and pairwise distinct rational primes $q_i \neq 3$ satisfying $q_i \equiv (d_k^e) \pmod{3}$ for $i = 1, \ldots, t$. Furthermore, the 3-exponent $e$ is restricted to the values 0, 2 if $d_k \equiv \pm 1 \pmod{3}$, and to the values 0, 1 if $d_k \equiv 3 \pmod{9}$, but no restrictions arise for $d_k \equiv -3 \pmod{9}$. An integer $f$ of this form will be called an admissible conductor for the quadratic discriminant $d_k$.

For any positive integer $f$, define the multiplicity $m(d_k, f)$ of $f$ with respect to $d_k$ to be the number of nonconjugate cubic fields $L|Q$ with coinciding discriminant $d_L = f^2 - d_k$. Then the following general formula for the recursive determination of multiplicities of cubic discriminants holds. This result will be proved in a more general context in §3.

**Theorem 1.1.** Let $f = 3^e \cdot q_1 \cdots q_t$ be an admissible conductor for the quadratic field $k$ with discriminant $d_k$ and 3-class rank $p$. Then

$$ \sum_{f'|f} m(d_k, f') = \frac{1}{2} (3^{p+l+w-\delta} - 1), $$

where the sum runs over all divisors $f'$ of $f$. Here,

$$ w = \begin{cases} 0 & \text{if } e = 0, \\ 2 & \text{if } e = 2 \text{ and } d_k \equiv -3 \pmod{9}, \\ 1 & \text{otherwise}, \end{cases} $$

and $\delta = \dim_{F_3}(I_{k,3}(f)/I_{k,3}(f) \cap (Q^\times(f)/k^\times(f)^3))$, where $k^\times(f)$ (resp. $Q^\times(f)$) denotes the numbers in $k^\times$ (resp. $Q^\times$) which are coprime to $f$, $k_f^* = \{ \gamma \in k^\times \mid \gamma \equiv 1 \pmod{f} \}$ is the group of generators of the ray mod $f$ of $k$, and $I_{k,3}(f) = I_{k,3} \cap k^\times(f)$ is the group of generators $\alpha \in k^\times$ of all principal ideals $\alpha F_k$ which are cubes of ideals of $k$.

2. Pure cubic discriminants

First, the complete solution of the multiplicity problem for the special case of pure cubic discriminants is obtained in a totally elementary way with the aid of the following well-known relationship [4] between the normalized radicand $D = m \cdot n^2$ of a pure cubic field $L = Q(\sqrt{D})$, where $m > n > 0$ are squarefree coprime integers, and the conductor $f$ of the corresponding relative extension $N|k$,

$$ f = \begin{cases} 3mn & \text{if } D \not\equiv \pm 1 \pmod{9} \text{ (field of Dedekind's 1st kind)}, \\ mn & \text{if } D \equiv \pm 1 \pmod{9} \text{ (field of Dedekind's 2nd kind)}. \end{cases} $$
Theorem 2.1. Let $f = 3^e \cdot q_1 \cdots q_t > 1$ be an admissible conductor for the special quadratic discriminant $d_k = -3$, i.e., $0 \leq e \leq 2$, $t \geq 0$, and $q_i \neq 3$ are arbitrary rational primes for $i = 1, \ldots, t$. Put

\[
u = \#\{1 \leq i \leq t \mid q_i \equiv \pm 1 \pmod{9}\},
\]

\[
u = \#\{1 \leq i \leq t \mid q_i \equiv \pm 2, \pm 4 \pmod{9}\}.
\]

Then the multiplicity $m(f) := m(-3, f)$ of the pure cubic discriminant $-3 \cdot f^2$ is given by

\[
m(3^e \cdot q_1 \cdots q_t) = \begin{cases} 2^t & \text{if } e = 2, \\ 2^\nu \cdot X_v & \text{if } e = 1, \\ 2^\nu \cdot X_{v-1} & \text{if } e = 0, \end{cases}
\]

where $X_j = \frac{1}{2}(2^j + (-1)^{j+1})$ for all $j \geq -1$.

Moreover, the multiplicities of conductors with 3-exponents $e = 0, 1$ satisfy the relation

\[
m(q_1 \cdots q_t) + m(3 \cdot q_1 \cdots q_t) = 2^{t-1}.
\]

Proof. To see that all claimed conductors are really admissible for $d_k = -3$, observe that $d_k \equiv -3 \pmod{9}$ and that the condition $q \equiv (\frac{d_k}{q}) \pmod{3}$ is satisfied by every prime $q$.

First, the case $e = 2$ is treated separately. The relation $f = 3^2 \cdot q_1 \cdots q_t$ is equivalent to $f = 3mn$, $3 \mid mn$, and thus also to $D \equiv 0 \pmod{3}$. In this case, there are $2t+1$ choices for the exponent systems $1 \leq w_0, w_1, \ldots, w_t \leq 2$ in cubefree radicands $D = q_1^{w_0} \cdot q_1^{w_1} \cdots q_t^{w_t}$ which all share the same value of $mn = q_1 \cdots q_t$. But only one of the two systems $(w_0, \ldots, w_t)$ and $(3-w_0, \ldots, 3-w_t)$ belongs to a normalized radicand. Hence,

\[
m(3^2 \cdot q_1 \cdots q_t) = \frac{1}{2}2^{t+1} = 2^t.
\]

Second, the cases $e = 1$ and $e = 0$ are investigated simultaneously. The relation $f = 3 \cdot q_1 \cdots q_t$ is equivalent to $f = 3mn$, $3 \mid mn$, and further to $D \equiv \pm 2, \pm 4 \pmod{9}$, whereas $f = q_1 \cdots q_t$ is equivalent to $f = mn$, $3 \nmid mn$, and also to $D \equiv \pm 1 \pmod{9}$. In both cases, there are $2^t$ choices for exponent systems $1 \leq w_1, \ldots, w_t \leq 2$ in cubefree radicands $D = q_1^{w_1} \cdots q_t^{w_t}$ which all share the same value of $mn = q_1 \cdot q_t$, but some of them $(D \equiv \pm 1 \pmod{9})$ give rise to conductor $f = mn$ and the others $(D \equiv \pm 2, \pm 4 \pmod{9})$ to conductor $f = 3mn$. Again, only one of the two systems $(w_1, \ldots, w_t)$ and $(3-w_1, \ldots, 3-w_t)$ belongs to a normalized radicand. (Both, the normalized and the nonnormalized radicand, are of the same Dedekind kind.) Therefore,

\[
m(q_1 \cdots q_t) + m(3 \cdot q_1 \cdots q_t) = \frac{1}{2}2^t = 2^{t-1}.
\]

To separate these two multiplicities, it is convenient to fix a value $u \geq 0$ of the number of prime divisors $q \equiv \pm 1 \pmod{9}$ of $D$ and to argue by induction with respect to the number $v \geq 0$ of prime divisors $q \equiv \pm 2, \pm 4 \pmod{9}$ of $D$. Obviously, $u + v = t$.

Induction start, $v = 0, 1$:

In the case $v = 0$, we have $mn = q_1 \cdots q_u$ with $u \geq 1$ and $D \equiv \pm 1 \pmod{9}$, whence

\[
Y_{-1} := m(q_1 \cdots q_u) = 2^{u-1},
\]

\[
Y_0 := m(3 \cdot q_1 \cdots q_u) = 0.
\]
In the case \( v = 1 \), we have \( mn = q_1 \cdots q_u \cdot q_{u+1} \) and \( D \equiv \pm 2, \pm 4 \pmod{9} \), whence

\[
m(q_1 \cdots q_u \cdot q_{u+1}) = 0 = Y_0,
Y_1 := m(3 \cdot q_1 \cdots q_u \cdot q_{u+1}) = 2^u.
\]

Induction step, \( v \to v + 1 \):

If the new prime factor \( q_{u+v+1} \equiv \pm 2, \pm 4 \pmod{9} \) and its square are multiplied by a radicand \( D \equiv \pm 1 \pmod{9} \), then there are generated two new radicands \( D \cdot q_{u+v+1}^{w_{u+v+1}} \equiv \pm 2, \pm 4 \pmod{9} \) with \( 1 \leq w_{u+v+1} \leq 2 \). However, if they are multiplied by a radicand \( D \equiv \pm 2, \pm 4 \pmod{9} \), then one of the two new radicands is congruent \( \pm 1 \pmod{9} \) (the one, where \( q_{u+v+1}^{w_{u+v+1}} \) represents the square of \( D \) in the group \( U(\mathbb{Z}/9\mathbb{Z})/\{\pm 1\} \simeq C(3) \)) and the other is congruent \( \pm 2, \pm 4 \pmod{9} \). Thus,

\[
m(q_1 \cdots q_{u+v+1}) = m(3 \cdot q_1 \cdots q_u \cdot q_{u+v}) = Y_v,
Y_{v+1} := m(3 \cdot q_1 \cdots q_{u+v+1}) = m(3 \cdot q_1 \cdots q_u \cdot q_{u+v}) + 2 \cdot m(q_1 \cdots q_{u+v})
= Y_v + 2Y_{v-1}.
\]

Consequently, the numbers \( Y_j \) \((j \geq -1)\) satisfy a binary linear recursion,

\[
Y_{j+1} = Y_j + 2Y_{j-1}
\]

for \( j \geq 0 \), with initial values \( Y_{-1} = 2^{u-1} \) and \( Y_0 = 0 \). This recursion can be solved by diagonalization of the corresponding matrix \( M \) in the equation

\[
\left( \begin{array}{c} Y_{j+1} \\ Y_j \end{array} \right) = \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right) \cdot \left( \begin{array}{c} Y_j \\ Y_{j-1} \end{array} \right).
\]

