WEIGHT FORMULAS FOR TERNARY MELAS CODES

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ABSTRACT. In this paper we derive a formula for the frequencies of the weights in ternary Melas codes and we illustrate this formula by computing a table of examples.

1. Introduction

Let $q = p^m$, where $p$ is a prime, and let $\alpha$ be a generator of the multiplicative group $\mathbb{F}_q^*$. Consider the cyclic code $C$ over $\mathbb{F}_q$ of length $q - 1$ with generator polynomial $(X - \alpha)(X - \alpha^{-1})$. The dual code $C^\perp$ is cyclic with zeros $1, \alpha, \alpha^2, \ldots, \alpha^{q-2}$, which are zeros of the polynomials

$$
\sum_{i=0}^{q-2} (a\alpha^i + b\alpha^{-i})X^i \in \mathbb{F}_q[X]/(X^{q-1} - 1) \quad \text{with} \quad a, b \in \mathbb{F}_q.
$$

This implies that the code

$$
D = \{(ax + b/x)_{x \in \mathbb{F}_q^*} : a, b \in \mathbb{F}_q\}
$$

satisfies $D = C^\perp$. The classical Melas code $M(q)$ is defined as the restriction to $\mathbb{F}_p$ of the code $C$ (see [5, 4]). By Delsarte's theorem [4, p. 208] we have

$$
\text{Tr}(C^\perp) = (C|\mathbb{F}_p)^\perp,
$$

where $\text{Tr}$ is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. If we substitute $C^\perp = D$ and $C|\mathbb{F}_p = M(q)$ in Delsarte's theorem, we find

$$
\{(\text{Tr}(ax + b/x))_{x \in \mathbb{F}_q^*} : a, b \in \mathbb{F}_q\} = M(q)^\perp.
$$

To ensure injectivity of the trace map, we require $2m + 1 < q$. Then the dual code $M(q)^\perp$ has dimension $2m$.

In [6, 1] we determined the weight distribution of $M(q)^\perp$ for $p = 2$ and 3. Then, by the MacWilliams identities and the Eichler-Selberg trace formula we derived a formula for the number $A_i$ of code words of weight $i$ in $M(q)$ involving traces of Hecke operators on certain spaces of cusp forms [6, Theorem 4.2; 1, Theorem 2.3]. Especially for $p = 3$, this was done in a rather concise way, only announcing results and further illustrations. In this paper we will work out the case $p = 3$ and illustrate the result by computing some weight formulas for ternary Melas codes.

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An outline of this paper is as follows. In §2 we derive an expression for traces of Hecke operators on $S_k(\Gamma_1(3))$. In §3 we prove the weight distribution theorem for ternary Melas codes. Then, in the next sections, we compute traces of Hecke operators, first for even $k$, then for odd $k$. Finally, in §6 we give a table of weight formulas for $M(q)$.

The references on coding theory can be found in the book of MacWilliams and Sloane [4]. For a systematic introduction to cusp forms and Hecke operators we refer to the books by S. Lang [2] and J.-P. Serre [8]. In [6, Theorem 2.2] the reader can find the precise form of the Eichler-Selberg trace formula, as we use it. Our notation in this paper links up with the notation in [6].

2. TRACES OF HECKE OPERATORS ON $S_k(\Gamma_1(3))$

For the space of cusp forms $S_k(\Gamma_1(3))$ we have

$$S_k(\Gamma_1(3)) = S_k(\Gamma_0(3), 1) \oplus S_k(\Gamma_0(3), \omega),$$

where $1$ is the trivial character on $(\mathbb{Z}/3\mathbb{Z})^*$ and $\omega$ is the quadratic character on $(\mathbb{Z}/3\mathbb{Z})^*$. Both characters have conductor 3, and we extend them to $\mathbb{Z}/3\mathbb{Z}$ by defining them 0 on the residue class of 0 modulo 3. Actually,

$$S_k(\Gamma_1(3)) = \begin{cases} 
S_k(\Gamma_0(3), 1) & \text{for even } k, \\
S_k(\Gamma_0(3), \omega) & \text{for odd } k.
\end{cases}$$

Now we can apply the Eichler-Selberg trace formula for $S_k(\Gamma_0(3), \chi)$, expressing traces of Hecke operators in class numbers of binary quadratic forms.

**Proposition 2.1.** Let $q = 3^m$ with $m \geq 1$, and denote by $\text{Tr} T_q$ the trace of the Hecke operator $T_q$ acting on the space of cusp forms $S_k(\Gamma_1(3))$. Then

$$\text{Tr} T_q = \begin{cases} 
-\sum_{\rho} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} H(t^2 - 4q) - 1 & \text{for } k = 2, \\
-\sum_{\rho} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} H(t^2 - 4q) - 1 + q & \text{for } k \geq 3,
\end{cases}$$

The summation variable $t$ runs over $\{t \in \mathbb{Z}: t^2 < 4q \text{ and } t \equiv 1 \pmod{3}\}$. The symbols $\rho$ and $\overline{\rho}$ indicate the zeros of the polynomial $X^2 - tX + q$, and $H(t^2 - 4q)$ is the Kronecker class number of $t^2 - 4q$.

