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DOUBLY CUSPIDAL COHOMOLOGY FOR PRINCIPAL CONGRUENCE SUBGROUPS OF $\mathrm{GL}(3, \mathbb{Z})$

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ABSTRACT. The cohomology of arithmetic groups is made up of two pieces, the cuspidal and noncuspidal parts. Within the cuspidal cohomology is a subspace—the f-cuspidal cohomology—spanned by the classes that generate representations of the associated finite Lie group which are cuspidal in the sense of finite Lie group theory. Few concrete examples of f-cuspidal cohomology have been computed geometrically, outside the cases of rational rank 1, or where the symmetric space has a Hermitian structure.

This paper presents new computations of the f-cuspidal cohomology of principal congruence subgroups $\Gamma(p)$ of $\mathrm{GL}(3, \mathbb{Z})$ of prime level p. We show that the f-cuspidal cohomology of $\Gamma(p)$ vanishes for all $p \leqslant 19$ with $p \neq 11$, but that it is nonzero for $p = 11$. We give a precise description of the f-cuspidal cohomology for $\Gamma(11)$ in terms of the f-cuspidal representations of the finite Lie group $\mathrm{GL}(3, \mathbb{Z}/11)$.

We obtained the result, ultimately, by proving that a certain large complex matrix M is rank-deficient. Computation with the SVD algorithm gave strong evidence that M was rank-deficient; but to prove it, we mixed ideas from numerical analysis with exact computation in algebraic number fields and finite fields.

INTRODUCTION

The cohomology of arithmetic groups has been much studied in recent years, especially because of its rich connections with number theory and geometry. The cohomology is made up of two pieces, the cuspidal and noncuspidal parts. The cuspidal part (at least in theory) is the harder to get hold of, while the noncuspidal part can be derived inductively from the cuspidal cohomology of lower-rank groups using the theory of Eisenstein cohomology.

Very few concrete examples are known of cuspidal cohomology for arithmetic subgroups of algebraic groups, outside the cases of rational rank 1, or where the symmetric space has a Hermitian structure. Such examples are of great interest because they provide tests for the validity of various conjectures, such as number-theoretical conjectures in the Langlands program [4, 8] and geometric conjectures in the theory of modular symbols [2]. Finding such examples is difficult, and it calls for an interesting mixture of geometrical and computational work.

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In this paper we concentrate on the general linear groups. The case of $\mathrm{GL}(2, \mathbb{Z})$ and its subgroups of finite index is classical and well known. The next case, that of $\mathrm{GL}(3, \mathbb{Z})$ and its subgroups of finite index, is already quite difficult. Computation of its cohomology was initiated by Soulé [18], and the first computer-generated examples of cuspidal cohomology appear in [3]. In the general case of $\mathrm{GL}(n, \mathbb{Z})$, the only known constructions of cuspidal cohomology arise from "lifting theorems" in the theory of automorphic forms, and they always yield "selfdual" automorphic representations. (For a discussion of these ideas, see [4, 8, and 14].)

The purpose of this paper is to present some new computations of the cuspidal cohomology of principal congruence subgroups $\Gamma(p)$ of $\mathrm{GL}(3, \mathbb{Z})$ of prime level p. Our main theorem (2.4) says that the cuspidal cohomology of $\Gamma(p)$ vanishes for all $p \leqslant 19$ with $p \neq 11$, but that it is nonzero for $p = 11$. We give a precise description of part of the cuspidal cohomology for $\Gamma(11)$, the f-cuspidal part (see (1.9)), in terms of the (isomorphism classes of) cuspidal representations of the finite Lie group $\mathrm{GL}(3, \mathbb{Z}/11)$. (One knows from lifting theory that the f-cuspidal part does not exhaust all the cuspidal cohomology—see [14].)

The cohomology classes we discover are interesting for several reasons. (1) They correspond to nonselfdual automorphic representations of $\mathrm{GL}(3)$, as was also the case with the classes in [3]. (2) They give rise to representations of the finite group $\mathrm{GL}(3, \mathbb{Z}/p)$, which are themselves cuspidal in the sense of finite Lie group theory (see below for the precise statement). This explains the phrase "doubly cuspidal" in our title. (3) They are not in the span of the maximal $(2, 1)$-modular symbols on the symmetric space for $\mathrm{GL}(3)$. This follows from (2) together with the main theorem in [2] and resolves a problem raised in that paper.

Some questions related to number theory are to work out the action of the Hecke operators on the cohomology classes we have found in this paper, to test the generalized Ramanujan conjecture on the Hecke eigenvalues, and to look for Galois representations associated with these packages of Hecke data. We would certainly also like to know why we found classes only at level 11 and at no other prime level $p \leqslant 19$.

From the point of view of computational mathematics, perhaps the most interesting aspect of our paper is the novel application of standard numerical methods to exact computations in algebraic number fields. Our main computational problem was to prove that a certain matrix M was rank-deficient. The usual algorithm for the singular value decomposition (SVD) told us that one singular value of M was nearly zero compared with the others; but of course this is not a proof of what we wanted. We could reduce M mod a prime and show the reduction was rank-deficient, but this only gives results modulo that prime. Our success came when we showed M was rank-deficient if and only if it was deficient mod a list of primes whose product was less than a certain bound b. The number b was found by making estimates using ideas from numerical analysis and the theory of the SVD decomposition, but where all the estimates were computed rigorously in exact arithmetic over a number field. See (5.5) for these results.

