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BLOSSOMING BEGETS B-SPLINE BASES BUILT BETTER BY B-PATCHES

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ABSTRACT. The concept of symmetric recursive algorithm leads to new, s-dimensional spline spaces. We present a general scheme for constructing a collection of multivariate B-splines with $k-1$ continuous derivatives whose linear span contains all polynomials of degree at most k. This scheme is different from the one developed earlier by Dahmen and Micchelli and, independently, by Höllig, which was based on combinatorial principles and the geometric interpretation of the B-spline. The new spline space introduced here seems to offer possibilities for economizing the computation for evaluating linear combinations of B-splines.

1. INTRODUCTION

Polar forms offer a unified approach to various algorithms for Bézier and B-spline curves [6, 7]. However, very little seems to be known about polar forms for surface representation. This question has been raised lately by several researchers. Some discouraging preliminary efforts seem to have cast doubt on the efficacy of this approach to recursive evaluation of spline surfaces.

Recently, in [8], one of us focused on symmetric recursive algorithms and polar forms for polynomial surfaces. This led to a new representation of a polynomial surface in terms of *B-patches*. A B-patch shares many properties in common with Bézier patches and includes them as special cases. Recursive evaluation of B-patch representations of a polynomial is given in [8]. Also, as for Bézier representations, the control points of a B-patch representation of a polynomial are obtained by evaluating its polar form at certain vectors which generate the B-patches.

The univariate analog of B-patches on an interval are no less than pieces of B-splines which have support on the interval with knots at the points used to generate the B-patches [7].

Hindsight makes it clear now that the term B-patch used in [8] was aptly chosen. For, in fact, we will first prove that the s-dimensional B-patch basis functions do agree with the restriction of certain s-dimensional multivariate B-splines to certain regions of $\mathbf{R}^s$. The exact prescription of the regions of agreement and the collection of knots for building the B-splines must be chosen with great care. However, when this has been done, we are led to the

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construction of a new multivariate spline space. As this space is rooted in symmetric recursive B-patch algorithms, there is hope that efficient evaluation of the multivariate splines will follow. The numerical implications of this construction will be given in a forthcoming paper.

The initial setup of our method follows a pattern that two of us have used earlier [2, 3]. Thus, we begin with a triangulation of $\mathbf{R}^s$ into simplices. With each vertex we associate a cloud of k additional vertices. A rule for selecting $\binom{k+s}{s}$ subsets of $k+s+1$ vertices from the $s+1$ clouds of a given simplex must be given. Each such subset then gives a multivariate B-spline of degree k which is generically C^{k-1}. The linear span of all the B-splines is then the spline space of interest.

Two of us developed a rule for knot selection based on the geometric interpretation of the multivariate B-spline [3] (see also [4]). Briefly, we associated with every nondescending path from $(0, 0)$ to (s, k) on the lattice points in $[0, s] \times [0, k]$ a knot set. The resulting spline space had a number of important desirable properties. Specifically, for clouds concentrated around the corresponding vertices, the B-splines were locally linearly independent, and provided a stable basis for the spline space. Moreover, the space contains polynomials of degree k, and also a spline projector with maximal order of convergence was constructed in [3].

In this paper we provide another spline space, by a totally different knot prescription, which accomplishes all the above properties while also achieving a simplicity not present earlier.

The new knot selection strategy is neither based on geometric nor combinatorial techniques but rather comes from matching the B-spline recursion to the symmetric recursive algorithm for B-patches. This suggests that the computation of linear combinations of these B-splines can be carried out more efficiently. Furthermore, we derive explicit representations of all polynomials of degree k in terms of B-splines of the same degree, which have an important advantage over the analogous formulas derived in [3]. In fact, the coefficient of each B-spline is formed by evaluating the polar form of the polynomial at the knots of the B-spline. Specializing this formula gives a natural normalization of the multivariate B-spline, so that they provide a partition of unity on $\mathbf{R}^s$. In addition, we construct linear projectors which have optimal approximation rates, and establish the stability of the B-spline basis.

2. B-PATCHES AND B-SPLINES

We will begin by formulating the concept of polar forms and B-patches considered in [8], for a general s-dimensional setting. In order to do so, we start with the concept of a simplicial algorithm that evaluates a polynomial from given control points through successive affine combinations (see Figure 2.1). Let $X = \{x^{l,\beta,j} : 1 \le l \le k,\ 0 \le j \le s,\ \beta \in \mathbf{Z}_+^{s+1}\} \subset \mathbf{R}^s$ be any collection of points such that for any $1 \le l \le k$ and any $\beta = (\beta_0, \dots, \beta_s) \in \Gamma_{k-l} := \{\beta \in \mathbf{Z}_+^{s+1} : |\beta| = \beta_0 + \cdots + \beta_s = k - l\}$ the (ordered) subsets

$$X^{l,\beta} := \{x^{l,\beta,j} : j = 0, \dots, s\} \tag{2.1}$$

are affinely independent. If we define for any ordered set $W = \{w^0, \dots, w^s\}$

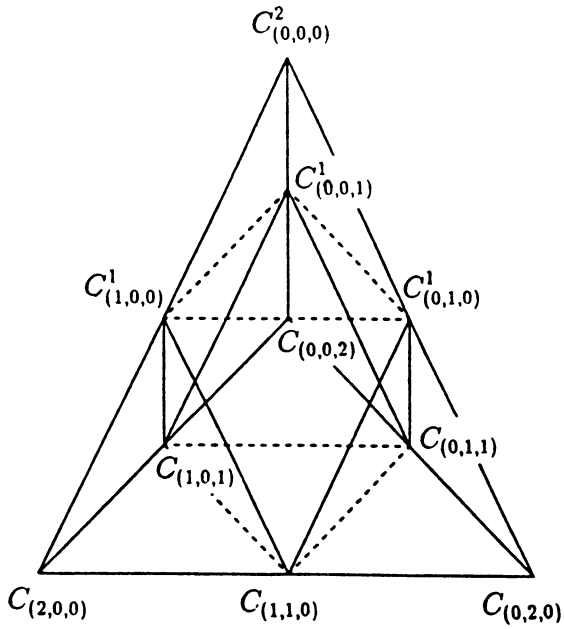

FIGURE 2.1. A simplicial algorithm for surfaces $(s = 2)$

of affinely independent points in $\mathbf{R}^s$

$$d(W) = \det\begin{pmatrix} 1 & \cdots & 1 \\ w^0 & \cdots & w^s \end{pmatrix} \tag{2.2}$$

and

$$d_j(W|x) = \det\begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w^0 & \cdots & w^{j-1} & x & w^{j+1} & \cdots & w^s \end{pmatrix}; \tag{2.3}$$

the barycentric coordinates of x with respect to the set $X^{l,\beta}$ are given by

$$\lambda_{l,\beta,j}(x) = d_j(X^{l,\beta}|x)/d(X^{l,\beta}), \tag{2.4}$$

i.e.,

$$x = \sum_{j=0}^{s} \lambda_{l,\beta,j}(x) \cdot x^{l,\beta,j}$$

and

$$\sum_{j=0}^{s} \lambda_{l,\beta,j}(x) = 1.$$

Also observe the simple fact that $\lambda_{l,\beta,j}(x^{l,\beta,m}) = \delta_{j,m}$, $j, m = 0, 1, \ldots, s$, which we will make use of later.