**Proof.** We start from the Eichler-Selberg trace formula as stated in [6, Theorem 2.2] and employ it for $S_k(\Gamma_0(3), \chi)$, where $\chi = 1$ for even $k$ and $\chi = \omega$ for odd $k$. In the notation of [6, Theorem 2.2], the contribution of $A_1$ is 0. As to the contribution of $A_2$, we notice that $\mu(t, f, n) = \chi(t)$. It follows that

$$A_2 = -\sum_{t \in \mathbb{Z}} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} H(t^2 - 4q)$$

by adding together terms with $t \equiv 1 \pmod{3}$ and $t \equiv 2 \pmod{3}$. Furthermore, $A_3 = -1$ in all cases, and $A_4 = q$ for $k = 2$ and $\chi = 1$, while $A_4 = 0$ in the other cases. Altogether, we get the above-mentioned formulas. □

The numbers $(\rho^{k-1} - \overline{\rho}^{k-1})/(\rho - \overline{\rho})$ are symmetric expressions in $\rho$ and $\overline{\rho}$, so they can be written as polynomials $Q_k(t, q)$ in $t = \rho + \overline{\rho}$ and $q = \rho \overline{\rho}$. 

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We have \( Q_0(t, q) = 1 \) and \( Q_1(t, q) = t \). From \( \rho^{k+1} - \rho^{k+1} = (\rho + \rho)(\rho^k - \rho^k) - \rho \rho (\rho^{k-1} - \rho^{k-1}) \) we get the recurrence relation

\[
Q_k(t, q) = tQ_{k-1}(t, q) - qQ_{k-2}(t, q) \quad \text{for } k \geq 2.
\]

The polynomial \( Q_k \) is, as a polynomial in \( \rho \) and \( \rho \), homogeneous of degree \( k \). Therefore, it is also homogeneous of degree \( k \) as a polynomial in \( t \) and \( q \), provided we assign a weight 1 to the variable \( t \) and a weight 2 to the variable \( q \). Note that \( Q_k \) is monic in \( t \), and has integer coefficients and terms \( q^r t^{k-2r} \), where \( 0 \leq r \leq \lfloor k/2 \rfloor \). It follows that we can write

\[
(2) \quad t^i = \sum_{j=0}^{i} \lambda_{i, j} Q_{i-j}(t, q) q^{j/2}.
\]

The \( \lambda_{i, j} \in \mathbb{Z} \) satisfy

\[
\lambda_{i, j} = 0 \quad \text{for } j \notin \{0, 1, \ldots, i\} \text{ or } j \text{ odd},
\]

\[
\lambda_{0, 0} = \lambda_{1, 0} = 1,
\]

while the recurrence relation for \( Q_k \) induces the recurrence relation

\[
(3) \quad \lambda_{i+1, j} = \lambda_{i, j-2} + \lambda_{i, j}.
\]

Now we rewrite the expressions for \( \text{Tr} T_q \) on \( S_k(\Gamma_1(3)) \) in Proposition 2.1 as

\[
(4) \quad \text{Tr} T_q = \begin{cases} 
-\sum_t Q_{k-2}(t, q) H(t^2 - 4q) - 1 & \text{for odd } k \text{ and even } k \geq 4, \\
-\sum_t H(t^2 - 4q) - 1 + q & \text{for } k = 2.
\end{cases}
\]

From the formula for \( \dim S_k(\Gamma_0(N), \chi) \) in [6] we easily derive:

\[
(5) \quad \dim S_k(\Gamma_1(3)) = \begin{cases} 
\dim S_k(\Gamma_0(3), \omega) = \lfloor k/3 \rfloor - 1 & \text{for odd } k, \\
\dim S_k(\Gamma_0(3), 1) = \lfloor k/3 \rfloor - 1 & \text{for even } k \geq 4, \\
0 & \text{for } k = 2.
\end{cases}
\]

Because \( \dim S_2(\Gamma_1(3)) = 0 \), one has

\[
\text{Tr} T_2 = -\sum_t H(t^2 - 4q) - 1 + q = 0.
\]

3. The weight distribution of ternary Melas codes

Let \( q = 3^m \) with \( m \geq 2 \). In [1] we derived the weight distribution of the dual ternary Melas code \( M(q) \parallel \):

The nonzero weights of \( M(q) \parallel \) are \( w_t = 2(q - 1 + t)/3 \), where \( t \in \mathbb{Z}, t^2 < 4q \), and \( t \equiv 1 \pmod{3} \). For \( t \neq 1 \) the frequency of \( w_t \) is \( (q-1)H(t^2 - 4q) \); the weight \( w_1 = 2q/3 \) has frequency \( (q-1)(H(1 - 4q) + 2) \).