We now outline the individual sections of the paper. In §1 we review facts about cuspidal representations and cuspidal cohomology. In §2 we state our

main theorem (2.4), after introducing some notation for describing the structure of cuspidal representations. Section 3 reviews an explicit algorithm for computing the cohomology of $\mathrm{GL}(3, \mathbb{Z})$ and its subgroups, with coefficients in any finite-dimensional complex representation. Section 4 reviews a few facts from numerical analysis and the theory of the SVD. We put all these ingredients together in the last two sections, where we prove our main theorem. The most interesting cases, where $p = 11$ and the cohomology is nonvanishing, are treated in §5; the other cases come in §6.

1. Background on cuspidal cohomology and cuspidal representations

(1.1) **Notation and conventions.** For any field K, let $\mathrm{SL}(n, K)^{\pm}$ be the group $\{\gamma \in \mathrm{GL}(n, K) \mid \det \gamma = \pm 1\}$. Let $\mathbb{F}_q$ be the finite field of q elements.

Let $\Gamma(m) = \{\gamma \in \mathrm{GL}(3, \mathbb{Z}) \mid \gamma \equiv I \pmod{m}\}$ be the principal congruence subgroup of level m. This is a normal subgroup of $\mathrm{GL}(3, \mathbb{Z})$ of finite index, and is torsion-free for $m \geqslant 3$. Let $G = \Gamma(p) \backslash \mathrm{GL}(3, \mathbb{Z})$ for a prime number p; this is isomorphic to $\mathrm{SL}(3, \mathbb{F}_p)^{\pm}$, and for $p > 2$ it is a normal subgroup of $\mathrm{GL}(3, \mathbb{F}_p)$ of index $\frac{1}{2}(p-1)$.

Throughout this paper, the word "representation" means representation on a finite-dimensional complex vector space.

(1.2) We will need some facts about induced representations (see [7, Chapters I and III]). Let A be a group, let $B \subseteq A$ be a subgroup, and let π be a representation of B on the vector space V. Then π naturally makes V into a left module over the group ring $\mathbb{Z}[B]$. We define the *induced module*

$$\mathrm{Ind}_B^A V = \{f : A \to V \mid f(ba) = \pi(b) \cdot f(a) \quad \forall a \in A,\ \forall b \in B\}.$$

This is a left $\mathbb{Z}[A]$-module via the action $a' \cdot f(a) = f(aa')$. Shapiro's Lemma gives a canonical isomorphism

$$H^*(B, V) \cong H^*(A, \mathrm{Ind}_B^A V).$$

Now assume π is the trivial representation (i.e., $V = \mathbb{C}$ with trivial B-action); we denote this by $\mathbb{C}$. Also assume B is a normal subgroup of A. Then $\mathrm{Ind}_B^A \mathbb{C}$ is just the vector space of all functions $B \backslash A \to \mathbb{C}$. This can be identified with the group ring $\mathbb{C}[B \backslash A]$, and hence it becomes a $\mathbb{C}[B \backslash A]$-bimodule. The right action of $\mathbb{C}[B \backslash A]$ makes $H^*(A, \mathrm{Ind}_B^A \mathbb{C})$ into a right $\mathbb{C}[B \backslash A]$-module; via the Shapiro isomorphism, $H^*(B, \mathbb{C})$ is a $\mathbb{C}[B \backslash A]$-module. This module structure is the same as that induced by the conjugation action of A on B.

(1.3) If $B \trianglelefteq A$ is normal of finite index, then $\mathrm{Ind}_B^A \mathbb{C}$ is just the regular representation of the finite group $B \backslash A$ (acting on the right). Let $\{R_i \mid i = 1, \ldots, h\}$ be a set of representatives of the irreducible finite-dimensional representations of $B \backslash A$, with $\dim R_i = n_i$. These extend to representations of A via the projection $A \twoheadrightarrow B \backslash A$; by abuse of notation we denote the extensions by R_i. The group ring breaks up as follows:

$$\mathbb{C}[B \backslash A] \cong \mathrm{Ind}_B^A \mathbb{C} \cong \bigoplus_{i=1}^{h} I_i,$$

where $I_i = \bigoplus_{j=1}^{n_i} R_{ij}$ is a two-sided ideal, and each R_{ij} is isomorphic to R_i. Thus $H^k(B, \mathbb{C}) \cong \bigoplus_{i=1}^{h} M_i$ as right $B\backslash A$-modules, where M_i is isomorphic to $\bigoplus_{j=1}^{n_i} H^k(A, R_{ij})$.

We apply these results below when $A = \mathrm{GL}(3, \mathbb{Z})$ and $B = \Gamma(p)$ is a principal congruence subgroup. The results mean that the cohomology of $\Gamma(p)$ can be found, in principle, by computing $H^k(\mathrm{GL}(3, \mathbb{Z}), R_i)$ for all the irreducible representations of $G = \Gamma(p)\backslash \mathrm{GL}(3, \mathbb{Z})$.