The characteristic polynomial of \( M \) is (denoting by \( I \) the identity matrix)

\[
det(x \cdot I - M) = x^2 - x - 2 = (x + 1) \cdot (x - 2).
\]

The eigenspaces of \( M \) with respect to the eigenvalues \(-1\) and \(2\) are

\[
\ker(M + I) = \mathbb{Q} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \ker(M - 2 \cdot I) = \mathbb{Q} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix},
\]

and \( M \) becomes diagonal under the transformation

\[
T^{-1} \cdot M \cdot T = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} := \Delta.
\]

Therefore, the solution of the recursion can be represented in the form

\[
\left( \begin{array}{c} Y_j \\ Y_{j-1} \end{array} \right) = M^j \cdot \left( \begin{array}{c} Y_0 \\ Y_{-1} \end{array} \right) = T \cdot \Delta^j \cdot T^{-1} \cdot \left( \begin{array}{c} Y_0 \\ Y_{-1} \end{array} \right)
= \frac{1}{3} 2^u \begin{pmatrix} (-1)^{j+1} + 2^j \\ (-1)^j + 2^{j-1} \end{pmatrix}
\]

for \( j \geq 0 \).

Hence, the solution is \( Y_j = 2^u \cdot X_j \) with \( X_j := \frac{1}{3}(2^j + (-1)^{j+1}) \) for all \( j \geq -1 \).

Remark. The proof of Theorem 2.1 used a very special elementary technique which cannot be applied to other types of cubic fields. In the next section, this result will be rederived as a particular instance of a much more general formula, which is deduced by completely different methods, using ring class groups of quadratic number fields.
Examples. As an application, the minimal occurrences of some higher multiplicities of pure cubic discriminants are determined. The first three cases appear in tables of complex cubic fields [1, 5, 12] which are ordered by discriminants. The further ones are constructed by means of Theorem 2.1, taking the smallest possible prime factors. These examples show that the normalized radicands associated with a fixed discriminant of higher multiplicity are spread rather widely in a table of pure cubic fields which is ordered by radicands, such as [11].

1. \( m(f) = 2 \) for \( f = 3^2 \cdot 2, d_L = -972 \), occurs in [1], with associated radicands \( D = 6, 12 \), and \( e = 2, u = 0, v = 1 \).
2. \( m(f) = 4 \) for \( f = 3^2 \cdot 2 \cdot 5, d_L = -24300 \), occurs in [12], with associated radicands \( D = 30, 60, 90, 150 \), and \( e = 2, u = 0, v = 2 \).
3. \( m(f) = 3 \) for \( f = 3 \cdot 2 \cdot 5 \cdot 7, d_L = -132300 \), occurs in [5], with associated radicands \( D = 70, 140, 490 \), and \( e = 1, u = 0, v = 3 \).
4. \( m(f) = 8 \) for \( f = 3^2 \cdot 2 \cdot 5 \cdot 7, d_L = -1190700 \), occurs in [11], with associated radicands \( D = 210, 420, 630, 1050, 1260, 1470, 2100, 2940 \), and \( e = 2, u = 0, v = 3 \).
5. \( m(f) = 5 \) for \( f = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 11, d_L = -16008300 \), occurs in [11], with associated radicands \( D = 770, 3850, 7700, 10780, 16940 \), and \( e = 1, u = 0, v = 4 \).
6. \( m(f) = 6 \) for \( f = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 17, d_L = -38234700 \), occurs in [11], with associated radicands \( D = 1190, 2380, 8330, 11900, 20230, 40460 \), and \( e = 1, u = 0, v = 3 \).
7. However, \( m(f) = 7 \) will never occur, since 7 is not a member of the sequence \((X_j)_{j \geq 1}\) in Theorem 2.1. The same is true for \( m(f) = 9, 13, 14, 15 \).
8. \( m(f) = 16 \) for \( f = 3^2 \cdot 2 \cdot 5 \cdot 7 \cdot 11, d_L = -144074700 \), occurs in [11], with associated radicands \( D = 2310, 4620, 6930, 11550, 13860, 16170, 23100, 25410, 32340, 34650, 48510, 50820, 69300, 76230, 80850, 97020 \), and \( e = 2, u = 0, v = 4 \).
9. \( m(f) = 11 \) for \( f = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 11 \cdot 13, d_L = -2705402700 \), with associated radicands \( D = 10010, 20020, 70070, 100100, 110110, 260260, 350350, 550550, 650650, 700700, 910910 \), and \( e = 1, u = 0, v = 3 \).
10. \( m(f) = 10 \) for \( f = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 11 \cdot 17, d_L = -4626398700 \), with associated radicands \( D = 13090, 65450, 130900, 183260, 222530, 287980, 458150, 719950, 1007930, 1112650 \), and \( e = 1, u = 1, v = 4 \).
11. \( m(f) = 12 \) for \( f = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 11 \cdot 19, d_L = -13802726700 \), with associated radicands \( D = 22610, 45220, 158270, 226100, 384370, 429590, 768740, 791350, 859180, 1582700, 2690590, 3007130 \), and \( e = 1, u = 2, v = 3 \).

3. Dihedral discriminants

As before, assume that \( k \) is a fixed quadratic field with discriminant \( d_k \). With \( p \) an odd rational prime, let \( N/k \) be a cyclic relative extension of degree \( p \) with conductor \( f \) and absolute Galois group \( \text{Gal}(N/Q) \simeq D_p \), the dihedral group of order \( 2p \). Then \( f \) is a rational integer and [2] must have the form

\[
f = p^e \cdot q_1 \cdots q_t
\]
with \(0 \leq e \leq 2\), \(t \geq 0\), and pairwise distinct rational primes \(q_i \neq p\) satisfying \(q_i \equiv \left(\frac{d_k}{q_i}\right) (\text{mod } p)\) for \(i = 1, \ldots, t\). Furthermore, the \(p\)-exponent \(e\) is restricted to the values 0, 2 if \(p\) is unramified in \(k\), and to the values 0, 1 if \(p | d_k\), except for the special configuration where \(p = 3\) and \(d_k \equiv -3 (\text{mod } 9)\). An integer \(f\) of this form will be called a \(p\)-admissible conductor for the quadratic discriminant \(d_k\).

For any positive integer \(f\), define the \(p\)-multiplicity \(m_p(d_k, f)\) of \(f\) with respect to \(d_k\) to be the number of nonisomorphic fields \(L|\mathbb{Q}\) of degree \(p\) sharing the same discriminant \(d_L = f^{p-1} \cdot d_k^{(p-1)/2}\). By the translation theorem of Galois theory, \(m_p(d_k, f)\) is also the number of cyclic extensions \(N|k\) of degree \(p\) with absolute group \(\text{Gal}(N|\mathbb{Q}) \simeq D_p\) sharing the common conductor \(f\).

**Remark.** If \(f\) is not a \(p\)-admissible conductor for \(d_k\), then certainly \(m_p(d_k, f) = 0\). But on the other hand, \(m_p(d_k, f)\) may be zero even for a \(p\)-admissible conductor \(f\) for \(d_k\), as will be shown for infinite families of fields in Corollaries 3.2 and 3.3.

Denote by \(\mathcal{I}_k(f)\) the group of (fractional) ideals of \(k\) coprime to \(f\), and by \(\mathcal{R}_{k,f} = \{\alpha \mathcal{O}_k \mid \alpha \in \mathbb{Q}^\times(f) \cdot k^\times_f\}\) the so-called ring mod \(f\) of \(k\), where \(\mathbb{Q}^\times(f)\) denotes the numbers in \(\mathbb{Q}^\times\) which are coprime to \(f\), and \(k^\times_f = \{\gamma \in k^\times \mid \gamma \equiv 1 (\text{mod } \mathfrak{f})\}\) is the group of generators of the ray mod \(f\) of \(k\). Then, by the Artin reciprocity law of class field theory [8], \(m_p(d_k, f)\) can be interpreted as the number of subgroups \(\mathcal{H}\) of index \(p\) of \(\mathcal{I}_k(f)\) which contain \(\mathcal{R}_{k,f}\) but not \(\mathcal{R}_{k,f'}\) for any proper divisor \(f' | f\), \(f' \neq f\). With the aid of this fact, the following formula for the recursive determination of multiplicities of dihedral discriminants can be derived.

**Theorem 3.1.** Let \(f = p^e \cdot q_1 \cdots q_t\) be a \(p\)-admissible conductor for a given quadratic field \(k\) with discriminant \(d_k\) and with \(p\)-class rank \(\rho = \rho_k(p)\). Then

\[
\sum_{f' \mid f} m_p(d_k, f') = \frac{1}{p - 1}(p^{\rho+t+w-\delta} - 1),
\]

where the sum runs over all divisors \(f'\) of \(f\). Here,

\[
w = \begin{cases} 
0 & \text{if } e = 0, \\
2 & \text{if } e = 2, \ p = 3, \ d_k \equiv -3 (\text{mod } 9), \\
1 & \text{otherwise},
\end{cases}
\]

and \(\delta = \delta(f) = \dim_{\mathbb{F}_p}(I_{k,p}(f)/I_{k,p}(f) \cap (\mathbb{Q}^\times(f) k^\times_f k^\times(f)^p)), \) where the numbers in \(k^\times\) which are coprime to \(f\) are denoted by \(k^\times(f)\), and \(I_{k,p}(f) = I_{k,p} \cap k^\times(f)\) with the group \(I_{k,p}\) of generators \(\alpha \in k^\times\) of all principal ideals \(\alpha \mathcal{O}_k\) which are \(p\)-th powers of ideals of \(k\).