Using the MacWilliams identities and the Eichler-Selberg trace formula, we obtain an expression for the weight distribution of \( M(q) \). We will elaborate the result announced in [1, Theorem 2.3].
Theorem 3.1. The number $A_i$ of code words of weight $i$ in the Melas code $M(q)$ is given by

$$q^2 A_i = \binom{q - 1}{i} 2^i + 2(q - 1) \sum_{s=0}^{i} (-1)^s \binom{2q/3}{s} \binom{q/3 - 1}{i - s} 2^{i-s} - (q - 1) \sum_{j=0}^{i} W_{i,j}(q)(1 + \tau_{j+2}(q)),$$

where the polynomials $W_{i,j}(q)$ are defined for $0 \leq j \leq i$ by

$$W_{0,0} = 1, \quad W_{1,0} = 0, \quad W_{1,1} = -2,$$

$$(i + 1)W_{i+1,j} = -iW_{i,j} - 2qW_{i,j+1} - 2W_{i,j-1} - 2(q - i)W_{i-1,j}$$

(otherwise, the $W_{i,j}$ are 0).

By $\tau_k(q)$ we denote for $k \geq 3$ the trace of the Hecke operator $T_q$ on $S_k(\Gamma_1(3))$. For convenience we let $\tau_2(q) = -q$.

Proof. This proof is a modification of the proof of the analogous theorem in [6]. For $0 \leq i \leq q - 1$, let $P_i(X)$ be the $i$th Krawtchouk polynomial

$$(6) \quad P_i(X; q - 1, 3) = \sum_{s=0}^{i} (-1)^s \binom{X}{s} \binom{q - 1 - X}{i - s} 2^{i-s}.$$ 

These polynomials satisfy the recurrence relation

$$(i + 1)P_{i+1}(X) = (2q - 2 - i - 3X)P_i(X) - 2(q - i)P_{i-1}(X).$$

We define $f_i(X) = P_i(2(q - 1 + X)/3)$; then

$$f_0(X) = P_0(2(q - 1 + X)/3) = 1, \quad f_1(X) = P_1(2(q - 1 + X)/3) = -2X$$

and the recurrence relation becomes

$$(7) \quad (i + 1)f_{i+1}(X) = (-i - 2X)f_i(X) - 2(q - i)f_{i-1}(X).$$

It follows that $f_i(X)$ has degree $i$, and we write

$$(8) \quad f_i(X) = \sum_{k=0}^{i} \pi_i(k)X^k.$$ 

Now $\pi_0(0) = 1$, $\pi_1(0) = 0$, $\pi_1(1) = -2$, and from (7) we derive

$$(9) \quad (i + 1)\pi_{i+1}(k) = -i\pi_i(k) - 2\pi_i(k - 1) - 2(q - i)\pi_{i-1}(k).$$

We define $\pi_i(k) = 0$ for cases other than $0 \leq k \leq i$. When we apply the MacWilliams identities to $M(q)^\perp$ and $M(q)$, we get

$$q^2 A_i = \sum_{t} \text{frequency}(w_t)P_i(2(q - 1 + t)/3) + P_i(0),$$

where $t$ runs over $\{t \in \mathbb{Z}: t^2 < 4q$ and $t \equiv 1 \pmod{3}\}$. Using the weight distribution of $M(q)^\perp$ and the polynomials $f_i$ introduced above, we find

$$\frac{q^2}{q - 1} A_i = \sum_{t} H(t^2 - 4q)f_i(t) + 2f_i(1) + \frac{P_i(0)}{q - 1}.$$
From definition (6) we see that  
\[ P_i(0) = \left( q^{-1} \right)^i \] 
and  
\[ f_i(1) = P_i \left( \frac{2q}{3} \right) = \sum_{s=0}^{i} (-1)^s \left( \frac{2q/3}{s} \right) \left( \frac{q/3 - 1}{i-s} \right) 2^{i-s}. \]

From (8) we obtain  
\[ \sum H(t^2 - 4q)f_i(t) = \sum_{j=0}^{i} \pi_i(j) \sum_t t^j H(t^2 - 4q). \]

By formula (2) this becomes  
\[ \sum_{j=0}^{i} \pi_i(j) \sum_{k=0}^{j} \lambda_j, k q^{k/2} \sum_{t} Q_{j-k}(t, q) H(t^2 - 4q). \]

Using (4) combined with the fact that, according to (5), \( \text{Tr } T_q = 0 \) on \( S_2(\Gamma_1(3)) \), and remembering our convention that \( x_2(q) = -q \), we get  
\[ (10) \sum_{j=0}^{i} \pi_i(j) \sum_{k=0}^{j} \lambda_j, k q^{k/2}(-1 - \tau_{j-k+2}(q)). \]

We define  
\[ W_{i,j}(q) = \sum_{k=0}^{i-j} \pi_i(k + j) \lambda_{k+j}, k q^{k/2}. \] 
By changing the horizontal summation in (10) into a diagonal summation, the expression (10) becomes  
\[ \sum_{j=0}^{i} W_{i,j}(q)(-1 - \tau_{j+2}(q)). \]

Putting all this together, we get the announced formula for \( q^2 A_i \).