(1.4) The irreducible representations of $\mathrm{GL}(n, \mathbb{F}_p)$ can be described inductively as follows (see [12]). All irreducible representations of $\mathrm{GL}(1, \mathbb{F}_p)$ are dubbed *cuspidal.* If $n > 1$, let P be any proper parabolic subgroup of $\mathrm{GL}(n, \mathbb{F}_p)$. Write $P = LU$, where L is a Levi component of P and U is the unipotent radical. Consider the set $I(P)$ of irreducible components of $\mathrm{Ind}_P^{\mathrm{GL}(n, \mathbb{F}_p)} \rho$, as ρ runs over all irreducible representations of L (extended trivially to U). Then an irreducible representation of $\mathrm{GL}(n, \mathbb{F}_p)$ is *cuspidal* if and only if it is not in the union $\bigcup_P I(P)$, taken over all proper parabolic subgroups P.

Intuitively, the cuspidal representations of $\mathrm{GL}(n, \mathbb{F}_p)$ for $n = 1, 2, \ldots$ are the basic building blocks for all the representations of these groups. For a given n they constitute what parts of the representation theory of $\mathrm{GL}(n, \mathbb{F}_p)$ cannot be induced from products of $\mathrm{GL}(m_i, \mathbb{F}_p)$'s with $m_i < n$.

(1.5) Now let $G = \mathrm{SL}(3, \mathbb{F}_p)^{\pm}$ as in (1.1). We call an irreducible representation of G *cuspidal* if it is a component of $\rho|_G$ for some cuspidal representation ρ of $\mathrm{GL}(3, \mathbb{F}_p)$.

Remark. If $p \equiv 2 \pmod 3$, one can easily show that the restriction of a cuspidal representation ρ to G is irreducible. This is true for $p = 11$, the most important value of p for this paper. If $p \equiv 1 \pmod 3$, then $\rho|_G$ will in general break up into three irreducible pieces.

(1.6) In (1.6)–(1.9) we motivate the study of cuspidal representations by describing their topological interpretation.

Let $X = O(3, \mathbb{R})\backslash \mathrm{SL}(3, \mathbb{R})^{\pm}$ be the symmetric space for $\mathrm{SL}(3, \mathbb{R})^{\pm}$; the group $\mathrm{GL}(3, \mathbb{Z})$ acts properly discontinuously on it on the right. Let Γ_0 be any torsion-free subgroup of finite index in $\mathrm{GL}(3, \mathbb{Z})$. Let $\overline{X}/\Gamma_0$ be the Borel-Serre compactification of X/Γ_0 (see [6]). Then, for trivial complex coefficients, the homology (resp. cohomology) of Γ_0 and of $\overline{X}/\Gamma_0$ are canonically isomorphic, and we shall identify them.

(1.7) Let $H_!^k(\Gamma_0, \mathbb{C})$ be the part of the cohomology of $\overline{X}/\Gamma_0$ that restricts to zero on the Borel-Serre boundary—that is, $H_!^k(\Gamma_0, \mathbb{C})$ is the kernel of the restriction map $r_k\colon H^k(\overline{X}/\Gamma_0, \mathbb{C}) \to H^k(\partial\overline{X}/\Gamma_0, \mathbb{C})$. The groups $H_!^k(\Gamma_0, \mathbb{C})$ are important because $H^k(\overline{X}/\Gamma_0, \mathbb{C})$ can be computed once we know (1) the groups $H_!^k(\Gamma_0, \mathbb{C})$, (2) some facts about the Tits building for $\mathrm{GL}(3)$ mod Γ_0, and (3) cuspidal cohomology groups for certain subgroups of $\mathrm{GL}(2, \mathbb{Z})$. (For a precise statement see [3, pp. 415–416], which is in turn based on [16].)

To compute $H_!^k(\Gamma_0, \mathbb{C})$ for all k, it suffices to compute $H_!^3(\Gamma_0, \mathbb{C})$ (see [3, §2]). So for the rest of the paper we focus our attention on cohomology in degree $k = 3$.

(1.8) We now look more closely at $H^3_!(\Gamma_0, \mathbb{C})$.

Proposition. *$H^3_!(\Gamma_0, \mathbb{C})$ is the subspace of $H^3(\overline{X}/\Gamma_0, \mathbb{C})$ consisting of those classes which can be represented by cuspidal differential forms.*

A proof of this proposition is given in [3, §2].

Remark. The notion of "cuspidal differential form" used here comes from the theory of automorphic forms on general Lie groups. It generalizes the classical notion of cusp form on the upper half plane. There is a "philosophy of automorphic cusp forms" analogous to that referred to above for finite groups; again the cuspidal contribution to the spectrum of $L^2(\mathrm{SL}(n, \mathbb{R})^{\pm}/\Gamma_0)$ is the "interesting part" that cannot be induced from smaller $\mathrm{GL}(n)$'s [5, 15]. The computation of cuspidal cohomology is interesting and challenging for this reason.