**Proof.** Since the ring class group mod \(f\) of \(k\), \(\mathcal{I}_k(f)/\mathcal{R}_{k,f}\), is an abelian group, its \(p\)-elementary subgroup is isomorphic to \(\mathcal{I}_k(f)/(\mathcal{R}_{k,f}, \mathcal{I}_k(f)^p)\). Hence, any subgroup \(\mathcal{H}\) of index \(p\) in \(\mathcal{I}_k(f)\) which contains \(\mathcal{R}_{k,f}\) must in fact be an intermediate group \(\mathcal{R}_{k,f} \cdot \mathcal{I}_k(f)^p < \mathcal{H} < \mathcal{I}_k(f)\). Therefore,

\[
\sum_{f' \mid f} m_p(d_k, f') = \#\{\mathcal{H} < \mathcal{I}_k(f) \mid (\mathcal{I}_k(f) : \mathcal{H}) = p, \ \mathcal{R}_{k,f} \cdot \mathcal{I}_k(f)^p < \mathcal{H}\}
\]
is exactly the number of the hyperplanes (subspaces of codimension 1) of the $p$-elementary ring class group mod $f$ of $k$, viewed as a vector space over the finite field $\mathbb{F}_p$. But this number equals $\frac{1}{p-1}(p^{\rho'} - 1)$, where $\rho' = \dim_{\mathbb{F}_p}(\mathcal{I}_k(f)/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p)$ is the $p$-rank of $\mathcal{I}_k(f)/\mathcal{R}_k,f$.

If $\mathcal{P}_k(f) = \mathcal{P}_k \cap \mathcal{I}_k(f)$ denotes the principal ideals of $k$ coprime to $f$, then the factorization relation of elementary abelian $p$-groups,

$$\mathcal{I}_k(f)/\mathcal{P}_k(f) \cdot \mathcal{I}_k(f)^p \simeq (\mathcal{I}_k(f)/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p) \times (\mathcal{P}_k(f) \cdot \mathcal{I}_k(f)^p)/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p)$$

is equivalent to a direct product relation

$$\mathcal{I}_k(f)/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p \simeq (\mathcal{I}_k(f)/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p) \times (\mathcal{P}_k(f) \cdot \mathcal{I}_k(f)^p)/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p),$$

where the first factor is isomorphic to the $p$-elementary class group of $k$. Further, the homomorphism $k^\times \rightarrow \mathcal{P}_k$, $\alpha \mapsto \alpha \mathcal{O}_k$ induces an isomorphism

$$\mathcal{P}_k(f) \cdot \mathcal{I}_k(f)^p/\mathcal{R}_k,f \cdot \mathcal{I}_k(f)^p \simeq k^\times(f)/(Q^\times(f) \cdot k^\times(f)^p \cdot I_k,p(f)).$$

Finally, it is well known that the local description of the congruence relation mod $\mathfrak{P}$ yields an isomorphism

$$k^\times(f)/Q^\times(f) \cdot k^\times(f)^p \simeq (k^\times(f)/k^\times(f)^p)/Q^\times(f)/(Q^\times(f) \cap k^\times(f)^p) \simeq U(\mathcal{O}_k,f\mathcal{O}_k) / U(\mathbb{Z},f\mathbb{Z})$$

and that the $p$-elementary subgroup of $U(\mathcal{O}_k,f\mathcal{O}_k)/U(\mathbb{Z},f\mathbb{Z})$, which is isomorphic to $k^\times(f)/Q^\times(f) \cdot k^\times(f)^p$, has $p$-rank equal $t + w$ [2, 6, 8], whence $p' = p + t + w - \delta$ with

$$\delta = \dim_{\mathbb{F}_p}(I_k,p(f)/I_k,p(f) \cap (Q^\times(f) \cdot k^\times(f)^p)). \Box$$

Remarks. 1. For totally real dihedral fields, a further decomposition of the essential index $p^\delta$ into two parts is useful for practical purposes:

$$p^\delta = (I_k,p(f) : I_k,p(f)\cap(Q^\times(f) \cdot k^\times(f)^p)) \cdot (U_k : U_k \cap (Q^\times(f) \cdot k^\times(f)^p)),$$

where the first part involves only the principal $p$th powers of ideals which represent $p$ generating classes of order $p$ of $k$, and the second part concerns exclusively the fundamental unit of $k$. Here, $U_k$ denotes the unit group of $k$.

2. The proof of Theorem 3.1 and the previous remark show that generally $\delta(f) \leq \min(p,t + w)$ (in particular, $\delta(f) = 0$ when $p = 0$) in the cases $d_k < -3$ or $p \geq 5$, $d_k = -3$, and $\delta(f) \leq \min(p + 1, t + w)$ in the cases $d_k > 0$ or $p = 3$, $d_k = -3$.

Note. It should be pointed out that the proof of Theorem 3.1 does not use the full ray class group mod $f$ of $k$ but only the ring class group mod $f$ of $k$, which is obtained from the former by factoring out the part invariant under the generating automorphism of $\text{Gal}(k|\mathbb{Q})$ [6, 8]. Otherwise, there would be counted all cyclic extensions $N|k$ of degree $p$ sharing the common conductor $f$, those with group $\text{Gal}(N|\mathbb{Q}) \simeq D_p$ as well as those with group $\text{Gal}(N|\mathbb{Q}) \simeq C(2) \times C(p)$ (composita of $k$ with cyclic extensions $K|\mathbb{Q}$ of degree $p$).
The only case where the sum in the formula of Theorem 3.1 consists of a single term is the unramified case \( f = 1 \), which was treated in [8, p. 581] and [14] already.

**Corollary 3.1.** The number of unramified cyclic extensions \( N|k \) of degree \( p \) is

\[
m_p(d_k, 1) = \frac{1}{p-1}(p^\rho - 1).
\]

**Proof.** If \( f = 1 \), that is, \( t = 0 \) and \( w = 0 \), then we have \( \delta(f) = 0 \), since \( \delta(f) \leq \min(p + 1, t + w) = 0 \), or also simply since \( k_f^\times = k^\times \). □

In all other cases, single multiplicities can be obtained by Moebius inversion.

**Theorem 3.2** (General multiplicity formula when \( w \leq 1 \)). Assume that \( f = p^e \cdot q_1 \cdots q_t > 1 \) with \( w \leq 1 \) is a \( p \)-admissible conductor for the quadratic field \( k \), put \( q_{t+1} = p^e \) when \( w = 1 \), and define \( \delta_{\text{max}} = \max\{\delta(q_i_1 \cdots q_{i_s}) \mid 0 \leq s \leq t + w, 1 \leq i_1 < \cdots < i_s \leq t + w\} \). Then

\[
m_p(d_k, q_1 \cdots q_{t+w}) = p^\rho \cdot \frac{1}{\delta_{\text{max}}} \left[ (p - 1)^{t+w-1} + \sum_{s=0}^{t+w} (-1)^{t+w-s} \cdot p^s \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} \frac{p^{\delta_{\text{max}} - \delta(q_{i_1} \cdots q_{i_s})} - 1}{p - 1} \right].
\]

**Proof.** Observe that, for fixed \( p \) and \( d_k \), the general formula in Theorem 3.1,

\[
\sum_{f' \mid f} m_p(d_k, f') = \frac{1}{p-1}(p^\rho + t+w - \delta - 1),
\]

can be viewed as the sum relation \( \sum_{f' \mid f} m(f') = n(f) \) between two integer-valued functions \( n(f) = \frac{1}{p-1}(p^\rho + t+w - \delta - 1) \) and \( m(f') = m_p(d_k, f') \). Hence, an application of the Moebius inversion formula yields an expression for a single multiplicity,

\[
m(q_1 \cdots q_{t+w}) = \sum_{s=0}^{t+w} \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} \mu\left(\frac{q_1 \cdots q_{t+w}}{q_{i_1} \cdots q_{i_s}}\right) \cdot n(q_{i_1} \cdots q_{i_s})
\]

\[
= \sum_{s=0}^{t+w} \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} (-1)^{t+w-s} \cdot \frac{1}{p-1} \left( p^\rho \cdot p^s \cdot \frac{1}{p^{\delta(q_{i_1} \cdots q_{i_s})} - 1} \right)
\]

\[
= \sum_{s=0}^{t+w} \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} (-1)^{t+w-s} \cdot \frac{1}{p-1} \cdot p^\rho \cdot p^s \cdot \frac{1}{p^{\delta(q_{i_1} \cdots q_{i_s})}}
\]

\[
- \frac{1}{p-1} \sum_{s=0}^{t+w} (-1)^{t+w-s} \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} 1.
\]
Now the second double sum equals $\sum_{s=0}^{t+w} (-1)^{t+w-s} \cdot \binom{t+w}{s} = 0$; hence,

$$
m(q_1 \cdots q_{t+w}) = \frac{1}{p-1} \cdot p^{\rho} \cdot \frac{1}{p^{\delta_{\text{max}}}} \cdot \left[ \sum_{s=0}^{t+w} (-1)^{t+w-s} \cdot p^s \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} 1 \\
+ \sum_{s=0}^{t+w} \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} (-1)^{t+w-s} \cdot p^s \cdot (p^{\delta_{\text{max}}} - \delta(q_1 \cdots q_{i_s}) - 1) \right],$$

where the first double sum equals $\sum_{s=0}^{t+w} (-1)^{t+w-s} \cdot p^s \cdot \binom{t+w}{s} = (p-1)^{t+w}$; thus,

$$
m(q_1 \cdots q_{t+w}) = \frac{1}{p-1} \cdot p^{\rho} \cdot \frac{1}{p^{\delta_{\text{max}}}} \cdot \left[ (p-1)^{t+w} \\
+ \sum_{s=0}^{t+w} \sum_{1 \leq i_1 < \cdots < i_s \leq t+w} (-1)^{t+w-s} \cdot p^s \cdot (p^{\delta_{\text{max}}} - \delta(q_1 \cdots q_{i_s}) - 1) \right].$$

This formula can be simplified further, since the indices for composed conductors, $\delta(q_1 \cdots q_{i_s})$ with $0 \leq s \leq t+w$ and $1 \leq i_1 < \cdots < i_s \leq t+w$, can be determined from those for prime conductors, $\delta(q_i)$ with $1 \leq i \leq t+w$.