Let $e^j = (\delta_{j,i})_{i=0}^s$, $j = 0, 1, \ldots, s$, denote the coordinate vectors. Given the sets $X^{l,\beta}$ above, a *simplicial algorithm* is a recursive algorithm of the form

$$C^0_\beta(x) := c_\beta, \qquad \beta \in \Gamma_k, \tag{2.5}$$

and

$$C^l_\beta(x) := \sum_{j=0}^{s} \lambda_{l,\beta,j}(x) C^{l-1}_{\beta+e^j}(x), \qquad \beta \in \Gamma_{k-l}, \tag{2.6}$$

that computes a polynomial $C^k_{(0,\dots,0)}(x)$ of degree k from given control points c_β, $\beta \in \Gamma_k$. Every simplicial algorithm has an *associated multiaffine version* defined as follows: given any $x^1, \dots, x^k \in \mathbf{R}^s$, set

$$c^0_\beta(\ ) := c_\beta, \qquad \beta \in \Gamma_k, \tag{2.7}$$

and

$$c^l_\beta(x^1, \dots, x^l) := \sum_{j=0}^{s} \lambda_{l,\beta,j}(x^l) c^{l-1}_{\beta+e^j}(x^1, \dots, x^{l-1}), \qquad \beta \in \Gamma_{k-l}. \tag{2.8}$$

The maps $c^l_\beta(x^1, \dots, x^l)$ are affine in every component and satisfy

$$c^l_\beta(x, \dots, x) = C^l_\beta(x).$$

Of particular interest are *symmetric* simplicial algorithms: A simplicial algorithm is symmetric if and only if the maps $c^l_\beta(x^1, \dots, x^l)$ that appear in the multiaffine version of the algorithm are symmetric for *all* control points c_β, $\beta \in \Gamma_k$. In this situation, $c^l_\beta(x^1, \dots, x^l)$ is the *polar form* or *blossom* of the polynomial $C^l_\beta(x)$. Recall that the polar form p of a polynomial P of total degree k on $\mathbf{R}^s$ is the unique symmetric k-affine function $p(x^1, \dots, x^k)$ defined for $x^1, \dots, x^k \in \mathbf{R}^s$ such that $p(x, \dots, x) = P(x)$, $x \in \mathbf{R}^s$. Examples of symmetric simplicial algorithms are the de Casteljau and de Boor algorithms for curves and surfaces (cf. [6, 7, 8]). The following proposition characterizes the form of these algorithms.

Proposition 2.1. *A simplicial algorithm is symmetric if and only if the points $x^{l,\beta,j}$, $j = 0, 1, \dots, s$, of the set $X^{l,\beta}$ only depend on β_j and not on l and $\beta_0, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_s$, that is,*

$$x^{l,\beta,j} = y^{j,\beta_j} \tag{2.9}$$

for some set $Y = \{y^{j,l} : 0 \le j \le s,\ 0 \le l \le k\}$.

Proof. Induction over l shows that the maps $c^l_\beta(x^1, \dots, x^l)$ are symmetric if and only if

$$c^l_\beta(x^1, \dots, x^{l-2}, y, z) = c^l_\beta(x^1, \dots, x^{l-2}, z, y) \tag{2.10}$$

holds for all $y, z \in \mathbf{R}^s$. Indeed, if (2.10) is valid for all $l \ge 2$, and for some $r > 2$ the functions $c^{r-1}_\beta(x^1, \dots, x^{r-1})$, $\beta \in \Gamma_{k-r+1}$, are symmetric, then $c^r_\beta(x^1, \dots, x^r)$ is symmetric by (2.8) and (2.10).

A repeated application of (2.8) yields the equation

$$\begin{aligned} &c^l_\beta(x^1, \dots, x^{l-2}, y, z) \\ &\quad = \sum_{i=0}^{s} \sum_{j=0}^{s} \lambda_{l,\beta,i}(z) \lambda_{l-1,\beta+e^i,j}(y) c^{l-2}_{\beta+e^i+e^j}(x^1, \dots, x^{l-2}). \end{aligned} \tag{2.11}$$

This identity is valid for all $x^1, \dots, x^{l-2} \in \mathbf{R}^s$ and all control points c_β, $\beta \in \Gamma_k$. Referring back to (2.8), we may use induction on l to verify that

$$\begin{aligned} &c^l_\beta(x^{1,\beta^{(l)},j_1}, \dots, x^{l,\beta^{(1)},j_l}) \\ &\quad = c_{\beta+e^{j_1}+\cdots+e^{j_l}}, \qquad \beta \in \Gamma_{k-l},\ 0 \le j_i \le s,\ i = 1, \dots, l, \end{aligned} \tag{2.12}$$

where $\beta^{(1)} := \beta$, and for $2 \le m \le l$, $\beta^{(m)} := \beta + e^{j_l} + \cdots + e^{j_{l-m+2}}$. Thus, if we fix a choice of β and $j_1, \ldots, j_{l-2}$, it is clear that the symmetric matrix $(c_{\beta+e^i+e^j+e^{j_1}+\cdots+e^{j_{l-2}}})_{i,j=1}^s$ is completely arbitrary. Thus we can assert that the symmetry relations (2.10) are equivalent to the system of equations

$$\begin{aligned}\lambda_{l,\beta,i}(z)\lambda_{l-1,\beta+e^i,j}(y) + \lambda_{l,\beta,j}(z)\lambda_{l-1,\beta+e^j,i}(y) \\ = \lambda_{l,\beta,i}(y)\lambda_{l-1,\beta+e^i,j}(z) + \lambda_{l,\beta,j}(y)\lambda_{l-1,\beta+e^j,i}(z),\end{aligned} \tag{2.13}$$

valid for all $l = 1, \ldots, k$, $i, j = 0, \ldots, s$, $\beta \in \Gamma_{k-l}$, and $y, z \in \mathbf{R}^s$.

Let us assume now that the algorithm is in fact symmetric, i.e., that the system (2.13) holds. We first show that this implies

$$\lambda_{l,\beta,i}(x^{l-1,\beta+e^j,h}) = 0 \quad \text{for } h \notin \{i, j\}. \tag{2.14}$$

We will distinguish two cases.

Case 1: $i = j$. For $i = j$, equation (2.13) simplifies to

$$\lambda_{l,\beta,i}(z)\lambda_{l-1,\beta+e^i,i}(y) = \lambda_{l,\beta,i}(y)\lambda_{l-1,\beta+e^i,i}(z),$$

and setting $y = x^{l-1,\beta+e^i,i}$ and $z = x^{l-1,\beta+e^i,h}$, we obtain

$$\lambda_{l-1,\beta+e^i,i}(y) = 1 \quad \text{and} \quad \lambda_{l-1,\beta+e^i,i}(z) = 0,$$

and hence

$$\lambda_{l,\beta,i}(x^{l-1,\beta+e^i,h}) = 0. \tag{2.15}$$

Case 2: $i \neq j$. Setting $y = x^{l-1,\beta+e^i,j}$ and $z = x^{l-1,\beta+e^j,h}$ in (2.13), we get

$$\lambda_{l-1,\beta+e^i,j}(y) = 1 \quad \text{and} \quad \lambda_{l-1,\beta+e^j,i}(z) = 0,$$

and by Case 1 we also have

$$\lambda_{l,\beta,j}(z) = \lambda_{l,\beta,i}(y) = 0.$$

Hence, equation (2.13) yields

$$\lambda_{l,\beta,i}(x^{l-1,\beta+e^j,h}) = 0,$$

which proves (2.14).

Now, fix $h \neq j$. From (2.14) and the fact that the barycentric coordinates for a given l, β sum to one, we infer $\lambda_{l,\beta,i}(x^{l-1,\beta+e^j,h}) = \delta_{ih}$. This, in turn, implies that

$$x^{l-1,\beta+e^j,h} = x^{l,\beta,h} \quad \text{for all } h \neq j.$$

To finish the proof, we observe that a repeated application of the above identity gives

$$x^{k-\beta_h,\beta_h e^h,h} = \cdots = x^{k-\beta_h-\beta_{j_1}-\cdots-\beta_{j_m},\beta_h e^h+\beta_{j_1}e^{j_1}+\cdots+\beta_{j_m}e^{j_m},h}$$

whenever $h \notin \{j_1, \ldots, j_m\}$. Consequently,

$$x^{k-\beta_h,\beta_h e^h,h} = x^{k-|\beta|,\beta,h} = x^{l,\beta,h}.$$

Setting $y^{j,l} := x^{k-l,le^j,j}$ proves the assertion.