Because of this proposition, we are justified in calling $H^3_!(\Gamma_0, \mathbb{C})$ the *cuspidal cohomology* in degree three; we denote it by $H^3_{\mathrm{cusp}}(\Gamma_0, \mathbb{C})$.

(1.9) Now set $\Gamma_0 = \Gamma(p)$, and let G and other notation be as in (1.1). At this point we have two notions of "cuspidal cohomology classes" for G. To distinguish them, we use the term *a-cuspidal* for a cuspidal cohomology class α for $\Gamma(p)$; this is the usage coming from (1.8). If α generates a subrepresentation of $H^*(\Gamma(p), \mathbb{C})$ for G all of whose components are cuspidal, α is called *f-cuspidal*; this comes from (1.4–1.5).

The two notions are related as follows:

Theorem. ("f-cuspidal $\Rightarrow$ a-cuspidal"). *Under the isomorphism of* (1.6), *the f-cuspidal cohomology for $\Gamma(p)$ pulls back to a subspace of $H^3_{\mathrm{cusp}}(\overline{X}/\Gamma(p), \mathbb{C})$.*

Proof. Clearly r_3 is a G-equivariant map. Also, $H^3(\partial\overline{X}/\Gamma_0)$ is a sum of induced representations of the form $\mathrm{Ind}_P^G \rho$, where P is a proper parabolic subgroup of G and ρ is a representation of its Levi factor, trivially extended to the unipotent radical (as in (1.4)) [16]. Hence, r_3 is an intertwining operator which intertwines the representation generated by α into a sum of parabolically-induced representations. Thus, if α is f-cuspidal, then $r_3(\alpha) = 0$. This implies α is a-cuspidal by (1.7). □

Remarks. (1) If V is an irreducible cuspidal representation of G and α is a nonzero class in $H^3(\mathrm{GL}(3, \mathbb{Z}), V)$, then α is f-cuspidal when viewed as an element of $H^3(\Gamma(p), \mathbb{C})$, since the G-representation generated by α is isomorphic to $\widehat{V}$ (the contragredient of V).

(2) In general, unfortunately, $H^3_{\mathrm{cusp}}(\overline{X}/\Gamma(p), \mathbb{C})$ is not spanned by cohomology coming from f-cuspidal representations, as shown by the classes found in [3].

2. Statement of results

In (2.4) we state our main theorems. To fix some necessary notation, we begin (2.1–2.3) by describing the structure of the cuspidal representations T^Λ of $\mathrm{GL}(3, \mathbb{F}_p)$, following the expositions in [13] and [10].

(2.1) Let U be the standard unipotent subgroup of $\mathrm{GL}(3, \mathbb{F}_p)$:

$$U = \left\{ \gamma \in \mathrm{GL}(3, \mathbb{F}_p) \;\middle|\; \gamma = \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let Q be the standard *mirabolic subgroup*

$$Q = \left\{ \gamma \in \mathrm{GL}(3, \mathbb{F}_p) \,\middle|\, \gamma = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let N be the index $[Q : U] = (p+1)(p-1)^2$.

The group $\mathrm{GL}(3, \mathbb{F}_p)$ is generated by the diagonal matrices

$$d(a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \qquad (a_1, a_2, a_3 \in \mathbb{F}_p^\times)$$

together with U and with the following generators of the Weyl group:

$$z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We will view $\mathrm{GL}(2, \mathbb{F}_p)$ as the subgroup of $\mathrm{GL}(3, \mathbb{F}_p)$ given by

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathrm{GL}(2, \mathbb{F}_p)$ is generated by $d(a_1, a_2, 1)$, z, and the following element of order three:

$$h = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By abuse of notation we will sometimes write

$$z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

and we will write

$$u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad d(a_1, a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

(2.2) In this subsection we give the classification of the cuspidal representations of $\mathrm{GL}(3, \mathbb{F}_p)$ up to isomorphism.

Let $e\colon \mathbb{F}_p \to \mathbb{C}$ be the additive character $e(x) = \exp(2\pi i x/p)$. Let $\chi\colon U \to \mathbb{C}^\times$ be the one-dimensional representation

$$\chi : \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix} \mapsto e(b_1 + b_3).$$

Then $\mathrm{Ind}_U^Q \chi$ is an irreducible representation t of Q on a vector space V of dimension N [10, Proposition 1.2].

Let $\Lambda\colon \mathbb{F}_{p^3}^\times \to \mathbb{C}^\times$ be a multiplicative character on the cubic extension field of $\mathbb{F}_p$. Assume $\Lambda \neq \Lambda^p$. One can define an irreducible character χ^Λ explicitly on $\mathrm{GL}(3, \mathbb{F}_p)$ in a way depending only on Λ and p (for the precise statement see [13, §2.3]). This is the character of a cuspidal representation T^Λ of $\mathrm{GL}(3, \mathbb{F}_p)$ on the vector space V of the preceding paragraph, and every cus-

pidal representation arises in this way. The restriction of every T^Λ to Q is just t [10, Theorem 2.3]. Representations T^Λ and $T^{\Lambda'}$ are isomorphic if and only if Λ' equals an element of the set $\{\Lambda, \Lambda^p, \Lambda^{p^2}\}$.