As to the polynomials \( W_{i,j}(q) \), we easily see that \( W_{0,0} = 1 \), \( W_{1,0} = 0 \), and \( W_{1,1} = -2 \). The recurrence relation for \( W_{i,j} \) follows by writing out the definition of \( (i+1)W_{i+1,j} \) and using the recurrence relations (9) and (3) for \( (i+1)\pi_{i+1}(k + j) \) and \( \lambda_{k+j}, k \).

We conclude this section by noticing that to obtain more explicit expressions for \( A_i \), we have to compute the traces of the Hecke operators \( \tau_k(q) \). This is the subject of the next two sections.

4. The computation of \( \tau_k(q) \) for \( k \) even, \( k \geq 4 \)

As always, we take \( q = 3^m \) with \( m \geq 2 \). By convention, we have that \( \tau_2(q) = -q \), while for \( k \geq 3 \) the trace of the Hecke operator \( T_q \) acting on the space \( S_k(\Gamma_1(3)) \) is indicated by \( \tau_k(q) \). For even \( k \), the space \( S_k(\Gamma_1(3)) = S_k(\Gamma_0(3), 1) \) and the theory of newforms of Atkin and Lehner [2] provides us with a decomposition  
\[ S_k(\Gamma_0(3), 1) = S_k(\Gamma_0(3))^\text{new} \oplus S_k(\Gamma_0(3))^\text{old}, \]
which is respected by the Hecke operators. The old part is spanned by the forms \( f(z) \) and \( f(3z) \), where \( f(z) \) runs over a basis of simultaneous eigenforms of \( S_k(\Gamma_0(1)) = S_k(\text{SL}_2(\mathbb{Z})). \)
Proposition 4.1. On $S_k(\Gamma_0(3))^{\text{old}}$ we have
\[
\begin{align*}
\text{Tr } T_1 &= 2 \dim S_k(\text{SL}_2(\mathbb{Z})), \\
\text{Tr } T_3 &= \text{Tr}(T_3 \text{ on } S_k(\text{SL}_2(\mathbb{Z}))), \\
\text{Tr } T_{3m} &= \text{Tr}(T_{3m} \text{ on } S_k(\text{SL}_2(\mathbb{Z}))) \\
&\quad - 3^{k-1} \text{Tr}(T_{3m-2} \text{ on } S_k(\text{SL}_2(\mathbb{Z}))) \quad \text{for } m \geq 2.
\end{align*}
\]

Proof. The subspace $S_k(\Gamma_0(3))^{\text{old}}$ is a direct sum of 2-dimensional complex vector spaces with basis $\{f(z), f(3z)\}$, where $f(z)$ is a simultaneous eigenform for all $T_n$ in $S_k(\text{SL}_2(\mathbb{Z}))$. The operator $T_1$ is the identity map, so
\[
\text{Tr } T_1 = \dim S_k(\Gamma_0(3))^{\text{old}} = 2 \dim S_k(\text{SL}_2(\mathbb{Z})).
\]

Let $f(z) = \sum_{m=1}^{\infty} a_m e^{2\pi imz}$; then by applying the formula for $T_n$ on $S_k(\Gamma_0(3), 1)$ (see [6]) we have
\[
T_3(f(z)) = \sum_{m \geq 1} a_{3m} e^{2\pi imz}.
\]

while on $S_k(\text{SL}_2(\mathbb{Z}))$ we have
\[
T_3(f(z)) = \lambda f(z) - 3^{k-1} f(3z) \quad \text{and} \quad T_3(f(3z)) = f(z).
\]

Then on $\langle f(z) \rangle \oplus \langle f(3z) \rangle$ the operator $T_3$ has eigenvalues $\alpha$ and $\beta$ with $\alpha + \beta = \lambda$ and $\alpha \beta = 3^{k-1}$. The eigenvalues of $T_3$ acting on $S_k(\Gamma_0(3))^{\text{old}}$ are precisely the $\alpha$ and $\beta$ for all possible eigenvalues $\lambda$ of $T_3$ acting on $S_k(\text{SL}_2(\mathbb{Z}))$. We conclude that
\[
\text{Tr } T_3 = \sum (\alpha + \beta) = \sum \lambda = \text{Tr}(T_3 \text{ on } S_k(\text{SL}_2(\mathbb{Z}))).
\]

From the product formula $T_n \cdot T_m = \sum_{d|n,m} d^{k-1} T_{mn/d^2}$ we derive
\[
T_{3m} = T_3 \cdot T_{3m-1} - 3^{k-1} T_{3m-2} \quad \text{for } m \geq 2
\]
on $S_k(\text{SL}_2(\mathbb{Z}))$. Thus, the eigenvalue $\lambda_{3m}$ of $T_{3m}$ on $S_k(\text{SL}_2(\mathbb{Z}))$ corresponding to $\lambda$ is
\[
\lambda \cdot \lambda_{3m-1} - 3^{k-1} \cdot \lambda_{3m-2}.
\]

While $\lambda = \alpha + \beta$ and $3^{k-1} = \alpha \beta$, it follows by induction that the eigenvalue of $T_{3m}$ on $S_k(\text{SL}_2(\mathbb{Z}))$ corresponding to $\lambda = \alpha + \beta$ is $\sum_{i=0}^{m} \alpha^i \beta^{m-i}$.