Corollary 3.2 (The special cases of $\delta_{\text{max}} = 0, 1$ when $w \leq 1$). Let $f = p^e \cdot q_1 \cdots q_t > 1$ with $w \leq 1$ be a $p$-admissible conductor for the quadratic field $k$, put $q_{t+1} = p^e$ when $w = 1$, suppose that $\delta_{\text{max}} < 2$, and put $u = \# \{1 \leq i \leq t+w \mid \delta(q_i) = 0\}$, i.e., $u := t+w-u$ is the number of "bad" primes.

1 (The case without constraints from principal $p$th ideal powers [8, p. 582]). If $u = t+w$, then $\delta_{\text{max}} = 0$ and

$$m_p(d_k, q_1 \cdots q_{t+w}) = p^\rho \cdot (p-1)^{t+w-1}. \tag{1}$$

2 (Restrictions caused by principal $p$th ideal powers). If $0 \leq u \leq t+w-1$, then $\delta_{\text{max}} = 1$ and

$$m_p(d_k, q_1 \cdots q_{t+w}) = p^\rho \cdot (p-1)^u \cdot \frac{(p-1)^{v-1} - (-1)^{v-1}}{p}. \tag{2}$$

In particular, $m_p(d_k, q_1 \cdots q_{t+w}) = 0$ for $v = 1$.

Proof. 1. If $u = t+w$, then $\delta(q_i) = 0$ for all $1 \leq i \leq t+w$, and consequently $\delta(q_1 \cdots q_{i_s}) = 0$ for all $0 \leq s \leq t+w$ and $1 \leq i_1 < \cdots < i_s \leq t+w$, whence $\delta_{\text{max}} = 0$. In this case, the formula of Theorem 3.2 immediately degenerates to

$$m_p(d_k, q_1 \cdots q_{t+w}) = p^\rho \cdot (p-1)^{t+w-1}. \tag{3}$$

2. If $0 \leq u \leq t+w-1$, then $\delta(q_i) = 1$ for some $1 \leq i \leq t+w$, and thus $\delta_{\text{max}} = 1$. In this case, it turns out that

$$\# \{1 \leq i_1 < \cdots < i_s \leq t+w \mid \delta(q_1 \cdots q_{i_s}) = 0\} = \binom{u}{s},$$

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for all $0 \leq s \leq t + w$, whence

$$m_p(d_k, q_1 \cdots q_{t+w})$$

$$= p^\rho \cdot \frac{1}{p^{\delta_{\text{max}}}} \cdot \left[ (p - 1)^{t+w - 1} + \sum_{s=0}^{t+w} (-1)^{t+w-s} \cdot p^s \sum_{1 \leq i_1 < \cdots < i_t \leq t+w} \frac{p^{\delta_{\text{max}} - 2 (q_{i_t} \cdots q_{i_1}) - 1}}{p - 1} \right]$$

$$= p^\rho \cdot \frac{1}{p} \cdot \left[ (p - 1)^{t+w - 1} + \sum_{s=0}^{t+w} (-1)^{t+w-s} \cdot p^s \cdot \binom{u}{s} \cdot \frac{p - 1}{p} \right]$$

$$= p^\rho \cdot \frac{1}{p} \cdot \left[ (p - 1)^{t+w - 1} + (-1)^{t+w-u} \sum_{s=0}^{u} \binom{u}{s} \cdot (-1)^{u-s} \cdot p^s \right]$$

$$= p^\rho \cdot \frac{1}{p} \cdot \left[ (p - 1)^{t+w - 1} + (-1)^{t+w-u} (p - 1)^u \right]$$

$$= p^\rho \cdot (p - 1)^u \cdot \frac{(p - 1)^{t+w - 1-u} + (-1)^{t+w-u}}{p} \quad \square$$

The following array shows the multiplicities $m_p(d_k, q_1 \cdots q_{t+w})$ in dependence on the total number $t + w > 0$ of prime divisors of the conductor $f > 1$, and on the number $0 \leq u \leq t + w$ of those "nice" primes which do not cause restrictions, for the special case $p = 3$ and $\rho = 0$. For positive 3-rank $\rho > 0$ of $k$, the numbers must only be multiplied by $3^\rho$, provided that still $\delta_{\text{max}} < 2$.

<table>
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<th>$u = 0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>2</td>
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</table>

**Example 1.** For pure cubic fields $L = \mathbb{Q}(\sqrt[3]{D})$, the formulas with $w = e \leq 1$ of Theorem 2.1 in the previous section, which were proved by elementary methods, can be reobtained here as the special case $p = 3$, $d_k = -3$, $\rho = 0$, and $u = \#\{1 \leq i \leq t + w \mid q_i \equiv \pm 1 \pmod{9}\}$, since for any $1 \leq i \leq t + w$ the condition $\delta(q_i) = 0$ is equivalent to $\zeta_3 \in \mathbb{Q}^\times(q_i) \cdot k_{q_i}^\times \cdot k^\times(q_i)^3$ and hence to $q_i \equiv \pm 1 \pmod{9}$:

$$m_3(-3, q_1 \cdots q_{t+w}) = 2^u \cdot \frac{2^{v+w-1} - (-1)^{v+w-1}}{3},$$

where $v := t - u$ and $v + w$ is the number of bad primes.

Finally, supplementary formulas must be established for the special case $p = 3$, $d_k \equiv -3 \pmod{9}$, $w = 2$. 

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Theorem 3.3 (General multiplicity formula when $w = 2$). Let $f = 3^2 \cdot q_1 \cdots q_t$ with $w = 2$ be a 3-admissible conductor for the quadratic field $k$. Then

$$m_3(d_k, 3^2 \cdot q_1 \cdots q_t)$$

$$= 3^\rho \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \sum_{1 \leq i_1 < \cdots < i_s \leq t} \frac{1}{2} (3^{2-\delta(3^2 \cdot q_{i_1} \cdots q_{i_s})} - 3^{1-\delta(3 \cdot q_{i_1} \cdots q_{i_s})}).$$

Proof. An application of the Möbius inversion formula to the sum relation $n(f) = \sum_{f' | f} m(f')$ in Theorem 1.1, where $n(f) = \frac{1}{3} (3^{\rho + t(f') + w(f') - \delta(f') - 1})$ and $m(f') = m_3(d_k, f')$, yields an expression for the single multiplicity of the conductor $f = 3^2 \cdot q_1 \cdots q_t$:

$$m(3^2 \cdot q_1 \cdots q_t) = \sum_{v=0}^{2} \sum_{s=0}^{t} \mu \left( \frac{3^2 \cdot q_1 \cdots q_t}{3^v \cdot q_{i_1} \cdots q_{i_s}} \right) \cdot n(3^v \cdot q_{i_1} \cdots q_{i_s}),$$

where the Möbius function takes the values

$$0 \text{ if } v = 0,$$

$$(-1)^{t-s+1} \text{ if } v = 1,$$

$$(-1)^{t-s} \text{ if } v = 2.$$

Consequently,

$$m(3^2 \cdot q_1 \cdots q_t) = \sum_{s=0}^{t} \sum_{1 \leq i_1 < \cdots < i_s \leq t} \left[ (-1)^{t-s+1} \cdot \frac{1}{2} \left( 3^\rho \cdot 3^{s+1} \cdot \frac{1}{3^{\delta(3^2 \cdot q_{i_1} \cdots q_{i_s})}} - 1 \right) 
+ (-1)^{t-s} \cdot \frac{1}{2} \left( 3^\rho \cdot 3^{s+2} \cdot \frac{1}{3^{\delta(3 \cdot q_{i_1} \cdots q_{i_s})}} - 1 \right) \right].$$

$$= 3^\rho \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \sum_{1 \leq i_1 < \cdots < i_s \leq t} \frac{1}{2} (3^{2-\delta(3^2 \cdot q_{i_1} \cdots q_{i_s})} - 3^{1-\delta(3 \cdot q_{i_1} \cdots q_{i_s})}).$$