To define T^Λ, it suffices to give its values on the generators of $\mathrm{GL}(3, \mathbb{F}_p)$ listed in (2.1). Its values on $d(a_1, a_2, 1)$ and z are the same as the values of t on these elements. One checks from the definition of χ^Λ that $T^\Lambda(d(a, a, a)) = \Lambda(a)$. Finally, Helversen-Pasotto uses [17] to give an explicit formula for $T^\Lambda(w)$ [13, p. 12]; we give her formula below in (2.8).

We pull T^Λ back to a representation of $\mathrm{GL}(3, \mathbb{Z})$ on V via the homomorphism $\mathrm{GL}(3, \mathbb{Z}) \to \mathrm{GL}(3, \mathbb{F}_p)$. The pullback is also called T^Λ.

(2.3) To classify the cuspidal representations, we must classify the multiplicative characters $\Lambda\colon \mathbb{F}_{p^3}^\times \to \mathbb{C}^\times$. We know Λ is a homomorphism into the subgroup of $\mathbb{C}^\times$ generated by a primitive (p^3-1)st root of unity ζ. Fix a generator η of the cyclic group $\mathbb{F}_{p^3}^\times$. Up to the action of the Galois group of $\mathbb{Q}(\zeta)/\mathbb{Q}$, every Λ is given by choosing a positive integer divisor λ of p^3-1 and setting $\Lambda(\eta^m) = \zeta^{\lambda m}$. The condition $\Lambda \neq \Lambda^p$ of (2.2) is equivalent to $(p^2+p+1) \nmid \lambda$; so in the rest of the paper we exclude values of λ that are divisible by p^2+p+1.

Note that if $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, then

$$\dim H^*(\mathrm{GL}(3, \mathbb{Z}), T^\Lambda) = \dim H^*(\mathrm{GL}(3, \mathbb{Z}), T^{\sigma(\Lambda)}).$$

This is because we can compute the dimensions using the $\mathbb{Q}(\zeta)$ forms for the representations, and σ gives a $\mathrm{GL}(3, \mathbb{Z})$-module isomorphism from one representation to the other. (However, the two representations are not equivalent, since this map is not $\mathbb{Q}(\zeta)$-linear.)

(2.4) We now state our main theorem.

Theorem. *Let p be a prime number with $2 \leqslant p \leqslant 19$. Let $\Lambda\colon \mathbb{F}_{p^3}^\times \to \mathbb{C}^\times$ be an arbitrary multiplicative character $\Lambda(\eta^m) = \zeta^{\lambda m}$ with λ a divisor of p^3-1 as in (2.3), and let T^Λ be the cuspidal representation of $\mathrm{GL}(3, \mathbb{Z})$ on V described in (2.2). Then $\dim H^3(\mathrm{GL}(3, \mathbb{Z}), V)$ is zero when λ is odd, and is given for even λ by the following table.*

p	λ	$\dim H^3(\mathrm{GL}(3, \mathbb{Z}), V)$
3	2	0
5	2, 4	0
7	2, 6, 18, 38	0
11	2, 10, 14, 70	0
11	38*, 190	1
13	2, 4, 6, 12, 18, 36, 122, 244	0
17	2*, 4*, 8*, 16*	0
19	2*, 6*, 18, 54, 254	0

Corollary. *The principal congruence subgroups $\Gamma(p) \subset \mathrm{GL}(3, \mathbb{Z})$ have no cuspidal cohomology for primes $p \leqslant 19$, except for the nonvanishing classes noted for $p = 11$.*

Remark. An asterisk beside a value of λ means that while we have strong numerical evidence for the result, we do not have a rigorous proof. These cases are explained in (6.2) and (6.3).

In §5 we prove the nonvanishing of the cohomology for $p = 11$, $\lambda = 190$. The rest of the proof occupies §6. Sections 3 and 4 contain preliminary material.

3. Computation of cuspidal cohomology

(3.1) When $n = 2, 3, 4$ we can (at least in principle) explicitly compute the cohomology of $\mathrm{GL}(n, \mathbb{Z})$ with coefficients in an arbitrary representation. Here we state the results for $n = 3$. A reference for the theorem in the trivial-coefficient case is [1]; it is straightforward to extend the proof in [1] to show that the theorem holds for arbitrary coefficients.

Theorem. *Let ρ_0 be an arbitrary representation of $\mathrm{GL}(3, \mathbb{Z})$ on a finite-dimensional vector space V_0. Then with notation as in* (2.1), *$H_3(\mathrm{GL}(3, \mathbb{Z}), V_0)$ is isomorphic to the subspace of all $v \in V_0$ satisfying the linear equations*

$$\rho_0(d(\pm 1, \pm 1, \pm 1)) \cdot v = v \qquad \text{(all eight choices of sign)}, \tag{3.1.1}$$

$$\rho_0(z) \cdot v = -v, \tag{3.1.2}$$

$$(\rho_0(I) + \rho_0(h) + \rho_0(h^2)) \cdot v = 0, \tag{3.1.3}$$

$$\rho_0(w) \cdot v = -v. \tag{3.1.4}$$

Remarks. 1. The theorem is a simplification of a formula of Soulé in [18].