Conversely, let us now assume that the point $x^{l,\beta,j}$ only depends on j and β_j, i.e.,

$$x^{l,\beta,j} = y^{j,\beta_j}.$$

Then each pair $(y, z) := (y^{i,\beta_i}, y^{j,\beta_j})$ satisfies (2.13). To see this, we set

$$G_{ijrh} := \lambda_{l,\beta,i}(y^{r,\beta_r})\lambda_{l-1,\beta+e^i,j}(y^{h,\beta_h})$$

and observe that $G_{ijrh} = \delta_{ir}\delta_{jh}$ if $r \neq h$. Hence (2.13) is satisfied for $r \neq h$ and also trivially for $r = h$. Since the points y^{i,β_i}, $i = 0, \dots, s$, are affinely independent, this carries over to arbitrary $y, z \in \mathbf{R}^s$, and thus the proof of Proposition 2.1 is complete. □

Proposition 2.1 has some important consequences. First of all, it shows that a symmetric simplicial algorithm is defined only by a collection $X = \{x^{i,j} : 0 \leq i \leq s,\ 0 \leq j \leq k\} \subset \mathbf{R}^s$ of points such that for any $\beta \in \Gamma_l$, $0 \leq l \leq k$, the subsets

$$X_\beta = \{x^{j,\beta_j} : j = 0, \dots, s\} \tag{2.16}$$

are affinely independent. For a symmetric simplicial algorithm the recursion (2.5), (2.6) then simplifies to

$$C^0_\beta(x) = c_\beta, \qquad \beta \in \Gamma_k, \tag{2.17}$$

and

$$C^l_\beta(x) = \sum_{j=0}^{s} \lambda_{\beta,j}(x) C^{l-1}_{\beta+e^j}(x), \qquad \beta \in \Gamma_{k-l}, \tag{2.18}$$

where

$$\lambda_{\beta,j}(x) = d_j(X_\beta|x)/d(X_\beta) \tag{2.19}$$

are the barycentric coordinates of x with respect to X_β.

The dual algorithm to (2.17), (2.18) is given by

$$B_{(0,\dots,0)}(x) = 1, \qquad x \in \mathbf{R}^s, \tag{2.20}$$

and

$$B_\beta(x) = \sum_{j=0}^{s} \lambda_{\beta-e^j,j}(x) B_{\beta-e^j}(x), \qquad |\beta| > 0, \tag{2.21}$$

so that for any $0 \leq l \leq k$,

$$C^k_{(0,\dots,0)}(x) = \sum_{\beta\in\Gamma_{k-l}} B_\beta(x) C^l_\beta(x). \tag{2.22}$$

The real-valued functions $B_\beta(x)$ are called *normalized B-weights*. Here we use the convention that the functions $\lambda_\beta(x)$ and $B_\beta(x)$ are set to zero whenever β has a negative component. As a consequence, the nonzero summands appearing on the right-hand side of (2.21) have a corresponding multi-index $\beta - e^j$ whose components are strictly less than k. Thus the normalized B-weights $B_\beta(x)$ are for all $\beta \in \Gamma_k$ actually independent of the points $x^{i,k}$, $i = 0, \dots, s$. These points $x^{i,k}$ will only become relevant later when we relate B-patches to B-splines.

Following the development in [8] for the bivariate case, we conclude that Proposition 2.1 implies that the normalized B-weights $B_\beta(x)$, $\beta \in \Gamma_k$, are linearly independent. Specifically, if $C^k_{(0,\dots,0)}(x) = 0$, $x \in \mathbf{R}^s$, then its polar

form is also zero, $c^k_{(0,\dots,0)}(x^1,\dots,x^k)=0$, $x^1,\dots,x^k\in\mathbf{R}^s$. Therefore, from (2.8) we get by induction on $k-l$ that $c^l_\beta(x^1,\dots,x^l)=0$, $x^1,\dots,x^l\in\mathbf{R}^s$, $\beta\in\Gamma_{k-l}$, and, in particular, $c_\beta=0$ for all $\beta\in\Gamma_k$. Since the cardinality of Γ_k agrees with the dimension $\binom{k+s}{s}$ of the space $\Pi_k(\mathbf{R}^s)$ of all polynomials of degree at most k on $\mathbf{R}^s$, every polynomial $P\in\Pi_k(\mathbf{R}^s)$ has a unique representation

$$P(x)=\sum_{\alpha\in\Gamma_k}c_\alpha B_\alpha(x). \tag{2.23}$$

Moreover, the control points c_α, $\alpha\in\Gamma_k$, in this representation are given by

$$c_\alpha=p(x^{0,0},\dots,x^{0,\alpha_0-1},\dots,x^{s,0},\dots,x^{s,\alpha_s-1}), \tag{2.24}$$

where p denotes the polar form of P.

Next, let us recall that for any set of points $v^0,\dots,v^n\in\mathbf{R}^s$ the B-spline $M(x|v^0,\dots,v^n)$ is defined by requiring that

$$\begin{aligned}\int_{\mathbf{R}^s} f(x)M(x|v^0,\dots,v^n)\,dx\\ =(n-s)!\int_{\Delta_n} f(\tau_0v^0+\cdots+\tau_nv^n)\,d\tau_1\cdots d\tau_n\end{aligned} \tag{2.25}$$

holds for every $f\in C(\mathbf{R}^s)$, where

$$\Delta_n=\left\{(\tau_0,\dots,\tau_n):\sum_{i=0}^n\tau_i=1,\ \tau_i\ge 0,\ i=0,\dots,n\right\}$$

denotes the standard n-simplex. $M(\cdot|v^0,\dots,v^n)$ is known to be a piecewise polynomial of degree $n-s$, supported on the convex hull $[v^0,\dots,v^n]$ of its knots v^i whenever this convex hull has nonvanishing s-dimensional volume. For more detailed information on the properties of the multivariate B-spline, the reader is referred to [2, 3, 4, 5]. Here we note that for the above normalization one has

$$M(x|v^0,\dots,v^s)=\chi_{[v^0,\dots,v^s]}(x)\Big/\left|\det\begin{pmatrix}1&\cdots&1\\ v^0&\cdots&v^s\end{pmatrix}\right|, \tag{2.26}$$

and for $V=\{v^0,\dots,v^n\}$, $n>s$,

$$M(x|V)=\sum_{j=0}^s\frac{d_j(W|x)}{d(W)}M(x|V\backslash\{v^{i_j}\}), \tag{2.27}$$

where $W=\{v^{i_0},\dots,v^{i_s}\}$ is any subset of affinely independent points in V [5].

Now let us assume throughout the following that the set

$$\Omega_k:=\operatorname{int}\left(\bigcap_{|\beta|\le k}[X_\beta]\right) \tag{2.28}$$

satisfies

$$\operatorname{vol}_s(\Omega_k)>0. \tag{2.29}$$

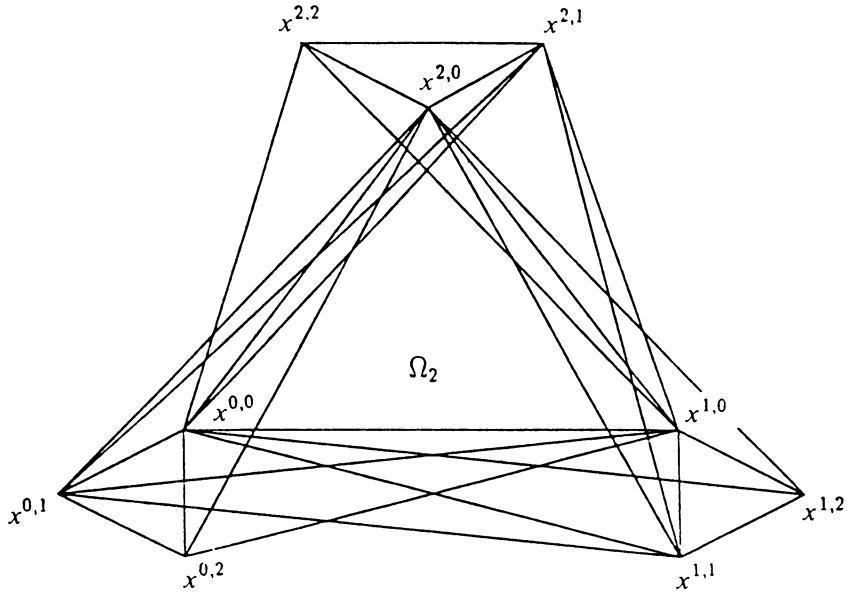

FIGURE 2.2. The set Ω_2

In order to show that each B-weight agrees with some B-spline on the region Ω_k, we will make use of the following simple facts.