Corollary 3.3 (The special cases of $\delta_{\text{max}} = 0, 1$ when $w = 2$). Let $f = 3^2 \cdot q_1 \cdots q_t$ with $w = 2$ be a 3-admissible conductor for the quadratic field $k$. Further, redefine $\delta_{\text{max}} = \max\{\delta(3^v \cdot q_{i_1} \cdots q_{i_s}) | 0 \leq v \leq 2, 0 \leq s \leq t, 1 \leq i_1 < \cdots < i_s \leq t\}$, suppose that $\delta_{\text{max}} < 2$, and put $u = \#\{1 \leq i \leq t | \delta(q_i) = 0\}$. Then

$$m_3(d_k, 3^2 \cdot q_1 \cdots q_t)$$

$$= \begin{cases} 
3^{\rho+1} \cdot 2^t & \text{if } \delta(3^2) = 0, \ u = t, \\
3^{\rho+1} \cdot 2^{u+1} \cdot \frac{1}{3} (2^{t-u-1} - (-1)^{t-u-1}) & \text{if } \delta(3^2) = 0, \ 0 \leq u \leq t - 1, \\
3^{\rho+1} \cdot 2^u \cdot \frac{1}{3} (2^{t-u} - (-1)^{t-u}) & \text{if } \delta(3) = 0, \ \delta(3^2) = 1, \\
3^{\rho} \cdot 2^t & \text{if } \delta(3) = 1.
\end{cases}$$

The first case is the unconstrained one [8, p. 582] where $\delta_{\text{max}} = 0$, in the second case the multiplicity is zero for $u = t - 1$, in the third case for $u = t$, and the fourth case is actually independent of $u$.

Proof. 1. If $\delta(3^2) = 0, \ u = t$, then clearly $\delta_{\text{max}} = 0$, and the formula in Theorem 3.3 becomes

$$m_3(d_k, 3^2 \cdot q_1 \cdots q_t) = 3^\rho \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \cdot \binom{t}{s} \cdot \frac{1}{2} (3^2 - 3) = 3^{\rho+1} \cdot 2^t.$$
2. However, if $\delta(3^2) = 0$ but $0 \leq u \leq t - 1$, then certainly $\delta_{\text{max}} = 1$, and $\delta(3^2 \cdot q_{i_1} \cdots q_{i_s}) = \delta(3 \cdot q_{i_1} \cdots q_{i_s}) = \delta(q_{i_1} \cdots q_{i_s})$ for all $0 \leq s \leq t$ and $1 \leq i_1 < \cdots < i_s \leq t$. In this case,

$$m_3(d_k, 3^2 \cdot q_{i_1} \cdots q_{i_t})$$

$$= 3^p \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \sum_{1 \leq i_1 < \cdots < i_s \leq t} \frac{1}{2} (3^2 - \delta(q_{i_1} \cdots q_{i_s}) - 3^1 - \delta(q_{i_1} \cdots q_{i_s}))$$

$$= 3^p \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \sum_{1 \leq i_1 < \cdots < i_s \leq t} \frac{1}{2} (3 - 1) \cdot 3^{1 - \delta(q_{i_1} \cdots q_{i_s}))}$$

$$= 3^p \left[ \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \cdot \left( \binom{t}{s} - (-1)^{t-u} \sum_{s=0}^{t} (-1)^{u-s} \cdot 3^s \cdot \binom{u}{s} \cdot (3 - 1) \right) \right]$$

$$= 3^p \cdot (2^t + (-1)^{t-u} \cdot 2^u \cdot 2) .$$

3. In the case $\delta(3) = 0$, $\delta(3^2) = 1$, we have $\delta(3 \cdot q_{i_1} \cdots q_{i_s}) = \delta(q_{i_1} \cdots q_{i_s})$ and $\delta(3^2 \cdot q_{i_1} \cdots q_{i_s}) = 1$ for all $0 \leq s \leq t$ and $1 \leq i_1 < \cdots < i_s \leq t$, whence $\delta_{\text{max}} = 1$ and

$$m_3(d_k, 3^2 \cdot q_{i_1} \cdots q_{i_t}) = 3^p \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \sum_{1 \leq i_1 < \cdots < i_s \leq t} \frac{1}{2} (3 - 1 - \delta(q_{i_1} \cdots q_{i_s}))$$

$$= 3^p \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \cdot \left[ \left( \binom{t}{s} - \left( \binom{u}{s} \cdot (3 - 1) \right) \right) \right]$$

$$= 3^p \left[ \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \cdot \left( \binom{t}{s} - (-1)^{t-u} \sum_{s=0}^{t} (-1)^{u-s} \cdot 3^s \cdot \binom{u}{s} \cdot (3 - 1) \right) \right]$$

$$= 3^p \cdot (2^t + (-1)^{t-u} \cdot 2^u \cdot 2) .$$

4. If, finally, $\delta(3) = 1$, then $\delta(3 \cdot q_{i_1} \cdots q_{i_s}) = 1$ and $\delta(3^2 \cdot q_{i_1} \cdots q_{i_s}) = 1$ for all $0 \leq s \leq t$ and $1 \leq i_1 < \cdots < i_s \leq t$, whence $\delta_{\text{max}} = 1$ and

$$m_3(d_k, 3^2 \cdot q_{i_1} \cdots q_{i_t}) = 3^p \sum_{s=0}^{t} (-1)^{t-s} \cdot 3^s \cdot \left( \binom{t}{s} \cdot \frac{1}{2} (3 - 1) = 3^p \cdot 2^t . \quad \square \right)$$

**Example 2.** For pure cubic fields $L = \mathbb{Q}(\sqrt{D})$, the formula with $w = e = 2$ in Theorem 2.1 of the previous section can be reobtained as the special case $d_k = -3$, $\rho = 0$, and $\delta(3) = 1$:

$$m_3(-3, 3^2 \cdot q_{i_1} \cdots q_{i_t}) = 2^t .$$

The various formulas of §3 are illustrated in the Supplements section at the end of this issue by a discussion of all known discriminants of multiplicity $\geq 5$ of totally real and complex cubic fields, which occur in the most extensive recent numerical tables [10], [5].

**Final Note.** It should be particularly emphasized that all the calculations which are necessary for the determination of the multiplicity $m_p(d_k, f)$ of a dihedral discriminant $d_k = f^{p-1}d_k^{(p-1)/2}$ can be carried out entirely in the underlying quadratic field $k$. 

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MULTIPLECTIES OF DIHEDRAL DISCRIMINANTS

Given a quadratic field $k$ with $p$-class rank $\rho = \rho_k(p)$ and discriminant $d_k$, and given a $p$-admissible conductor $f$ for $d_k$, we have to find $\rho$ principal $p$th powers $\alpha_1, \ldots, \alpha_\rho$ of ideals prime to $f$ which represent $\rho$ generating classes of order $p$ of the $p$-elementary class group of $k$, and additionally a fundamental unit $\eta$ of $k$ if $d_k > 0$ or $d_k = -3$. Then $I_{k,f}(f) = \langle \alpha_1, \ldots, \alpha_\rho, \eta \rangle \cdot (f)^p$.

Next, we must examine successively, if $\eta \in \mathbb{Q}^\times(f) \cdot k^\times \cdot k^\times(f)^p$ and if $\alpha_i \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle \cdot U_k \cdot \mathbb{Q}^\times(f) \cdot k^\times \cdot k^\times(f)^p$ for $i = 1, \ldots, \rho$,

whence we will be able to determine $\delta(f')$ for all divisors $f' | f$ and finally the multiplicity $m_p(d_k, f)$ of $d_L = f^{p-1}d_k^{(p-1)/2}$, by means of Theorem 3.2 or 3.3.

In this manner it is possible to construct complete tables of complex or totally real dihedral discriminants up to a given bound, $|d_k| < B$, and for a fixed value of the prime $p$, by the computation of the $p$-ranks of ring class groups mod $f$ of quadratic fields $k$, varying the discriminants $d_k$ and the $p$-admissible conductors $f$ for each discriminant $d_k$.

In particular, approximations can be determined for the asymptotic densities of dihedral discriminants for various primes $p \geq 5$, similar to the densities of Davenport and Heilbronn [3] for $p = 3$. As D. Shanks pointed out in his review [13] of [1], the convergence of these approximations to the asymptotic limits would be rather slow, because the really high multiplicities, which contribute the essential part to the limit, unfortunately occur in very high ranges of discriminants.

A drawback of the proposed method is that it does not seem to be suitable for obtaining generating polynomials for the non-Galois subfields $L$ of dihedral fields.

**Example.** For any two positive integers $m$ and $B$, let $n_p(m, B)$ denote the number of complex dihedral discriminants $d_l = f^{p-1}d_k^{(p-1)/2}$ of multiplicity $m_p(d_k, f) = m$, which are bounded by $|d_L| < B^{(p-1)/2}$. (Observe that dihedral discriminants are always complete ($(p - 1)/2)$th powers.)

To make the ideas in the final note more concrete, we have computed the 303968 quadratic discriminants in the range $-10^6 < d_k < 0$ and determined the class numbers and $p$-class ranks $\rho = \rho_k(p)$ of the corresponding imaginary quadratic fields $k$ for $p = 3, 5, 7$. With the aid of this information, we can calculate the exact numbers $n_p(m, B)$ for $m = 1, p - 1, p + 1$, and for any given upper bound $B < 10^6$.