2. The proof of the theorem uses the *well-rounded retract* $W \subset X$, which is described in [1]. This is a $\mathrm{GL}(n, \mathbb{Z})$-equivariant deformation retract of X. For any $\Gamma_0 \subseteq \mathrm{GL}(n, \mathbb{Z})$ of finite index, W/Γ_0 is compact. W has the structure of a locally finite regular cell complex, and Γ_0 preserves this cell structure. The conditions on the v's are derived from the combinatorics of the top-dimensional cells of W and how their faces meet.

3. We worked with cohomology in §1, but Theorem (3.1) deals with homology. The difference is unimportant. Our representations are over $\mathbb{C}$; this means $H_i(\mathrm{GL}(3, \mathbb{Z}), V)^*$ is isomorphic to $H^i(\mathrm{GL}(3, \mathbb{Z}), V^*)$ for any representation V we are considering and for any i (with $(\ \ )^*$ denoting linear dual). In particular, the homology and cohomology groups of degree i with coefficients in V have the same dimension, because T^{Λ^*} equals $T^{c(\Lambda)}$, where c denotes complex conjugation. We will work with homology for the rest of the paper.

4. Numerical facts about matrices

In this section we collect for later use some facts about matrix norms and the singular value decomposition (SVD). A reference for this material is [11, §§2.2–2.3].

(4.1) Let $\|\cdot\|$ be the standard norm $\sqrt{|z_1|^2 + \cdots + |z_k|^2}$ on $\mathbb{C}^k$. If $A = (a_{ij})$ is an $m \times n$ matrix over $\mathbb{C}$, we define the *2-norm*

$$\|A\|_2 = \sup\left\{ \frac{\|Ax\|}{\|x\|} \,\middle|\, x \in \mathbb{C}^n,\ x \neq 0 \right\}$$

and the *Frobenius norm*

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} \quad .$$

These are norms on the complex vector space of $m \times n$ matrices. For matrices A of size $m \times n$ and B of size $n \times q$, we have the relations

$$\|AB\|_2 \leqslant \|A\|_2 \cdot \|B\|_2, \tag{4.1.1}$$

$$\|A\|_2 \leqslant \|A\|_F \leqslant \sqrt{n} \cdot \|A\|_2. \tag{4.1.2}$$

The values of $\|A\|_2$ and $\|A\|_F$ do not change if A is multiplied by a unitary matrix on either side. If A is a block-diagonal matrix,

$$A = \begin{pmatrix} A_1 & 0 & \cdots \\ 0 & A_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

then the Pythagorean theorem implies

$$\|A\|_2 = \max_i \|A_i\|_2. \tag{4.1.3}$$

(4.2) If B is any $m \times n$ matrix over $\mathbb{C}$, there exist unitary matrices U, V of size $m \times m$ and $n \times n$ respectively, and an $m \times n$ matrix $S = (s_{ij})$ with

$$s_{ij} = \begin{cases} \sigma_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (\sigma_i \in \mathbb{R},\ \sigma_i \geqslant 0)$$

such that

$$B = U \cdot S \cdot V.$$

This is the *singular value decomposition* of B. The σ_i are called the *singular values*; they are unique up to rearrangement.

We remark that if B is an invertible $n \times n$ matrix, then the singular value decomposition is the familiar KAK decomposition in the Lie group $\mathrm{GL}(n, \mathbb{C})$. Here A is the group of positive real diagonal matrices, and K is the unitary group $U(n)$.

For any B we have

$$\|B\|_F^2 = \sum_{i=1}^{\min(m,n)} \sigma_i^2. \tag{4.2.1}$$

(4.3) Let A be an $m \times n$ matrix over $\mathbb{C}$. A *maximal square submatrix* of A is any submatrix of size $\min(m, n) \times \min(m, n)$.

Lemma. *The determinant of any maximal square submatrix of A is bounded in absolute value by the product of the singular values of A.*

The proof is an exercise in linear algebra.

5. Proof of the main part of the theorem

In this section we prove the most interesting case of our main theorem (2.4). We show that if $p = 11$ and V is a representation of $\mathrm{GL}(3, \mathbb{F}_p)$ isomorphic to the particular cuspidal representation T^Λ with $\lambda = 190$ (see §2 for the notation), then the cuspidal cohomology $H^3(\mathrm{GL}(3, \mathbb{F}_p), V)$ is nonvanishing and in fact has dimension 1. We show this by systematically working out the dimension of the subspace defined in Theorem 3.1, using a mixture of hand and machine computation.

(5.1) Let $\mathscr{O}_k = \mathbb{Z}[\zeta_k]$ be the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_k)$, for ζ_k a primitive kth root of unity. The cyclotomic field is of degree $\phi(k)$ over $\mathbb{Q}$, where ϕ is the Euler phi-function.

We use the notation of (2.1–2.3). For the p and λ in the part of the proof we are doing now, the representation t is defined over $\mathbb{Q}(\zeta_{11})$, and T^Λ is defined over $\mathbb{Q}(\zeta_{11}, \zeta_{\frac{11^3-1}{190}}) = \mathbb{Q}(\zeta_{77})$.