Lemma 2.1. *If* (2.29) *holds, then there exist* $\sigma \in \{-1, 1\}$ *such that*

$$\sigma_\beta := \operatorname{sgn} d(X_\beta) = \sigma \quad \textit{for all } \beta \in \Gamma_l, \ 0 \le l \le k. \tag{2.30}$$

Proof. Suppose for some $\beta \in \Gamma_k$ and some i, $0 \le i \le s$, we have $\beta_i > 0$. By (2.29), $[X_\beta] \cap [X_{\beta - e^i}]$ contains some neighborhood in $\mathbf{R}^s$. Hence, the vertices x^{i,β_i} and $x^{i,\beta_i - 1}$ are both located on the same side relative to the hyperplane spanned by the vertices x^{j,β_j}, $j = 0, 1, \dots, i-1, i+1, \dots, s$. Therefore, $\sigma_\beta = \sigma_{\beta - e^i}$. Applying this fact repeatedly to each positive coordinate of β implies $\sigma_\beta = \sigma_0$, which finishes the proof of Lemma 2.1. □

Defining for $\beta \in \Gamma_l$, $l \le k$, the sets $V_\beta := \{x^{i,j} : j = 0, \dots, \beta_i, \ i = 0, \dots, s\}$, we have $V_{\beta - e^j} = V_\beta \backslash \{x^{j,\beta_j}\}$ whenever $\beta_j > 0$. Thus, it will be consistent to use the convention that $V_{\beta - e^j}$ does not contain any knot from the group $x^{j,l}$, $l = 0, \dots, k$, when $\beta_j = 0$.

Lemma 2.2. *Suppose that for some* $\beta \in \Gamma_k$ *we have* $\beta_j = 0$. *If* (2.29) *holds, then* $\Omega_k \cap [V_{\beta - e^j}] = \varnothing$.

Proof. Define

$$W_i := [\{x^{j,l} : l = 0, \dots, k, \ j = 0, \dots, s, \ j \ne i\}]$$

as well as

$$C_i := [\{x^{i,m} : m = 0, \dots, k\}], \qquad i = 0, \dots, s.$$

We will show first that

$$C_i \cap W_i = \varnothing, \qquad i = 0, \dots, s. \tag{2.31}$$

To see this, observe first that $y \in W_i$ implies that there exists a hyperplane H containing y such that

$$H \cap C_j \neq \varnothing \quad \text{for } j \neq i. \tag{2.32}$$

Without loss of generality, let $i = 0$ and suppose that

$$y := \sum_{n=0}^{k} \eta_n x^{0,n} \in W_0 \cap C_0 \tag{2.33}$$

for some $\eta_j \geq 0$, $\sum_{j=0}^{k} \eta_j = 1$. This implies the existence of nonnegative reals $\mu_{j,n}$ such that

$$y = \sum_{j=1}^{s} \sum_{n=0}^{k} \mu_{j,n} x^{j,n}, \qquad \sum_{j=1}^{s} \sum_{n=0}^{k} \mu_{j,n} = 1.$$

Thus, defining

$$\lambda_j := \sum_{n=0}^{k} \mu_{j,n} \tag{2.34}$$

and

$$y^j := \begin{cases} (\sum_{n=0}^{k} \mu_{j,n} x^{j,n})/\lambda_j & \text{if } \lambda_j \neq 0, \\ x^{j,n_j} & \text{if } \lambda_j = 0, \end{cases}$$

for some $n_j \leq k$, we clearly have $y^j \in C_j$ and

$$y, y^j \in H := \left\{ x = \sum_{j=1}^{s} \nu_j y^j : \sum_{j=1}^{s} \nu_j = 1 \right\}, \qquad j = 1, \dots, s,$$

confirming (2.32). Next, we will show that one can find a hyperplane H containing y and satisfying (2.32) so that, in particular,

$$y^j = x^{j,n_j}, \qquad j = 1, \dots, s, \ j \neq l, \tag{2.35}$$

for some $l \in \{1, \dots, s\}$ and some $n_j \leq k$. Let $K := \{j : 1 \leq j \leq s, y^j = x^{j,n_j}\}$. If $\#K = s-1$, we are finished. So assume that H does not satisfy (2.35) yet. In fact, K could be empty initially. So let $p, q \notin K$ and pick some point $x^{p,r}$ in the pth group of knots which minimizes the distance from H. Let $z \in H \cap C_p$ be the closest point to $x^{p,r}$, and denote for $t \in [0, 1]$ the hyperplane passing through y, y^l, $l = 1, \dots, s$, $l \neq p, q$, and through $y^p(t) := tz + (1-t)x^{p,r}$ by $H(t)$. Clearly, by construction, $H(1)$ still intersects all the sets C_j, $j = 1, \dots, s$, $j \neq q$. If $H(1)$ also still intersects C_q, we replace H by $H(1)$, where the cardinality of the corresponding new set K has increased by at least one. If $H(1)$ does not intersect C_q anymore, there must exist some $t_0 \in (0, 1)$ such that $H(t_0)$ is a supporting hyperplane of C_q, and hence must contain some extreme point x^{q,n_q} of C_q. Thus, replacing H by $H(t_0)$ in this case, we have again increased K. We may repeat this process until $\#K = s-1$, which is (2.35). So we may assume that H satisfies (2.35). We consider now two cases: Suppose first that the point $y \in C_0$ already agrees with some point $x^{0,m}$. Note that, by (2.16), H does then not contain any of the points $x^{l,m}$. Thus, H must intersect the interior of C_l, which means

that there exist two points x^{l,m_1}, x^{l,m_2} located on different sides relative to H. But by (2.16), the two simplices $[\{x^{0,m}, x^{l,m_i}, x^{j,n_j} : j = 1, \dots, s, j \neq l\}]$, $i = 1, 2$, must both contain Ω_k. However, since by construction, their respective interiors are separated by H, we arrive at a contradiction. If y does not agree with any knot in the set $\{x^{0,0}, x^{0,1}, \dots, x^{0,k}\}$, we argue similarly as above and pick a point $x^{0,m}$ which minimizes the distance to H. Let again $z \in H \cap C_0$ be closest to $x^{0,m}$, and define for $t \in [0, 1]$ the hyperplanes spanned by the points $y(t) := tz + (1-t)x^{0,m}$, x^{j,n_j}, $j = 1, \dots, s$, $j \neq l$. If $H(1)$ still intersects C_l, we replace H by $H(1)$, obtaining a contradiction by the first case. Otherwise, there must be a $t_0 \in (0, 1)$ such that $H(t_0)$ contains some extreme point x^{l,n_l} from C_l. Interchanging the roles of 0 and l, we are again in the situation described by the first case and obtain a contradiction. This proves (2.31).

By (2.31), for any $j \in \{0, \dots, s\}$, there exist exactly two common supporting hyperplanes H_1, H_2 of the sets C_i, $i \neq j$, which 'sandwich' the set W_j. Suppose they are spanned by the sets $Y_1 := \{x^{i,n_i} : i \neq j\}$, $Y_2 := \{x^{i,m_i} : i \neq j\}$, respectively. Fixing $x^{j,m}$, we claim that one of the hyperplanes must separate $x^{j,m}$ from W_j. If this were not the case, the two simplices $\delta_1 := [\{x^{j,m}\} \cup Y_1]$ and $\delta_2 := [\{x^{j,m}\} \cup Y_2]$ are by (2.16) nondegenerate and intersect only in $x^{j,m}$, contradicting (2.29). So we may assume that H_1 separates $x_{j,m}$ from W_j. Since $[V_{\beta-e^j}] \subseteq W_j$ whenever $\beta_j = 0$, and since δ_1 contains Ω_k, the proof of Lemma 2.2 is complete. □

We are now ready to state the main result of this section.