Furthermore, it holds that $T_{3m} = (T_3)^m$ on $S_k(\Gamma_0(3), 1)$, so $T_{3m}$ has eigenvalues $\alpha^m$ and $\beta^m$ on $\langle f(z) \rangle \oplus \langle f(3z) \rangle$. Adding up the relation
\[
\alpha^m + \beta^m = \sum_{i=0}^{m} \alpha^i \beta^{m-2-i} - \alpha \beta \sum_{i=0}^{m-2} \alpha^i \beta^{m-2-i}
\]
for all pieces of $S_k(\Gamma_0(3))^{\text{old}}$, we obtain the stated result for $T_{3m}$, $m \geq 2$.

Remark 4.2. From the dimension formula [6, Corollary 2.3] we conclude
\[
\dim S_2(\text{SL}_2(\mathbb{Z})) = 0,
\]
\[
\dim S_k(\text{SL}_2(\mathbb{Z})) = \begin{cases} 
[k/12] & \text{for } k \not\equiv 2 \pmod{12}, \\
[k/12] - 1 & \text{for } k \equiv 2 \pmod{12}, \quad k \geq 4.
\end{cases}
\]

Next we derive a formula for $\text{Tr } T_q$ on $S_k(\Gamma_0(3))^{\text{new}}$.
Proposition 4.3. On $S_k(\Gamma_0(3))^{\text{new}}$ we have

$$\text{Tr } T_q = \left\{ \begin{array}{ll}
\dim S_k(\Gamma_0(3))^{\text{new}} \cdot q^{k/2-1} & \text{for } m \text{ even}, \\
q^{k/2-1} & \text{for } m \text{ odd, } k \equiv 2, 6 \pmod{12}, \\
-q^{k/2-1} & \text{for } m \text{ odd, } k \equiv 0, 8 \pmod{12}, \\
0 & \text{for } m \text{ odd, } k \equiv 4, 10 \pmod{12}.
\end{array} \right.$$ 

Proof. First we consider $T_3$. The eigenvalues of $T_3$ acting on $S_k(\Gamma_0(3))^{\text{new}}$ are $\pm 3^{k/2-1}$ (see [3, Theorem 3]). In order to find the multiplicities of the eigenvalues, we compute

$$\text{Tr } T_3 \text{ on } S_k(\Gamma_0(3))^{\text{new}} = \text{Tr } T_3 \text{ on } S_k(\Gamma_0(3), 1) - \text{Tr } T_3 \text{ on } S_k(\Gamma_0(3))^{\text{old}}$$

$$= \text{Tr } T_3 \text{ on } S_k(\Gamma_0(3), 1) - \text{Tr } T_3 \text{ on } S_k(\text{SL}_2(\mathbb{Z})).$$

By the Eichler-Selberg formula we find

$$\text{Tr } T_3 \text{ on } S_k(\Gamma_0(3), 1)$$

(13)

$$= -\left\{ \frac{\rho_1^{k-1} - \bar{\rho}_1^{k-1}}{\rho_1 - \bar{\rho}_1} h_w(-11) + \frac{\rho_2^{k-1} - \bar{\rho}_2^{k-1}}{\rho_2 - \bar{\rho}_2} h_w(-8) + 1 \right\},$$

where $\rho_1, \bar{\rho}_1$ are the zeros of $X^2 - X + 3$ and $\rho_2, \bar{\rho}_2$ are the zeros of $X^2 - 2X + 3$. Applying the same formula for $\text{Tr } T_3 \text{ on } S_k(\text{SL}_2(\mathbb{Z}))$, we find

(13) and the extra terms

$$-\left( \frac{\rho_3^{k-1} - \bar{\rho}_3^{k-1}}{\rho_3 - \bar{\rho}_3} h_w(-3) \right) - \frac{1}{2} \left( \frac{\rho_4^{k-1} - \bar{\rho}_4^{k-1}}{\rho_4 - \bar{\rho}_4} \right) (h_w(-12) + h_w(-3)),$$

where $\rho_3, \bar{\rho}_3$ are the zeros of $X^2 - 3X + 3$ and $\rho_4, \bar{\rho}_4$ are the zeros of $X^2 + 3$.