(a) According to Corollary 3.1, the first component of $n_p(1, B)$ is the number of quadratic fields $k$ with $p$-rank $\rho = 1$ and discriminant

$$|d_k| = |d_L|^{2/(p-1)} < B,$$

i.e., single unramified cyclic extensions (absolute class fields) $N|k$ of degree $p$ with conductor $f = 1$.

By Corollary 3.2.1, the second component is the number of ramified cyclic extensions (ring class fields) $N$ of degree $p$ over quadratic fields $k$ with $p$-rank $\rho = 0$ (and thus $\delta = 0$) and discriminant

$$|d_k| = |d_L/f^{p-1}|^{2/(p-1)} = |d_L|^{2/(p-1)}/f^2 < B/f^2$$
with a single prime divisor \( q_1 \) of the conductor \( f \), such that \( t + w = 1 \). To compute this number, we examine each of the relevant quadratic discriminants \( d_k \) for admissible prime conductors \( q_1 \equiv \pm 1 \pmod{p} \), also taking into consideration the possibility of \( p|f \), \( w = 1 \).

In the case of \( p = 3 \), we must further add the number of pure cubic discriminants (\( d_k = -3 \)) of multiplicity 1, which can be evaluated by counting primes and products of up to three primes below certain bounds, according to Theorem 2.1, taking into account the parameter values

\[
(e, t) = (2, 0) \quad \text{and} \quad \quad (e, u, v) = (1, 0, 1), (1, 0, 2), (0, 1, 0), (0, 0, 2), (0, 0, 3).
\]

<table>
<thead>
<tr>
<th>( p )</th>
<th>( (p - 1)/2 )</th>
<th>( B )</th>
<th>( n_p(1, B) )</th>
<th>( f = 1 )</th>
<th>( f &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( d_k &lt; -3 )</td>
<td>( d_k = -3 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( 10^3 )</td>
<td>125</td>
<td>84</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^4 )</td>
<td>1396</td>
<td>1034</td>
<td>343</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^5 )</td>
<td>14565</td>
<td>11286</td>
<td>3215</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^6 )</td>
<td>149204</td>
<td>118455</td>
<td>30559</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>( 10^3 )</td>
<td>58</td>
<td>52</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^4 )</td>
<td>665</td>
<td>617</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^5 )</td>
<td>7099</td>
<td>6686</td>
<td>413</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^6 )</td>
<td>73252</td>
<td>69365</td>
<td>3887</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>( 10^3 )</td>
<td>31</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^4 )</td>
<td>435</td>
<td>411</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^5 )</td>
<td>4709</td>
<td>4503</td>
<td>206</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 10^6 )</td>
<td>49607</td>
<td>47595</td>
<td>2012</td>
</tr>
</tbody>
</table>

The result \( n_3(1, 10^6) = 149204 \), in comparison to the number 148709 claimed in [5, Table 5.2, p. 321], shows that 495 complex cubic discriminants of multiplicity 1 were missing from the original version of [5].

The series of single complex dihedral discriminants starts with \( -23 \) for \( p = 3 \), with \(+2209 = (-47)^2 \) for \( p = 5 \), and with \(-357911 = (-71)^3 \) for \( p = 7 \). The minimal discriminants of arbitrary quintic fields with signature \((1, 2)\), resp. septic fields with signature \((1, 3)\), are somewhat smaller and not \(((p - 1)/2)\)th powers: \(+1609\), resp. \(-184607\). Unramified extensions are dominating, 79.5\% for \( p = 3 \), 94.7\% for \( p = 5 \), and 95.9\% for \( p = 7 \), and this effect even seems to increase with the value of the prime \( p \). The minimal examples of the rare ramified extensions are \( d_k = -11 \) with \( f = 2 \) for \( p = 3 \), \( d_k = -15 \) with \( f = 5 \) for \( p = 5 \), and \( d_k = -7 \) with \( f = 7 \) for \( p = 7 \).

(b) By Corollary 3.2.1, \( n_p(p - 1, B) \) is the number of \((p - 1)\)-tuplets of ramified cyclic extensions \( N \) of degree \( p \) over quadratic fields \( k \) of \( p \)-rank \( \rho = 0 \) and discriminant \( |d_k| < B/f^2 \) with two prime divisors \( q_1, q_2 \) of the conductor \( f \), such that \( w \leq 1 \).

But for the special case \( p = 3 \), the number of pure cubic discriminants of multiplicity 2 must be added. According to Theorem 2.1, we determine this number by counting primes and products of at most four primes up to certain
bounds, taking into consideration the parameter values

\[(e, t) = (2, 1) \quad \text{and} \quad (e, u, v) = (1, 1, 1), (1, 1, 2), (0, 2, 0), (0, 1, 2), (0, 1, 3).\]

It might be of interest to list explicitly the root discriminants \(|d_L|^{2/(p-1)}\) for the few encountered \((p-1)\)-tuplets of dihedral fields with two prime divisors of the conductor \(f\), if \(p = 5, 7\) (for \(p = 3\), they start with the well-known 1836 [1]).

There are 20 cases for \(p = 5\):

- 105,875, 121,000, 180,500, 287,375, 305,767, 315,375, 315,875, 360,375, 363,000, 496,375, 529,375, 589,875, 605,000, 650,375, 771,375, 786,500, 814,088, 840,500, 841,000, 902,500,

and 7 cases for \(p = 7\):


<table>
<thead>
<tr>
<th>(p)</th>
<th>((p-1)/2)</th>
<th>(B)</th>
<th>(n_p(p-1, B))</th>
<th>(f &gt; 1) (d_k &lt; -3)</th>
<th>(d_k = -3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>10^3</td>
<td>1 = 0 + 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10^4</td>
<td>12 = 10 + 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10^5</td>
<td>167 = 157 + 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10^6</td>
<td>1,683 = 1,639 + 44</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10^6</td>
<td>20 = 20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>10^5</td>
<td>1 = 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10^6</td>
<td>7 = 7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, the authors of [5] announce the value \(n_3(2, 10^6) = 1,762\) instead of 1,683. In fact, the multiplicities of the superfluous 79 discriminants must necessarily be greater than 2.

(c) The determination of \(n_p(p, B)\) is much more difficult, since it involves the complete treatment of various cases of \(p\)-tuplets of ramified cyclic extensions \(N\) of degree \(p\) over quadratic fields \(k\) of \(p\)-rank \(p \geq 1\) and discriminant \(|d_k| < B/f^2\) with conductor \(f > 1\), according to Corollary 3.2. If \(p = 3\), then further contributions arise from Corollary 3.3. Here, \(\delta \geq 1\) is possible, and thus the principal \(p\)th powers \(\alpha_1, \ldots, \alpha_p\) of ideals of \(k\) must be examined for each of the relevant quadratic discriminants \(d_k\).

(d) By Corollary 3.1, \(n_p(p+1, B)\) is the number of quadratic fields \(k\) with \(p\)-rank \(p = 2\) and discriminant \(|d_k| < B\), i.e., \((p+1)\)-tuplets of unramified cyclic extensions \(N/k\) of degree \(p\) with conductor \(f = 1\).

In the case of \(p = 3\), where exceptionally not only \((p^2 - 1)/(p - 1) = p + 1\) but also \((p - 1)^2 = p + 1\), we must additionally consider the number of ramified cyclic extensions \(N\) of degree \(p\) over quadratic fields \(k\) of \(p\)-rank \(p = 0\) and discriminant \(|d_k| < B/f^2\) with three prime divisors \(q_1, q_2, q_3\) of the conductor \(f\), such that \(w \leq 1\).

A further additive component for \(p = 3\) is the number of pure cubic discriminants of multiplicity 4, which can again be evaluated with the aid of Theorem 2.1 by counting products of up to five primes below certain bounds, taking into
account the parameter values

\[(e, t) = (2, 2)\] and \[(e, u, v) = (1, 2, 1), (1, 2, 2), (0, 3, 0), (0, 2, 2), (0, 2, 3).\]

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\frac{p - 1}{2})</th>
<th>(B)</th>
<th>(n_p(p + 1, B))</th>
<th>(f = 1)</th>
<th>(d_k &lt; -3)</th>
<th>(f &gt; 1)</th>
<th>(d_k = -3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>(10^4)</td>
<td>7</td>
<td>7</td>
<td>0</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10^5)</td>
<td>216</td>
<td>214</td>
<td>0</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10^6)</td>
<td>3216</td>
<td>3190</td>
<td>15</td>
<td>+</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(10^5)</td>
<td>39</td>
<td>39</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10^6)</td>
<td>398</td>
<td>398</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>(10^5)</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10^6)</td>
<td>97</td>
<td>97</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The result \(n_3(4, 10^6) = 3216\), in comparison to the number 3189 given in [5], reveals the lack of 27 complex cubic discriminants of multiplicity 4 in the original version of [5].

Finally it should be mentioned, that the authors of [5] announced the extremely exciting number \(n_3(5, 10^6) = 7\), which caused a lot of confusion. However, our Example 5 after Theorem 2.1 shows that the minimal occurrence \(d_L = -16008300\) of multiplicity 5 (which is only possible for pure and certain totally real cubic fields) lies considerably outside the range of the computations in [5].

**Bibliography**

12. ——, *Table of simply real cubic number fields with discriminants between \(-30000\) and 0*, Univ. Graz, 1989.


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4. Numerical examples

Generally, \( p = 3 \) in all examples. The information about the 14 totally real cubic discriminants with multiplicities 4, 6 has been taken from [10]. Data for the 69 complex cubic discriminants with multiplicities 6, 9 has kindly been made available by the authors of [5] in personal communication.