(5.2) First we choose an explicit basis of V, following [13, §2.2]. Recall from (2.2) that $V \cong \operatorname{Ind}_U^Q \chi$ as representations of Q, meaning V may be identified with the space of functions $f\colon \mathrm{GL}(2, \mathbb{F}_p) \to \mathbb{C}$ satisfying $f(u(b) \cdot x) = e(b) \cdot f(x)$ for all $b \in \mathbb{F}_p$ and all $x \in \mathrm{GL}(2, \mathbb{F}_p)$. Let $\mathscr{R}$ be the set

$$\{d(a_1, a_2) \mid a_i \in \mathbb{F}_p^\times\} \cup \{z \cdot u(b) \cdot d(a_1, a_2) \mid b \in \mathbb{F}_p, a_i \in \mathbb{F}_p^\times\}.$$

(Note that $\mathscr{R}$ is a set of coset representatives for $U\backslash Q$, where as in (2.1) we have identified $\mathrm{GL}(2, \mathbb{F}_p) \subset Q \subset \mathrm{GL}(3, \mathbb{F}_p)$.) For $x \in \mathscr{R}$, define $[x] \in V$ by

$$[x](u(b)y) = \begin{cases} e(b) & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \qquad \forall y \in \mathscr{R},\ \forall b \in \mathbb{F}_p.$$

Then $\{[x] \mid x \in \mathscr{R}\}$ forms a vector space basis of V. One can write down a formula for the representation $t = T^\Lambda|_Q$ with respect to this basis.

Observe that if $q \in Q$ and $[x]$ for $x \in \mathscr{R}$ is a basis element of V, then $q \cdot [x]$ is just a pth root of unity times another basis element $[x']$ with $x' \in \mathscr{R}$. We express this by saying q permutes the basis elements of V up to multiplication by roots of unity.

(5.3) To prove the theorem, we want to compute $H_3(\mathrm{GL}(3, \mathbb{Z}), V)$ using Theorem 3.1. In this subsection we begin by finding the common solution to (3.1.1), (3.1.2), and (3.1.3) with $\rho_0 = t$. Since this involves elements of $\mathrm{GL}(3)$ that generate the embedded $\mathrm{GL}(2)$, we call this "solving the $\mathrm{GL}(2)$ problem."

First we use the basis $\{[x] \mid x \in \mathscr{R}\}$ to find the common solution set of (3.1.1) and (3.1.2). We know $T^\Lambda(-I)$ is the scalar $\Lambda(-1) = 1$ by our choice of Λ, so (3.1.1) holds trivially for $T^\Lambda(-I)$. The subgroup $G_2 \subset \mathrm{GL}(3, \mathbb{F}_p)$ generated by $\mathrm{diag}(-1, 1, 1)$, $\mathrm{diag}(1, -1, 1)$ and z is of order eight. There is a unique way to partition $\mathscr{R}$ into subsets, each of order eight, so that G_2 preserves the decomposition. (That is, if E is an eight-element set in the decomposition, then G_2 acts transitively on E up to multiples by pth roots of unity.) It follows that the common solution to (3.1.1) and (3.1.2) has a basis consisting of $\frac{1}{8}N$ vectors, each of which is a sum of exactly eight basis elements with coefficients in the roots of unity. We represent this basis as the columns of an $N \times \frac{1}{8}N$ matrix M_2, which is zero except in a sequence of 8×1 blocks down the diagonal, where its entries are pth roots of unity.

Second, the group $G_3 \subset \mathrm{GL}(3, \mathbb{F}_p)$ generated by G_2 and h permutes the basis elements (up to multiplication by pth roots of unity), and its orbits are the $\frac{1}{2}(p-1) = 5$ sets $\{[x] \mid x \in \mathscr{R}, \det x = \pm\beta\}$ for $\beta \in \mathbb{F}_p^\times$. We used a computer to write down $t(h)$. We then used Bareiss' algorithm [9, p. 86] and found explicitly on the computer a matrix M_3 whose columns represent a basis for the kernel of

$$(I + t(h) + t(h)^2) \cdot M_2.$$

The common solution to (3.1.1), (3.1.2), and (3.1.3) has as basis the columns of $M_2 \cdot M_3$.

The computations of M_2 and M_3 were done in exact arithmetic over $\mathscr{O}_{11}$. The set $\mathscr{R}$ can be ordered so that $t(h)$, and therefore M_3 itself, breaks into five diagonal blocks $M_{3,1}, \ldots, M_{3,5}$ corresponding to the five G_3-orbits in $\mathscr{R}$.

(5.4) To finish our proof, it suffices to show that $(T^{\Lambda}(w)+I)\cdot M_2 \cdot M_3$ (which is a 1200×50 matrix) has rank 49. In this subsection we give some details concerning what $T^{\Lambda}(w)$ is.

Helversen-Pasotto gives a formula [13, p. 12] for $T^{\Lambda}(w)$ as an $N \times N$ matrix. The rows and columns of the matrix are indexed by the elements of $\mathscr{R}$ that are shown underlined; the appropriate matrix entry is given in the body of the table.