Theorem 2.1. *Suppose* (2.29) *holds. Then*

$$B_\beta(x) = \sigma_\beta\, d(X_\beta) M(x|V_\beta) \quad \textit{for all } x \in \Omega_k, \ \beta \in \Gamma_k,$$

where $V_\beta = \{x^{i,j} : j = 0, \dots, \beta_i, \ i = 0, \dots, s\}$.

Proof. We will proceed by induction on k. Since $\Omega_0 = [X_0]$, the assertion for $k = 0$ readily follows from (2.20) and (2.26).

Suppose it holds for $k-1$. By Lemma 2.2 we know that the interiors of the sets $[V_{\beta-e^j}]$ and Ω_k are disjoint whenever $\beta_j = 0$. Consequently, $M(x|V_{\beta-e^j})$ vanishes for $x \in \Omega_k$, and, by convention, so does $B_{\beta-e^j}(x)$ (in fact for all $x \in \mathbf{R}^s$). Thus, we may rewrite the right-hand side of (2.21) for any $\beta \in \Gamma_k$ and $x \in \Omega_k$ as

$$B_\beta(x) = \sum_{\substack{j=0 \\ \beta_j > 0}}^{s} \frac{d_j(X_{\beta-e^j}|x)}{d(X_{\beta-e^j})} \sigma_{\beta-e^j} d(X_{\beta-e^j}) M(x|V_{\beta-e^j}). \tag{2.36}$$

But since $d_j(X_{\beta-e^j}|x) = d_j(X_\beta|x)$ whenever $\beta_j > 0$, the right-hand side of (2.36) reduces, in view of (2.30), for $x \in \Omega_k$ to

$$\sigma_\beta d(X_\beta) \sum_{j=0}^{s} \lambda_{\beta,j}(x) M(x|V_\beta \backslash \{x^{j,\beta_j}\}) = \sigma_\beta d(X_\beta) M(x|V_\beta),$$

where we have used (2.27) in the last step. This completes the proof of Theorem 2.1. □

Setting $c_\beta = 1$ in (2.23) readily gives, in view of (2.18) and (2.22),

$$1 = \sum_{\beta \in \Gamma_k} B_\beta(x)$$

and hence, by Theorem 2.1,

$$\sum_{\beta \in \Gamma_k} \sigma_\beta d(X_\beta) M(x|V_\beta) = 1, \qquad x \in \Omega_k .$$

This suggests introducing the following normalized B-splines:

$$N_\beta(x) := \sigma_\beta d(X_\beta) M(x|V_\beta), \qquad \beta \in \Gamma_k, \tag{2.37}$$

which therefore form a partition of unity on Ω_k. Also, the linear independence of the polynomials B_β, $\beta \in \Gamma_k$, readily assures the following fact.

Corollary 2.1. *The B-splines N_β, $\beta \in \Gamma_k$, are locally linearly independent on every subdomain of Ω_k.*

3. A POLYNOMIAL IDENTITY FOR B-SPLINES

The previous discussion suggests the following scheme for constructing linear combinations of B-splines on all of $\mathbf{R}^s$ (or on some bounded domain in $\mathbf{R}^s$). Given points $X = \{x^{i,j},\ i \in \mathbf{Z},\ j = 0, \dots, k\}$, let $T = \{\Delta(I) = [x^{i_0,0}, \dots, x^{i_s,0}] : I = (i_0, \dots, i_s) \in \mathscr{T} \subseteq \mathbf{Z}_+^{s+1}\}$ define a triangulation of $\mathbf{R}^s$. This means that $\mathbf{R}^s = \bigcup_{I \in \mathscr{T}} \Delta(I)$, while for any two $I, J \in \mathscr{T}$, $\Delta(I) \cap \Delta(J)$ is empty or is a common face of $\Delta(I)$ and $\Delta(J)$. In particular, any $(s-1)$-dimensional facet of some $\Delta(I) \in T$ is the common face of exactly two simplices $\Delta(I)$ and $\Delta(J)$, say. We will adhere to the notation of the previous section, indicating the reference to a particular simplex $\Delta(I)$ throughout by an additional sub- or superscript I. For instance, $V_\beta^I := \{x^{i_j,l} : l = 0, 1, \dots, \beta_j,\ j = 0, 1, \dots, s\}$ is the set that determines the (normalized) B-splines $N_\beta^I(x)$, $\beta \in \Gamma_k$, associated with the simplex $\Delta(I)$, $I \in \mathscr{T}$.

Note that the polar form of the polynomial $(a + \omega \cdot x)^k$ with respect to the set $V_\beta^I = \{x^{i_j,l} : l = 0, \dots, \beta_j,\ j = 0, \dots, s\}$ is given by

$$\Psi_{I,\beta}(a, \omega) = \prod_{j=0}^{s} \prod_{l=0}^{\beta_j - 1} (a + \omega \cdot x^{i_j,l}), \tag{3.1}$$

where, in agreement with our previous convention on the set $V_{\beta - e^j}$ when $\beta_j = 0$, it is to be understood that the factor $\prod_{l=0}^{\beta_j - 1}(a + \omega \cdot x^{i_j,l})$ is interpreted as one. Hence, we readily conclude from Theorem 2.1 and the properties of B-patches derived earlier that the identity

$$(a + \omega \cdot x)^k = \sum_{\beta \in \Gamma_k} \Psi_{I,\beta}(a, \omega) N_\beta^I(x) \tag{3.2}$$

holds for all $x \in \Omega_{I,k}$, where we will assume throughout the following that

$$\mathrm{vol}_s(\Omega_{I,k}) > 0, \qquad I \in \mathscr{T}. \tag{3.3}$$

It is also clear that every element

$$S(x) = \sum_{I \in \mathscr{T}} \sum_{\beta \in \Gamma_k} c_{I\beta} N_\beta^I(x)$$

of the space

$$\mathscr{S}_k(X) := \operatorname{span}\{N_\beta^I : \beta \in \Gamma_k\,,\ \ I \in \mathscr{T}\}$$

on $\Omega_{I,k}$ reduces to $\sum_{\beta\in\Gamma_k} c_{I\beta} N_\beta^I(x)$. This suggests the following claim.

Proposition 3.1. *Suppose* (3.3) *holds. Then for any* $a \in \mathbf{R}$ *and* $x, \omega \in \mathbf{R}^s$ *we have*

$$(a + \omega\cdot x)^k = \sum_{I\in\mathscr{T}} \sum_{\beta\in\Gamma_k} \Psi_{I,\beta}(a, \omega) N_\beta^I(x)\,. \tag{3.4}$$

Under the assumption (3.3) it is also clear, in view of Theorem 2.1, that any element of $\mathscr{S}_k(X)$ can be evaluated efficiently by means of the recursive scheme (2.17) and (2.18) on any of the regions $\Omega_{I,k}$. The proof of Proposition 3.1 is based on studying to what extent this recursive evaluation persists for x outside the regions $\Omega_{I,k}$. The essential observation may be formulated as follows.