A common feature of all examples is that the 3-rank of the ring class group \( \text{mod} / \) of the quadratic field \( k \) is always \( p + t + w - 6(f) = 3 \), whence the sum of all partial multiplicities equals \( \frac{1}{2}(3p + t + w - 4 - f) = 13 \).

Both parts (\( w \leq 1 \) and \( w = 2 \)) begin with the unconstrained cases and are arranged according to increasing quadratic 3-class rank \( \rho \) of \( k \). Correspondingly, the values of \( t + w \) (resp. \( t \)) and of \( u \) decrease.

For any integer \( n \geq 1 \), denote by \( \mathcal{O}_n \) the suborder \( \mathbb{Z} \oplus \mathbb{Z}[\omega] \) of the maximal order \( \mathcal{O}_k \), where \( \omega = \sqrt{d_k} \). Then the unit group \( U(\mathcal{O}_n) \) is exactly the intersection \( U_k \cap \mathbb{Q}^*(n) \cdot k^*. \)

In the totally real case, the fundamental unit of \( k \) is denoted by \( \eta \). If \( p = 0 \), then the condition \( \eta \in \mathcal{O}_n \) is certainly sufficient for \( \delta(n) = 0 \).

Part 1. Applications of Corollary 3.2 to conductors with \( w \leq 1 \).

a) A totally real cubic field with
\[
\begin{array}{c}
\rho = 0, t = 2, w = 1, \delta_{\text{max}} = 0, u = 3 + t + w, \\
f = q_1^2 \cdot q_2 \cdot q_3 \\
q_1 = 2, q_2 = 7, q_3 = 11.
\end{array}
\]

Here, we have \( I_k(f) = U_k \cdot k^*(f)^3 \). There are no constraints from the fundamental unit \( \eta \) of \( k \), since \( \delta(q_1) = 0, \delta(q_2) = 0 \) and \( \delta(q_3) = 0 \). The quadratic discriminant \( d_k \) is simultaneously congruent \(-3(\text{mod} 9)\) and congruent \(5(\text{mod} 8)\), since \( 2 \mid f \), \( f^2 \equiv 2 \equiv -1(\text{mod} 3) \).

The partial multiplicities \( m_3(d_k, f) \) of divisors \( f \) of the conductor \( f \) can be determined by means of Corollary 3.2.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\text{No.} & d_k & d_4 & \eta & m_3(d_k, f) \\
\hline
1 & 8250228 & 4677 & 2796250463 + 40887672w & 1 & q_1 & q_2 & 1 & 1 & 1 & 2 & 2 & 2 & 4
\end{array}
\]

b) 14 complex cubic fields with
\[
\begin{array}{c}
\rho = 1, t + w = 2, \delta_{\text{max}} = 0, u = 2 + t + w, \\
f = q_1 \cdot q_2 \cdot q_3 \\
q_1 = 2.
\end{array}
\]

In this case, \( I_k(f)/k^*(f)^3 = (\alpha) \). There do not arise restrictions from the single generating principal ideal cube \( \alpha \) of \( k \), i.e., \( \alpha \in \mathbb{Q}^*(f) \cdot k^* \cdot k^*(f)^2 \), since \( \delta(q_1) = 0 \) and \( \delta(q_2) = 0 \). All quadratic discriminants \( d_k \) are congruent \( 5(\text{mod} 8) \), since \( 2 \mid f \). The class numbers of all cubic fields \( L \) are divisible by 9.

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\[ 0025-5519/92 $1.00 + $.25 per page \]
\[
\begin{array}{cccc}
\text{No.} & d_L & d_k & q_2 \ w \\
1 & -225612 & -6267 & \equiv -3(9) & 3 \ 1 \\
2 & -439628 & -2243 & \equiv +1(3) & 7 \ 0 \\
3 & -483948 & -13443 & \equiv +3(9) & 3 \ 1 \\
4 & -593676 & -16491 & \equiv -3(9) & 3 \ 1 \\
5 & -649612 & -307 & \equiv -1(3) & 23 \ 0 \\
6 & -729708 & -3723 & \equiv +3(9) & 7 \ 0 \\
7 & -762156 & -21171 & \equiv -3(9) & 3 \ 1 \\
8 & -772300 & -7723 & \equiv -1(3) & 5 \ 0 \\
9 & -789868 & -547 & \equiv -1(3) & 19 \ 0 \\
10 & -803628 & -22323 & \equiv -3(9) & 3 \ 1 \\
11 & -812300 & -8123 & \equiv +1(3) & 5 \ 0 \\
12 & -968300 & -9683 & \equiv -1(3) & 5 \ 0 \\
13 & -990700 & -9907 & \equiv -1(3) & 5 \ 0 \\
\end{array}
\]

\[
f' = m_3(d_k, f') = 1 \quad q_1 \quad q_2 \quad q_1 q_2 \quad q_1 q_2
\]

\[
m_3(d_k, f') = 1 \quad 3 \quad 3 \quad 6
\]

c) 2 totally real cubic fields with

\[
\rho = 1, \ t + w = 1, \ \delta_{max} = 0, \ u = 2 = t + w.
\]

\[
f = q_1 \cdot q_2 \ \text{with} \ q_1 = 2, \ q_2 = 3.
\]

Here, \( I_k(3)f/f_k^*(f)^3 = (\alpha) \). Constraints neither arise from the fundamental unit \( \eta \) of \( k \) nor from the single generating principal ideal cube \( \alpha \) of \( k \), because \( \delta(q_1) = 0, \delta(q_2) = 0 \), whence \( \eta \in Q^*(f) \cdot k^* \cdot k^*(f)^3 \). The quadratic discriminants \( d_k \) are congruent \( 5 \bmod(8) \), since \( 2 \mid f \).

\[
\begin{array}{cccc}
\text{No.} & d_L & d_k & q_2 \ w \\
10 & 3054132 & 84837 & \equiv +3(9) \\
11 & 4735476 & 131541 & \equiv -3(9) \\
\end{array}
\]

\[
f' = m_3(d_k, f') = 1 \quad 3 \quad 3 \quad 6
\]

d) 8 complex cubic fields with

\[
\rho = 2, \ t + w = 1, \ \delta_{max} = 0, \ u = 1 = t + w.
\]

In this case, \( I_k(3)f/f_k^*(f)^3 = (\alpha, \beta) \) and there are no restrictions from the two generating principal ideal cubes \( \alpha, \beta \) of \( k \), because \( \delta(q_1) = 0, \alpha, \beta \in Q^*(f) \cdot k^* \cdot k^*(f)^3 \). All quadratic discriminants \( d_k \) are congruent \( 5 \bmod(8) \). The class numbers of all cubic fields \( L \) are divisible by 9.

\[
\begin{array}{cccc}
\text{No.} & d_L & d_k & q_2 \ w \\
12 & -406700 & -83 & \equiv 1(3) & 5 \ 7 \ 0 \ 1 \ 0 \ 1 \\
13 & -507999 & -339 & \equiv 1(3) & 7 \ 3 \ 1 \ 0 \ 1 \ 1 \\
14 & -672300 & -83 & \equiv 1(3) & 5 \ 3 \ 2 \ 1 \ 1 \ 0 \ 1 \\
15 & -866700 & -107 & \equiv 1(3) & 5 \ 3 \ 2 \ 1 \ 1 \ 1 \ 0 \\
16 & -936684 & -59 & \equiv 1(3) & 7 \ 3 \ 2 \ 1 \ 1 \ 0 \ 1 \\
\end{array}
\]

\[
f' = m_3(d_k, f') = 1 \quad q_1 \quad q_2 \quad q_1 q_2 \quad q_1 q_2 \quad q_1 q_2 \quad q_1 q_2
\]

\[
m_3(d_k, f') = 1 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 6 \\
14 \quad 1 \quad 3 \quad 0 \quad 0 \quad 0 \quad 3 \quad 6 \\
15 \quad 1 \quad 0 \quad 3 \quad 0 \quad 0 \quad 3 \quad 6 \\
16 \quad 1 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 6 \\
27 \quad 1 \quad 0 \quad 3 \quad 0 \quad 0 \quad 3 \quad 6
\]

e) 5 complex cubic fields with

\[
\rho = 1, \ t + w = 3, \ \delta_{max} = 1, \ u = 1 = t + w.
\]

\[
f = q_1 \cdot q_2 \ \text{with} \ q_1 = 2, \ q_2 = 3.
\]

Restrictions arise from the single generating principal ideal cube \( \alpha \) of \( k \), where \( I_k(3)f/f_k^*(f)^3 = (\alpha) \), but \( \alpha \notin Q^*(f) \cdot k^* \cdot k^*(f)^3 \). All quadratic discriminants \( d_k \) are congruent \( 5 \bmod(8) \), since \( 2 \mid f \). The class numbers of all cubic fields \( L \) are divisible by 9, and two of the six fields with discriminant \( d_L = -936648 \) even have 3-class rank 3.