For $\Delta = -4$, the $h_w(\Delta)$ are class numbers and $h_w(-3) = 1/3$.

Note that in the case of $\text{SL}_2(\mathbb{Z})$, the character involved is the principal character, which is 1 on all of $\mathbb{Z}$ and has conductor 1.

Substituting the zeros of $X^2 - 3X + 3$ and $X^2 + 3$, we get

$$\text{Tr } T_3 \text{ on } S_k(\Gamma_0(3))^{\text{new}} = 2.3^{k/2-2}(\sin(k - 1)\pi/6 + \sin(k - 1)\pi/2)$$

(14)

$$= \left\{ \begin{array}{ll}
0 & \text{for } k \equiv 4, 10 \pmod{12}, \\
3^{k/2-1} & \text{for } k \equiv 2, 6 \pmod{12}, \\
-3^{k/2-1} & \text{for } k \equiv 0, 8 \pmod{12}.
\end{array} \right.$$ 

Denoting the multiplicities of the eigenvalues $3^{k/2-1}$ and $-3^{k/2-1}$ by $A$ and $B$, respectively, we now know $A - B$, while $A + B = \dim S_k(\Gamma_0(3))^{\text{new}}$. Because $T_3^m = (T_3)^m$ on $S_k(\Gamma_0(3), 1)$, the eigenvalues of $T_3^m$ on $S_k(\Gamma_0(3))^{\text{new}}$ are $(3^{k/2-1})^m$ and $(-3^{k/2-1})^m$, while their multiplicities are known as well. From (14) we easily confirm the required result. □

The dimension of $S_k(\Gamma_0(3))^{\text{new}}$ for even $k \geq 4$ can be computed explicitly. From the decomposition

$$S_k(\Gamma_0(3), 1) = S_k(\Gamma_0(3))^{\text{new}} \oplus S_k(\Gamma_0(3))^{\text{old}}$$

we see that $\dim S_k(\Gamma_0(3))^{\text{new}} = \dim S_k(\Gamma_1(3)) - 2 \dim S_k(\text{SL}_2(\mathbb{Z})).$ Combining
For $k = 12, 16, 18, 10, 22$, the $t_{k,m}$ are respectively given by $t_{k,0} = 2$, $t_{k,1} = 252, -3348, -4284, 50652, -128844$ and

$$t_{k,m} = t_{k,1} \cdot t_{k,m-1} - 3^{k-1} t_{k,m-1} \quad \text{for } m \geq 2.$$ 

Proof. For $k = 4, 6, 8, 10, 14$, the spaces $S_k(SL_2(\mathbb{Z}))$ are zero, therefore $\text{Tr} T_q$ on $S_k(\Gamma_0(3))^{\text{old}}$ is zero, and our formulas follow easily.

For $k = 12, 16, 18, 20, 22$, the spaces $S_k(SL_2(\mathbb{Z}))$ are one-dimensional. If $\lambda$ is the eigenvalue of $T_3$ on $S_k(SL_2(\mathbb{Z}))$, we have $\lambda = \alpha + \beta$ and $3^{k-1} = \alpha \beta$, where $\alpha$ and $\beta$ are the corresponding eigenvalues of $T_3$ on $S_k(\Gamma_0(3))^{\text{old}}$ (see the proof of Proposition 4.1). Now $t_m = \text{Tr} T_3^m$ on $S_k(\Gamma_0(3))^{\text{old}}$ satisfies the recurrence relation

$$t_m = \alpha^m + \beta^m = \lambda t_{m-1} - 3^{k-1} t_{m-2} \quad \text{for } m \geq 2,$$
while \( t_1 = \lambda \) and \( t_0 = 2 \). We calculate \( \lambda = \text{Tr} T_3 \) on \( S_k(\text{SL}_2(\mathbb{Z})) \) by the trace formula. The result is

\[
\lambda = -\sum_{t=0}^{3} r_t Q_{k-2}(t, 3) - 1,
\]

with \( r_0 = \frac{3}{5} \), \( r_1 = r_2 = 1 \), and \( r_3 = \frac{1}{3} \). Combining these observations with Proposition 4.3, we obtain our formulas. \( \square \)

Note that for \( k = 12 \), the eigenvalue of \( T_q \) on \( S_{12}(\text{SL}_2(\mathbb{Z})) \) is \( \tau(q) \), where \( \tau \) is the Ramanujan \( \tau \)-function. Then \( t_m = \tau(q) - 3^{x^2(q/9)} \) for \( q = 3^m \) with \( m \geq 2 \) and \( t_1 = 252 \).