Lemma 3.1. *Suppose the sequence* $\{c_{I\beta}\}_{I\in\mathscr{T},\beta\in\Gamma_k}$ *has the following property. For any two adjacent simplices* $\Delta(I)$ *and* $\Delta(J) \in T$ *whose common face is*

$$\begin{aligned}&[x^{i_0,0}, \dots, x^{i_{m-1},0}, x^{i_{m+1},0}, \dots, x^{i_s,0}]\\ &\quad = [x^{j_0,0}, \dots, x^{j_{q-2},0}, x^{j_q,0}, \dots, x^{j_s,0}],\end{aligned} \tag{3.5}$$

and for any $\beta, \gamma \in \Gamma_k$ *such that*

$$\begin{aligned}&\gamma_j = \beta_j\,, \qquad j = 0, \dots, m-1\,,\ \ j = q, \dots, s\,,\\ &\gamma_{q-1} = \beta_m = 0\,,\\ &\gamma_j = \beta_{j+1}\,, \qquad j = m, \dots, q-2\,,\end{aligned} \tag{3.6}$$

one has

$$c_{I,\beta} = c_{J,\gamma}\,. \tag{3.7}$$

Then

$$\sum_{I\in\mathscr{T}} \sum_{\beta\in\Gamma_k} c_{I,\beta} N_\beta^I(x) = \sum_{I\in\mathscr{T}} \sum_{\beta\in\Gamma_{k-1}} c_{I,\beta}^{(1)}(x) N_\beta^I(x)\,, \tag{3.8}$$

where

$$c_{I,\beta}^{(1)}(x) := \sum_{j=0}^{s} \lambda_\beta^I(x) c_{I,\beta+e^j}\,. \tag{3.9}$$

Proof. By (2.37) and (2.27) we obtain

$$\begin{aligned}\sum_{\beta\in\Gamma_k} c_{I,\beta} N_\beta^I(x) &= \sum_{j=0}^{s} \sum_{\beta\in\Gamma_k} \sigma_{I,\beta} d(X_\beta^I) c_{I,\beta} \lambda_{\beta,j}^I(x) M(x|V_{\beta-e^j}^I)\\ &= \sum_{j=0}^{s} \Bigg(\sum_{\substack{\beta\in\Gamma_k\\ \beta_j>0}} + \sum_{\substack{\beta\in\Gamma_k\\ \beta_j=0}}\Bigg) c_{I,\beta}\sigma_{I,\beta} d(X_\beta^I) \lambda_{\beta,j}^I(x) M(x|V_{\beta-e^j}^I)\\ &= \sum_{\beta\in\Gamma_{k-1}} \Bigg(\sum_{j=0}^{s} \sigma_{I,\beta+e^j} d(X_{\beta+e^j}^I) c_{I,\beta+e^j} \lambda_{\beta+e^j,j}^I(x)\Bigg) N(x|V_\beta^I)\\ &\quad + \sum_{j=0}^{s} R_{I,j}(x)\,.\end{aligned}$$

Here, $R_{I,j}(x)$ is defined to be

$$R_{I,j}(x) := \sum_{\substack{\beta\in\Gamma_k \\ \beta_j=0}} c_{I,\beta}\sigma_{I,\beta} d(X^I_\beta)\lambda^I_{\beta,j}(x) M(x|V^I_{\beta-e^j}).$$

Noting that by (3.3) and Lemma 2.1, $\sigma_{\beta+e^j} = \sigma_\beta$, $\beta \in \Gamma_{k-1}$, we obtain

$$\begin{aligned}
&\sum_{j=0}^{s} \sigma_{I,\beta+e^j} d(X^I_{\beta+e^j}) c_{I,\beta+e^j} \lambda^I_{\beta+e^j,j}(x) \\
&\quad = \sigma_{I,\beta} \sum_{j=0}^{s} c_{I,\beta+e^j} d(X^I_{\beta+e^j}) \frac{d_j(X^I_{\beta+e^j}|x)}{d(X^I_{\beta+e^j})} \\
&\quad = \sigma_{I,\beta} \sum_{j=0}^{s} c_{I,\beta+e^j}\, d_j(X^I_{\beta+e^j}|x) = \sigma_{I,\beta} \sum_{j=0}^{s} c_{I,\beta+e^j}\, d_j(X^I_\beta|x) \\
&\quad = \sigma_{I,\beta} d(X^I_\beta) \sum_{j=0}^{s} c_{I,\beta+e^j} \lambda_{\beta,j}(x),
\end{aligned}$$

which, in view of (2.37), proves that

$$\sum_{\beta\in\Gamma_k} c_{I,\beta} N^I_\beta(x) = \sum_{\beta\in\Gamma_{k-1}} c^{(1)}_{I,\beta}(x) N^I_\beta(x) + \sum_{j=0}^{s} R_{I,j}(x). \tag{3.10}$$

Hence, to finish the proof of Lemma 3.1, we have to show that, under the assumption (3.7), (3.6),

$$\sum_{I\in\mathcal{T}} \left(\sum_{j=0}^{s} R_{I,j}(x) \right) = 0. \tag{3.11}$$

To this end, note that for $0 \le m \le s$ the term $R_{I,m}(x)$ consists of summands of the form

$$c_{I,\beta}\sigma_{I,\beta} d_m(X^I_\beta|x) M(x|V^I_{\beta-e^m})$$

for some $\beta \in \Gamma_k$, where $\beta_m = 0$. Hence, there exists exactly one adjacent simplex $\Delta(J)$ in T such that the common face of $\Delta(I)$ and $\Delta(J)$ does neither contain $x^{i_m,0}$ nor $x^{j_{q-1},0} \in \Delta(J)$. Thus, for this $\beta \in \Gamma_k$, with $\beta_m = 0$, we choose $\gamma \in \Gamma_k$ as defined by (3.6). Then the sum $R_{J,q-1}(x)$ contains the summand

$$c_{J,\gamma}\sigma_{J,\gamma} d_{q-1}(X^J_\gamma|x) M(x|V^J_{\gamma-e^{q-1}}).$$

Since $V^J_{\gamma-e^{q-1}} = V^I_{\beta-e^m}$, we have $M(x|V^J_{\gamma-e^{q-1}}) = M(x|V^I_{\beta-e^m})$, and also the determinants $d_{q-1}(X^J_\gamma|x)$ and $d_m(X^I_\beta|x)$ have the same absolute value, since the sets $X^J_\gamma\backslash\{x^{j_{q-1},0}\}$ and $X^I_\beta\backslash\{x^{i_m,0}\}$ are identical. These functions are zero only on the hyperplane H spanned by the set $X^I_\beta\backslash\{x^{i_m,0}\}$. Moreover, we observe next that

$$\sigma_{J,\gamma} d_{q-1}(X^J_\gamma|x) = -\sigma_{I,\beta} d_m(X^I_\beta|x). \tag{3.12}$$

In fact, it follows from Lemma 2.2 that $x^{i_m,0}$ and $x^{j_{q-1},0}$ are separated by the hyperplane H defined above. Hence,

$$\lambda^J_{\gamma,q-1}(x^{i_m,0}) = \frac{\sigma_{J,\gamma} d_{q-1}(X^J_\gamma | x^{i_m,0})}{\sigma_{J,\gamma} d(X^J)} < 0. \tag{3.13}$$

On the other hand, since $\beta_m = 0$, one has

$$1 = \lambda^I_{\beta,m} = \frac{\sigma_{I,\beta} d_m(X^I_\beta | x^{i_m,0})}{\sigma_{I,\beta} d(X^I)}. \tag{3.14}$$

Since the denominators in (3.13) and (3.14) are both positive, (3.12) must hold. Thus, (3.11) follows by (3.7), $c_{J,\gamma} = c_{I,\beta}$, thereby completing the proof of Lemma 3.1. □

We are now in a position to complete the proof of Proposition 3.1. By virtue of (2.26), (2.37), and Theorem 2.1, the assertion holds trivially for $k = 0$. Suppose it has been verified for $k-1$. Note that for any $I, J \in \mathscr{T}$ such that $\Delta(I)$ and $\Delta(J)$ share an $(s-1)$-face, and for any $\beta, \gamma \in \Gamma_k$, related by (3.6), the coefficients $c_{I,\beta} := \Psi_{I,\beta}(a, \omega)$ and $c_{J,\gamma} := \Psi_{J,\gamma}(a, \omega)$, in view of (3.1) and by our above convention, are both composed of the same factors involving only knots from the groups corresponding to $I \cap J$. Hence, $\Psi_{J,\gamma}(a, \omega)$ and $\Psi_{I,\beta}(a, \omega)$ coincide under the assumption (3.6) and thus satisfy (3.7). Moreover, for any $\beta \in \Gamma_{k-1}$,

$$\sum_{j=0}^{s} \lambda^I_\beta(x) \Psi_{I,\beta+e^j}(a, \omega) = (a + \omega \cdot x) \Psi_{I,\beta}(a, \omega),$$

so that (3.4) is an immediate consequence of Lemma 3.1.