\[
\begin{array}{cccc}
\text{No.} & d_L & d_k & q_2 \ w \\
17 & -301675 & -12067 & \equiv -1(3) & 5 \ 0 \\
18 & -414508 & -103027 & \equiv -1(3) & 2 \ 0 \\
19 & -429676 & -107419 & \equiv -1(3) & 2 \ 0 \\
20 & -706540 & -176635 & \equiv -1(3) & 2 \ 0 \\
21 & -738572 & -184643 & \equiv +1(3) & 2 \ 0 \\
22 & -795180 & -189705 & \equiv -3(9) & 2 \ 0 \\
23 & -821452 & -205363 & \equiv -1(3) & 2 \ 0 \\
24 & -864243 & -96027 & \equiv +3(9) & 3 \ 1 \\
\end{array}
\]

\[
f' = m_3(d_k, f') = 1 \quad q_1 \quad q_2 \quad q_1 q_2 \quad q_1 q_2
\]

\[
m_3(d_k, f') = 1 \quad 4 \quad 9
\]

f) A single complex cubic field with

\[
\rho = 2, \ t + w = 1, \ \delta_{max} = 1, \ u = 0 = t + w.
\]

\[
f = q_1 \cdot q_2 \ \text{with} \ q_1 = 2, \ q_2 = 3.
\]

Restrictions arise from the two generating principal ideal cubes \( \alpha, \beta \) of \( k \), where \( I_k(3)f/f_k^*(f)^3 = (\alpha, \beta) \) and \( \alpha \notin Q^*(f) \cdot k^* \cdot k^*(f)^3 \). However, the other ideal cube is not independent from \( \alpha, i, c, \beta \in (n) \cdot Q^*(f) \cdot k^* \cdot k^*(f)^3 \). Here, we have \( \delta(q_1) = 1 \) and \( \delta(q_2) = 1 \). The quadratic discriminant \( d_k \) is simultaneously
congruent \(-3 \pmod{9}\) and congruent \(5 \pmod{8}\), since \(2 \mid f\). The class numbers of all cubic fields \(L\) are divisible by 9.

\[
\begin{array}{c|cc}
\text{No} & d_L & d_k \\
\hline
28 & -941004 & -26139 \\
\end{array}
\]

\[
f'(s) = \frac{m(s, d_L, f')}{4} \left(1 \quad q_1 \quad q_1 \quad q_2 \right)
\]

\[
\begin{array}{c|cccc}
\text{No} & d_L & d_k & d_k & d_k \\
\hline
4 & 796788 & 30937 \\
\end{array}
\]

Part 2. Applications of Corollary 3.3 to conductors with \(w = 2\).

All quadratic discriminants \(d_k\) are congruent \(-3 \pmod{9}\), since \(w = 2\).

\(a\) \(30\) complex cubic fields with

\[
\rho = 0, \quad t = 1, \quad w = 2, \quad \delta_{\text{max}} = 0, \quad u = 1 = t, \quad f = 3^2 \cdot q_1.
\]

There are no possible constraints, since \(I_k(s) = k^*(f)^3\). Therefore, \(\delta(3^2) = 0\) and \(\delta(q_1) = 0\). In general, the 3-class groups of all cubic fields \(L\) are cyclic.

\[
\begin{array}{c|cc|cccc}
\text{No} & d_L & d_k & q_1 & \text{No} & d_L & d_k \\
\hline
29 & -70956 & -219 & 2 & 44 & -537516 & -1659 & 2 \\
30 & -94284 & -291 & 2 & 45 & -560844 & -1731 & 2 \\
31 & -140940 & -435 & 2 & 46 & -584172 & -1803 & 2 \\
32 & -187596 & -579 & 2 & 47 & -630828 & -1947 & 2 \\
33 & -210924 & -651 & 2 & 48 & -654156 & -2019 & 2 \\
34 & -234252 & -723 & 2 & 49 & -700812 & -2163 & 2 \\
35 & -257580 & -795 & 2 & 50 & -747468 & -2307 & 2 \\
36 & -304236 & -939 & 2 & 51 & -753390 & -2372 & 5 \\
37 & -370575 & -183 & 5 & 52 & -770796 & -2370 & 2 \\
38 & -374220 & -1155 & 2 & 53 & -794124 & -2451 & 2 \\
39 & -397548 & -1227 & 2 & 54 & -826200 & -408 & 5 \\
40 & -430876 & -1299 & 2 & 55 & -864108 & -2662 & 2 \\
41 & -440559 & -111 & 7 & 56 & -887436 & -2730 & 2 \\
42 & -461700 & -226 & 5 & 57 & -910764 & -2811 & 2 \\
43 & -467532 & -1443 & 2 & 58 & -912951 & -39 & 17 \\
\end{array}
\]

The partial multiplicities \(m(s, d_k, f')\) of divisors \(f'\) of the conductor \(f\) can be determined with the aid of Corollary 3.3.

\[
f'(s) = \frac{m(s, d_L, f')}{4} \left(1 \quad q_1 \quad 3 \quad q_1 \quad 3^2 \quad 3^2 \quad q_1 \right)
\]

\[
\begin{array}{c|cccc}
\text{No} & d_L & d_k & d_k & d_k \\
\hline
4 & 796788 & 30937 \\
\end{array}
\]

\(b\) A totally real cubic field with

\[
\rho = 0, \quad t = 1, \quad w = 2, \quad \delta_{\text{max}} = 0, \quad u = 1 = t, \quad f = 3^2 \cdot q_1, \quad \text{with } q_1 = 2.
\]

Here, we have \(I_k(s) = U_k \cdot k^*(f)^3\) without constraints from the fundamental unit \(\eta\) of \(k\), i.e., \(\delta(3^2) = 0\) and \(\delta(q_1) = 0\). The quadratic discriminant \(d_k\) is congruent \(5 \pmod{8}\), since \(2 \mid f\).

\[
\begin{array}{c|cc|cccc}
\text{No} & d_L & d_k & q_2 & \text{No} & d_L & d_k & \eta & \delta(q_1) & \delta(q_1) \\
\hline
5 & 1725300 & 213 & 5 & (73 + 5\omega)/2 & \in \mathbb{Q}_L & 1 & 0 & 1 \\
6 & 2238516 & 141 & 7 & 95 + 8\omega & \in \mathbb{Q}_L & 0 & 1 & 1 \\
7 & 2891700 & 357 & 5 & (19 + \omega)/2 & \in \mathbb{Q}_L & 1 & 0 & 1 \\
8 & 4441300 & 573 & 5 & 583 + 16\omega & \in \mathbb{Q}_L & 1 & 1 & 1 \\
9 & 5807700 & 717 & 5 & (241 + 9\omega)/2 & \in \mathbb{Q}_L & 1 & 1 & 1 \\
10 & 6810804 & 429 & 7 & (145 + 7\omega)/2 & \in \mathbb{Q}_L & 1 & 0 & 1 \\
11 & 7557300 & 933 & 5 & 75203 + 246\omega & \in \mathbb{Q}_L & 0 & 1 & 1 \\
12 & 7953870 & 501 & 7 & (28225 + 1261\omega)/2 & \in \mathbb{Q}_L & 1 & 1 & 0 \\
13 & 8723700 & 1077 & 5 & (361 + 11\omega)/2 & \in \mathbb{Q}_L & 1 & 1 & 0 \\
\end{array}
\]

It is illuminating to compare the various multiplicities \(m(s, d_k, f')\) of divisors \(f'\) of the conductor \(f\).
e) 9 complex cubic fields with

\[ p = 1, t = 1, w = 2, b_{\text{max}} = 1, u \leq t. \]

Restrictions arise from the single generating principal ideal cube \( \alpha \) of \( k \), where

\[ I_k(f)/k^*(f)^3 = (\alpha) \] but \( \alpha \notin \mathbf{Q}^*(f) \cdot k^*(f)^3. \] Here, always \( \theta(3^2) = 1. \) In general, the 3-class groups of all cubic fields \( L \) are cyclic, but the class numbers of the six fields with discriminant \( d_L = -174140 \) are divisible by 9.

\[
\begin{array}{cccccc}
\text{No.} & d_L & d_k & q_1 & q_2 & \theta(q_1) & \theta(3) \\
61 & -32556 & -1011 & 2 & 0 & 1 & 1 \\
62 & -44429 & -1371 & 2 & 0 & 1 & 1 \\
63 & -49080 & -1515 & 2 & 0 & 1 & 1 \\
64 & -60217 & -327 & 5 & 0 & 1 & 1 \\
65 & -67748 & -2091 & 2 & 0 & 1 & 0 \\
66 & -72410 & -2335 & 2 & 0 & 1 & 0 \\
67 & -84070 & -2555 & 2 & 0 & 1 & 1 \\
68 & -93420 & -2955 & 2 & 0 & 1 & 0 \\
69 & -96074 & -3077 & 2 & 0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{No.} & f' & 1 & q_1 & 3 & q_2 & 3^2 q_1 \\
61 & m_3(d_k, f') = 1 & 0 & 0 & 3 & 3 & 6 \\
62 & 1 & 0 & 0 & 3 & 3 & 6 \\
63 & 1 & 0 & 0 & 3 & 3 & 6 \\
64 & 1 & 0 & 0 & 3 & 3 & 6 \\
65 & 1 & 0 & 3 & 0 & 0 & 9 \\
66 & 1 & 3 & 0 & 0 & 3 & 6 \\
67 & 1 & 0 & 0 & 3 & 3 & 6 \\
68 & 1 & 0 & 3 & 0 & 0 & 9 \\
69 & 1 & 0 & 0 & 3 & 3 & 6 \\
\end{array}
\]

f) Finally, a single totally real cubic field with

\[ \rho = 1, t = 1, w = 2, b_{\text{max}} = 1, u = 0 < t. \]

\[ f = 3^2 \cdot q_1 \text{ with } q_1 = 2. \]