5. THE COMPUTATION OF \( \tau_k(q) \) FOR ODD \( k \geq 3 \)

By (5), we have for odd \( k \) that

\[
\dim S_k(\Gamma(3)) = \dim S_k(\Gamma_0(3), \omega) = [k/3] - 1.
\]

Since the action of the character \( \omega \) on \((\mathbb{Z}/3\mathbb{Z})^*\) differs from the action of the principal character, the space of cusp forms \( S_k(\Gamma_0(3), \omega) \) consists entirely of newforms. Therefore, the eigenvalues \( \lambda \) of \( T_3 \) acting on \( S_k(\Gamma_0(3), \omega) \) have absolute values \( 3^{(k-1)/2} \) (see [3, Theorem 3]). This implies that the monic polynomial \( F_k(X) \) with roots \( \lambda/3^{(k-1)/2} \) is reciprocal. So, to determine \( F_k(X) \), which has degree \([k/3] - 1\), we only have to know the first \( ([k/3] - 1)/2 + 1 \) coefficients, provided they are not 0.

Since \( T_q = (T_3)^m \) on \( S_k(\Gamma_0(3), \omega) \), we have \( \text{Tr} T_q = \sum_2 \lambda^m \), and from the Newton identities for power sums we can derive some elementary symmetric functions of the eigenvalues \( \lambda \) from \( \text{Tr} T_3, \text{Tr} T_9 \), etc. We only need a few \( \text{Tr} T_q \) to fix \( F_k(X) \), bearing in mind that \( F_k(X) \) is reciprocal. From \( F_k(X) \) we obtain the characteristic polynomial of \( T_3 \) and from that the eigenvalues \( \lambda \) of \( T_3 \). Then we can compute \( \tau_k(q) = \sum_2 \lambda^m \) for odd \( k \geq 3 \) and \( q = 3^m \) with \( m \geq 2 \).

**Proposition 5.1.** The trace \( \tau_k(q) \) of the Hecke operator \( T_q \) with \( q = 3^m \) and \( m \geq 2 \), acting on \( S_k(\Gamma_0(3), \omega) \), is for \( k = 3, 5, 7, 9, 11, 13, 15, \) and 17 given by the following table:

\[
\begin{align*}
\tau_3(q) &= \tau_5(q) = 0, \\
\tau_7(q) &= (-1)^m q^3, \\
\tau_9(q) &= q^4 \cdot \text{Trace}(\alpha q^9), \\
\tau_{11}(q) &= q^5 \cdot \text{Trace}(\alpha q^{11}), \\
\tau_{13}(q) &= q^6 \cdot \{1 + \text{Trace}(\alpha q^{13})\}, \\
\tau_{15}(q) &= q^7 \cdot \text{Trace}(\alpha q^{15}), \\
\tau_{17}(q) &= q^8 \cdot \text{Trace}(\alpha q^{17}).
\end{align*}
\]
The \( \alpha_i \) are algebraic numbers of absolute value 1 given by

\[
\begin{align*}
\alpha_9 &= \frac{5 + 2\sqrt{-14}}{9}, \quad \alpha_{11} = \frac{-1 + 4\sqrt{-5}}{9}, \quad \alpha_{13} = \frac{-25 + 2\sqrt{-26}}{27}, \\
\alpha_{15} &= \frac{61 - 16\sqrt{91} + 4\sqrt{-2002} - 122\sqrt{91}}{243}, \\
\alpha_{17} &= \frac{-19 + 2\sqrt{8089} + 2\sqrt{-6583} - 19\sqrt{8089}}{243}.
\end{align*}
\]

In this table, the Trace of an algebraic number is the sum of all its conjugates.

Proof. Since \( \dim S_k(\Gamma_0(3), \omega) = 0 \) for \( k = 3, 5 \), we have that \( \tau_3(q) = \tau_5(q) = 0 \). For the other values of \( k \) we compute the eigenvalues of \( T_3 \) acting on \( S_k(\Gamma_1(3)) = S_k(\Gamma_0(3), \omega) \) in the way indicated above. The trace formula (4) gives us that for weight \( k \):

\[
\begin{align*}
\text{Tr } T_3 &= -Q_{k-2}(1, 3)H(-11) - Q_{k-2}(-2, 3)H(-8) - 1 \\
\text{Tr } T_9 &= -1 - \sum_{t^2 \equiv 1 \pmod{3}} Q_{k-2}(t, 9)H(t^2 - 36).
\end{align*}
\]

Using the recurrence relations for the polynomial \( Q_{k-2} \) and a small table of class numbers from [7], we get the entries of the table below:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \text{Tr } T_3 )</th>
<th>( \text{Tr } T_9 )</th>
<th>( F_k(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>-27</td>
<td>729</td>
<td>( X + 1 )</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>-5022</td>
<td>( X^2 - \frac{10}{9}X + 1 )</td>
</tr>
<tr>
<td>11</td>
<td>-54</td>
<td>-115182</td>
<td>( X^2 + \frac{7}{9}X + 1 )</td>
</tr>
<tr>
<td>13</td>
<td>-621</td>
<td>1291059</td>
<td>((X - 1)(X^2 + \frac{50}{27}X + 1))</td>
</tr>
<tr>
<td>15</td>
<td>2196</td>
<td>-1624860</td>
<td>( X^4 - \frac{244}{243}X^3 + \frac{1474}{2187}X^2 - \frac{244}{243}X = 1 )</td>
</tr>
<tr>
<td>17</td>
<td>-2052</td>
<td>18618660</td>
<td>( X^4 + \frac{76}{243}X^3 - \frac{122}{729}X^2 + \frac{76}{243}X + 1 )</td>
</tr>
</tbody>
</table>