Since any polynomial of degree at most k can be written as a linear combination of polynomials of the form $(a+\omega\cdot x)^k$ for appropriate choices of $a \in \mathbf{R}$ and $\omega \in \mathbf{R}^s$, Proposition 3.1 leads us to the main result of this section:

Theorem 3.1. *Let P be any polynomial of degree k, and let p denote its polar form. Then the following identity holds for all $x \in \mathbf{R}^s$:*

$$P(x) = \sum_{I \in J} \sum_{\beta \in \Gamma_k} p(x^{i_0,0}, \dots, x^{i_0,\beta_0-1}, \dots, x^{i_s,0}, \dots, x^{i_s,\beta_s-1}) N^I_\beta(x). \tag{3.15}$$

4. Approximation properties

In this section we derive further consequences of Proposition 3.1. Specifically, we will provide approximation properties of the spaces $\mathscr{S}_k(X)$ and also prove the stability of the B-spline bases constructed above.

Setting $\omega = 0$, $a = 1$ in (3.4) gives

$$1 = \sum_{I \in \mathscr{T}} \sum_{\beta \in \Gamma_k} N^I_\beta(x), \qquad x \in \mathbf{R}^s,$$

i.e., the B-splines form a partition of unity. More generally, differentiating both sides of (3.4) with respect to ω, evaluating at $\omega = 0$, and putting $a = 1$ yields for every $\alpha \in \mathbf{Z}^s_+$, $|\alpha| \le k$,

$$k \cdots (k - |\alpha| + 1) x^\alpha = \sum_{I \in \mathscr{T}} \sum_{\beta \in \Gamma_k} (D^\alpha \Psi_{I,\beta}(1, \omega)|_{\omega=0}) N^I_\beta(x), \tag{4.1}$$

which also confirms that

$$\Pi_k(\mathbf{R}^s) \subseteq \mathscr{S}_k(X). \tag{4.2}$$

Next, we will discuss a multivariate version of the Schoenberg operator. For any affine function $L \in \Pi_1(\mathbf{R}^s)$, its polar form (as a polynomial of degree k) is $l(x^1, \dots, x^k) := L(\frac{1}{k}(x^1 + \cdots + x^k))$, and so, specializing Theorem 3.1 to affine functions, yields

$$L(x) = \sum_{I \in J} \sum_{\beta \in \Gamma_k} L(\xi_{I,\beta}) N_\beta^I(x), \qquad x \in \mathbf{R}^s, \tag{4.3}$$

where

$$\xi_{I,\beta} = \frac{1}{k} \sum_{j=0}^{s} \left(\sum_{m=0}^{\beta_{j-1}} x^{i_j, m} \right), \qquad \beta \in \Gamma_k. \tag{4.4}$$

Consequently, a standard argument implies that the operator

$$(Sf)(x) := \sum_{I \in \mathscr{T}} \sum_{\beta \in \Gamma_k} f(\xi_{I,\beta}) N_\beta^I(x)$$

realizes for any compact set Ω estimates of type

$$\|Sf - f\|_\infty(\Omega) \le Ch^2 \max_{|\alpha|=2} \|D^\alpha f\|_\infty(\Omega_T),$$

where $h := \max_{I \in \mathscr{T}} \operatorname{diam} \Delta(I)$ and $\Omega_T := \bigcup\{[X_\beta^I] : \beta \in \Gamma_k,\ \Delta(I) \cap \Omega \neq \varnothing\}$.

The usual way of obtaining better approximation rates is to construct operators that reproduce higher-order polynomials. We briefly sketch this procedure in our case by again exploiting the identity (3.4).

Expanding

$$\Psi_{I,\beta}(a - \omega \cdot \tau_{I,\beta}, \omega) = \sum_{|\alpha| \le k} a^{k-|\alpha|} C_\alpha^{I,\beta} \omega^\alpha,$$

where $\tau_{I,\beta}$ is some fixed point in $\Omega_{I,k}$, we define the functional $\mu_{I,\beta}$ on $C^k(\Omega_{I,k})$ by

$$\mu_{I,\beta}(f) := \sum_{|\alpha| \le k} C_\alpha^{I,\beta} \frac{(k - |\alpha|)!}{k!} D^\alpha f(\tau_{I,\beta}) \tag{4.5}$$

and observe that

$$\mu_{I,\beta}((a + \omega \cdot \bullet)^k) = \sum_{|\alpha| \le k} C_\alpha^{I,\beta} \omega^\alpha (a + \omega \cdot \tau_{I,\beta})^{k-|\alpha|} = \Psi_{I,\beta}(a, \omega).$$

Hence, the operator

$$(Qf)(x) := \sum_{I \in \mathscr{T}} \sum_{\beta \in \Gamma_k} \mu_{I,\beta}(f) N_\beta^I(x)$$

reproduces all polynomials in $\Pi_k(\mathbf{R}^s)$.

Moreover, since for $I \neq J$, by Lemma 2.2 and (2.31), $\operatorname{supp}(N_\gamma^J) \cap \Omega_{I,k} = \varnothing$ and $\mu_{I,\beta}$ is supported in $\Omega_{I,k}$, so that $\mu_{I,\beta}(N_\gamma^J) = 0$, $J \neq I$, $\beta, \gamma \in \Gamma_k$,

and since by Corollary 2.1 the B-splines N_β^I, $\beta \in \Gamma_k$, are linearly independent over $\Omega_{I,k}$, one readily concludes

$$\mu_{I,\beta}(N_\gamma^J) = \delta_{IJ}\delta_{\beta\gamma}, \qquad I, J \in \mathscr{T}, \ \beta, \gamma \in \Gamma_k, \tag{4.6}$$

i.e., the $\mu_{I,\beta}$ establish a dual basis for $\mathscr{S}_k(X)$.

The main step needed to characterize the approximation properties of the spaces $\mathscr{S}_k(X)$ and to confirm the stability of the B-spline basis $\{N_\beta^I : \beta \in \Gamma_k, I \in J\}$ is to bound the dual functionals $\mu_{I,\beta}$ (4.5). This can be done by following well-established lines. However, the normalization (2.37) of the B-splines arising in the present setting provides us with two results, one on approximation and the other on stability, which are valid under less restrictive assumptions than for the corresponding B-spline spaces of [3, 4]. For the convenience of the reader, we sketch the main steps leading to these facts.

To this end, let $D_{I,\beta}$ denote the smallest ball containing the knot set V_β^I. Define, as usual, the norm

$$\|f\|_p(\Omega) = \begin{cases} \operatorname{ess\,sup}_{x\in\Omega} |f(x)|, & p = \infty, \\ (\int_\Omega |f(x)|^p \, dx)^{1/p}, & 1 \le p < \infty, \end{cases}$$

and $L_p(\Omega)$, the corresponding space of functions f over Ω with $\|f\|_p(\Omega) < \infty$.

Lemma 4.1. *There exists a constant c depending only on s and k such that*

$$|\mu_{I,\beta}(P)| \le c \operatorname{vol}_s(D_{I,\beta})^{-1/p} \|P\|_p(D_{I,\beta}) \tag{4.7}$$

for all $P \in \Pi_k(\mathbf{R}^s)$, $\beta \in \Gamma_k$, and $I \in J$.

To prove (4.7), one observes first that

$$\begin{aligned} j! \sum_{|\alpha|=k-j} C_\alpha^{I,\beta} \omega^\alpha &= \left(\frac{d}{da}\right)^j \Psi_{I,\beta}(a - \omega\cdot\tau_{I,\beta}, \omega)|_{a=0} \\ &= \sum_r \prod_l \omega\cdot(x^{i_r,l} - \tau_{I,\beta}), \end{aligned}$$

where the number of summands depends on s, k, and j, and each summand is a product of $k-j$ factors $\omega\cdot(x^{i_r,l} - \tau_{I,\beta})$. Therefore, it follows immediately that

$$|C_\alpha^{I,\beta}| \le C h_{I,\beta}^{|\alpha|}, \tag{4.8}$$

where $h_{I,\beta} := \max\{\|\tau_{I,\beta} - x^{i_r,l}\| : r = 0, \dots, s, \ 0 \le l \le \beta_r\}$ and the constant C depends only on s and k. Using the equivalence of norms on finite-dimensional spaces, standard scaling arguments, and Markov's inequality, one obtains that

$$\|D^\alpha P\|_\infty(D_{I,\beta}) \le C\rho^{-|\alpha|-s/p} \|P\|_p(D_{I,\beta}) \tag{4.9}$$

for all $P \in \Pi_k(\mathbf{R}^s)$, where $\rho := \operatorname{diam} D_{I,\beta}$ and the constant C depends only on s and k. Since $h_{I,\beta} \le \rho$, (4.8) and (4.9) yield

$$\begin{aligned} |\mu_{I,\beta}(P)| &\le \sum_{|\alpha|=k} |C_\alpha^{I,\beta}| \frac{(k-|\alpha|)!}{k!} |D^\alpha P(\tau_{I,\beta})| \\ &\le C\rho^{-s/p} \|P\|_p(D_{I,\beta}), \end{aligned}$$

confirming (4.7).

By the Hahn-Banach Theorem, $\mu_{I,\beta}$ has a norm-preserving extension to all of $L_p(\Omega)$, which we also denote by $\mu_{I,\beta}$. Thus, the operator Q defined above is a projector from $L_p(\mathbf{R}^s)$ onto $\mathscr{S}_k(X)$ for $1 \le p \le \infty$.

One expects that the approximation properties of Q depend on how far the knots $x^{i_j,l}$ are allowed to deviate from $\Delta(I)$. To describe this in detail, let

$$\mathscr{T}(I) = \{J \in \mathscr{T} : D_{J,\beta} \cap \Delta(I) \neq \varnothing \text{ for some } \beta \in \Gamma_k\}$$

and

$$U(I) := \bigcup \{D_{J,\beta} \cap \Delta(I) \neq \varnothing,\ J \in \mathscr{T},\ \beta \in \Gamma_k\}.$$

Of course, keeping the elements of V_β^I sufficiently close to $\Delta(I)$ would assure that $J \in \mathscr{T}(I)$ corresponds to an adjacent simplex $\Delta(J)$ for which $J \cap I \neq \varnothing$. In general, we will assume

$$N := \sup_{I \in \mathscr{T}} \#\mathscr{T}(I) < \infty, \tag{4.10}$$

where $\#\mathscr{T}(I)$ denotes the cardinality of $\mathscr{T}(I)$.

We are now in a position to prove the following result.

Theorem 4.1. *Suppose* (2.29) *holds. For any* $f \in L_p(\mathbf{R}^s)$ *one has*

$$\|f - Qf\|_p(\Delta(I)) \le C \inf_{P \in \Pi_k(\mathbf{R}^s)} \|f - P\|_p(U(I)), \tag{4.11}$$

where for $p = \infty$ *the constant* C *depends only on* s *and* k. *For* $1 \le p < \infty$, *it also depends on* N *given by* (4.10), *which is assumed to be finite.*

Using standard estimates for local polynomial approximation, one derives from (4.11) estimates in terms of moduli of continuity or Sobolev seminorms, confirming optimal (local or global) convergence rates $\mathscr{O}(h^k)$, $h = \max\{\operatorname{diam}\Delta(I) : I \in \mathscr{T}\}$, whenever the kth-order derivatives of f are bounded in $L_p(\mathbf{R}^s)$.

To prove (4.11), one may use the estimate [1]

$$\|M(\cdot\,|V_\beta^I)\|_p \le C \operatorname{vol}_s([V_\beta^I])^{-1+1/p}, \tag{4.12}$$

where C depends only on s and k.

Hence, by (2.37), one concludes

$$\|N_\beta^I\|_p \le C \operatorname{vol}_s([V_\beta^I])^{1/p},$$

giving

$$\begin{aligned} \|Qf\|_p(\Delta(I)) &\le \sum_{J \in \mathscr{T}(I)} \sum_{|\beta|=k} |\mu_{J,\beta}(f)|\, \|N_\beta^I\|_p \\ &\le C\|f\|_p(U(I)), \end{aligned} \tag{4.13}$$

where now C depends also on N when $p < \infty$. Since Q reproduces all polynomials in $\Pi_k(\mathbf{R}^s)$, we may apply the local boundedness of Q (4.13) to

$$\|f - Qf\|_p(\Delta(I)) \le \|f - P\|_p(\Delta(I)) + \|Q(f - P)\|_p(\Delta(I)),$$

which yields (4.11).

The second important application of Lemma 4.1 is concerned with the stability of the B-spline basis. To this end, one needs first an appropriate L_p-normalization of the B-splines, namely

$$N^I_{p,\beta}(x) := |d(X^I_\beta)|^{-1/p} N^I_\beta(x), \tag{4.14}$$

so that $N^I_{\infty,\beta}(x) = N^I_\beta(x)$ and $N^I_{1,\beta}(x) = M(x|V^I_\beta)$.

Defining for $c = \{c_{I,\beta}\}_{I\in\mathcal{T},\beta\in\Gamma_k}$,

$$\|c\|_{l_p} := \left(\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} |c_{I,\beta}|^p\right)^{1/p},$$

one obtains the following result.

Theorem 4.2. *Suppose* (3.3) *holds. Then for any sequence* $c = \{c_{I,\beta}\}_{I\in\mathcal{T},\beta\in\Gamma_k}$, *the estimate*

$$\gamma\|c\|_{l_p} \le \left\|\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} c_{I,\beta} N^I_{p,\beta}\right\|_p \le \|c\|_{l_p}$$

holds for some constant γ *which for* $p = \infty$ *depends only on* s *and* k, *and in addition on* N, *see* (4.10), *when* $1 \le p < \infty$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then (4.14) yields

$$\begin{aligned}
&\left\|\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} c_{I,\beta} N^I_{p,\beta}\right\|_p^p \\
&\quad = \int_{\mathbf{R}^s} \left(\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} c_{I,\beta} |d(X^I_\beta)|^{-1/p} N^I_\beta(x)^{1/p} N^I_\beta(x)^{1/q}\right)^p dx \\
&\quad \le \int_{\mathbf{R}^s} \left(\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} |c_{I,\beta}|^p |d(X^I_\beta)|^{-1} N^I_\beta(x)\right) \left(\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} N^I_\beta(x)\right)^{p/q} dx.
\end{aligned}$$

Since the N^I_β form a partition of unity, the right-hand side reduces to

$$\sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} |c_{I,\beta}|^p |d(X^I_p)|^{-1} \int_{\mathbf{R}^s} N^I_\beta(x)\,dx = \|c\|_{l_p}, \tag{4.15}$$

where we have used again the fact that $\int_{\mathbf{R}^s} M(x|V)\,dx = 1$ and the normalization (2.37).

Conversely, given $S(x) = \sum_{I\in\mathcal{T}} \sum_{\beta\in\Gamma_k} c_{I,\beta} N^I_{p,\beta}(x)$, one has

$$\begin{aligned}
|\mu_{I,\beta}(S)| &= \left|\sum_{J\in\mathcal{T}} \sum_{\nu\in\Gamma_k} c_{J,\nu} |d(X^J_\nu)|^{-1/p} \mu_{I,\beta}(N^J_\nu)\right| \\
&= |c_{I,\beta}|\,|d(X^I_\beta)|^{-1/p}.
\end{aligned}$$

Since, on the other hand, Lemma 4.1 says that

$$|\mu_{I,\beta}(S)| \le C\,\mathrm{vol}_s(D_{I,\beta})^{-1/p} \|S\|_p(D_{I,\beta}),$$

we obtain

$$|c_{I,\beta}|^p \leq C \left(\frac{|d(X_\beta^I)|}{\mathrm{vol}_s(D_{I,\beta})} \right) \|S\|_p^p(D_{I,\beta}) . \tag{4.16}$$

In view of (4.15), the assertion follows now from (4.16). □

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