The eigenvalues \( \lambda \) of \( T_3 \) on \( S_k(\Gamma_0(3), \omega) \) are \( 3^{(k-1)/2} \cdot \alpha \), where \( \alpha \) runs over the zeros of \( F_k(X) \) and

\[
\tau_k(q) = \sum_{\lambda} \lambda^m = q^{(k-1)/2} \sum_{\{\alpha : F_k(\alpha) = 0\}} \alpha^m,
\]

which provides us with our formulas for \( k = 7, 9, 11, 13, 15, 17 \). \( \square \)

Note that by computing the trace of more \( T_3^m \) for \( m \geq 3 \) we can easily extend Proposition 5.1. Adding \( \text{Tr } T_{27} \), for instance, will get us to \( k = 23 \).

6. Weight Formulas for \( M(q) \)

When we combine Theorem 3.1 with Propositions 4.4 and 5.1, we get explicit formulas for the frequencies \( A_i \) of words of weight \( i \) in \( M(q) \). To obtain these formulas, we used the symbolic manipulation language MACSYMA.

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We conclude by giving a table of weight formulas. In this table, Ramanujan's \( \tau \)-function is denoted by \( \tau \) and the numbers \( t_k \) denote \( \text{Trace}(\alpha_k^m) \) as in Proposition 5.1.

**Table 6.1**

*Frequencies \( A_i \) of small weights \( i \) in the Melas codes \( M(q) \)*

\[
egin{align*}
A_1 &= A_3 = 0, \\
A_2 &= q - 1, \\
A_4 &= (q - 1)(q - 3)/2, \\
A_5 &= 4(q - 1)(q^2 + ((-1)^m - 14)q + 36)/15, \\
A_6 &= (q - 1)(8q^3 - 165q^2 + (1240 - 68(-1)^m)q - 2655)/90, \\
A_7 &= 2(q - 1)(4q^4 - 108q^3 + (4t_9 - 18(-1)^m + 1215)q^2 \\
&\quad + (399(-1)^m - 6744)q + 12884)/315, \\
A_8 &= (q - 1)(16q^5 - 560q^4 + 8225q^3 - (224t_9 - 880(-1)^m + 66255)q^2 \\
&\quad - (16296(-1)^m - 298263)q - 517825)/2520, \\
A_9 &= (q - 1)(16q^6 - 704q^5 + 13216q^4 \\
&\quad - (160t_9 - 16t_{11} - 216(-1)^m + 138656)q^3 \\
&\quad + (3816t_9 - 13776(-1)^m + 895209)q^2 \\
&\quad - (3470238 - 187593(-1)^m)q + 5597820)/11340, \\
A_{10} &= (q - 1)(32q^7 - 1728q^6 + 40512q^5 - 540519q^4 \\
&\quad + (6240t_9 - 720t_{11} - 6120(-1)^m + 4529826)q^3 \\
&\quad + (-110280t_9 + 360000(-1)^m - 24851277)q^2 \\
&\quad + (85643448 - 4448871(-1)^m)q - 129806479 \\
&\quad - 32(\tau(q) - 177147(\tau(q/9)))/2))/113400, \\
A_{11} &= (q - 1)(32q^8 - 2080q^7 + 59520q^6 - 985920q^5 \\
&\quad + (2288t_9 + 32t_{13} - 560t_{11} - 440(-1)^m + 10453958)q^4 \\
&\quad + (-136840t_9 + 16720t_{11} + 110220(-1)^m - 74203966)q^3 \\
&\quad + (1705506t_9 - 5122359(-1)^m + 358627785)q^2 \\
&\quad + (57077625(-1)^m - 112429735)q + 1617492524 \\
&\quad + 880(\tau(q) - 177147(\tau(q/9)))/2))/623700, \\
A_{12} &= (q - 1)(64q^9 - 4928q^8 + 168960q^7 - 3400320q^6 \\
&\quad + 44564751q^5 - (2112t_{13} - 33440t_{11} + 115808t_9 \\
&\quad + 398775397 + 16720(-1)^m)q^4 - (664400t_{11} \\
&\quad - 5020400t_9 + 2939640(-1)^m - 2486674179)q^3 \\
&\quad - (52961436t_9 - 145879734(-1)^m + 10845159710)q^2 \\
&\quad + (31412188148 - 1550485266(-1)^m)q - 43190708055 \\
&\quad + (\tau(q) - 177147(\tau(q/9)))(1408q^4 - 46992q^3)/q^5)/7484400.
\end{align*}
\]
BIBLIOGRAPHY


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