	$\underline{d(c_1, c_2)}$	$\underline{z \cdot u(c_0) \cdot d(c_1, c_2)}$
$\underline{d(a_1, a_2)}$	0	$p^{-2} \cdot \delta(a_1 - c_1 a_2) \cdot \Lambda(a_2) \cdot F_{\Lambda}(a_2 c_2)$
$\underline{z \cdot u(a_0) \cdot d(a_1, a_2)}$	$p^{-2} \cdot \delta(c_1 - a_1 c_2) \cdot \Lambda^{-1}(c_2) \cdot \overline{F}_{\Lambda}(a_2 c_2)$	$p^{-3} \cdot \Lambda(a_1 c_1^{-1}) \cdot e(a_0 a_1^{-1} a_2 c_1 - c_0 c_1^{-1} c_2 a_1) \cdot H_{\Lambda}(a_1^{-1} a_2 c_1, c_1^{-1} c_2 a_1)$

Here the letters z, u, d, and e are as in (2.1–2.2). The function $\delta(x)$ is 1 if $x = 0$ and 0 otherwise. The letters $a_0, c_0 \in \mathbb{F}_p$ and $a_i, c_i \in \mathbb{F}_p^{\times}$ (for $i = 1, 2$) are arbitrary elements. The functions $F_{\Lambda}\colon \mathbb{F}_p^{\times} \to \mathbb{C}$ and $H_{\Lambda}\colon \mathbb{F}_p^{\times} \times \mathbb{F}_p^{\times} \to \mathbb{C}$ are defined by

$$F_{\Lambda}(x) = \sum_{\eta \in \mathbb{F}_{p^3}^{\times}} \delta(x - N(\eta)) \cdot \Lambda^{-1}(\eta) \cdot e(\mathrm{Tr}(\eta)),$$

$$\begin{aligned} H_{\Lambda}(x_1, x_2) = {} & -p \cdot \delta(x_1 + x_2) \\ & + \sum_{\eta \in \mathbb{F}_{p^3}^{\times}} \Bigg[\delta(x_1 N(\eta) + x_2) \cdot \Lambda^{-1}(\eta) \\ & \qquad \cdot \sum_{a \in \mathbb{F}_p^{\times}} e(a^{-1} + a(x_1 \mathrm{Tr}(\eta) + x_2 \mathrm{Tr}(\eta^{-1}) - (x_1 + x_2))) \Bigg], \end{aligned}$$

where $\mathrm{Tr}(\eta)$ and $N(\eta)$ are respectively the trace and norm of $\eta \in \mathbb{F}_{p^3}^{\times}$ for the field extension $\mathbb{F}_{p^3}/\mathbb{F}_p$.

Define

$$M_4 = p^3 \cdot (T^{\Lambda}(w) + I) \cdot M_2 \cdot M_3.$$

The formula for $T^{\Lambda}(w)$ shows that the entries of M_4 lie in $\mathscr{O}_{77}$. As we mentioned at the start of this subsection, we want to show that the 1200×50 matrix M_4 has rank 49.

(5.5) The proof of the next lemma is the technical heart of our result.

Recall that for any cyclotomic field $\mathbb{Q}(\zeta_k)$ there are exactly $\frac{1}{2}\phi(k)$ distinct complex embeddings $\mathbb{Q}(\zeta_k) \to \mathbb{C}$ modulo complex conjugation.

Lemma. *Choose any maximal square submatrix of M_4, and denote its determinant by $D \in \mathscr{O}_{77}$. Choose any complex embedding $\alpha : \mathbb{Q}(\zeta_{77}) \to \mathbb{C}$. Then*

$$|\alpha(D)| < 10^{265}.$$

Proof. $T^{\Lambda}(w)$ is unitary (with respect to a natural inner product on V) and is different from $\pm I$, so conjugation by a unitary matrix makes $T^{\Lambda}(w) + I$ into a diagonal matrix with 2's and 0's on the diagonal. Thus $\|T^{\Lambda}(w) + I\|_2 = 2$.

Applying (4.1.3) to M_2, we see that $\|M_2\|_2$ equals the 2-norm of an 8×1 matrix of roots of unity. That is, $\|M_2\|_2 = \sqrt{8}$.

By machine we computed $\|M_{3,j}\|_F^2$ in exact arithmetic over $\mathscr{O}_{11}$, obtaining for $j = 1, \dots, 5$ the values

$$\begin{aligned}
&128 - 67\theta^{(2)} - 119\theta^{(3)} - 146\theta^{(4)} - 160\theta^{(5)},\\
&294 + 153\theta^{(2)} + 18\theta^{(3)} + 95\theta^{(4)} + 45\theta^{(5)},\\
&294 + 35\theta^{(2)} + 168\theta^{(3)} - 18\theta^{(4)} + 95\theta^{(5)},\\
&259 - 53\theta^{(2)} + 60\theta^{(3)} + 133\theta^{(4)} - 35\theta^{(5)},\\
&191 - 77\theta^{(2)} - 91\theta^{(3)} - 50\theta^{(4)} + 58\theta^{(5)}
\end{aligned}$$

(where $\theta^{(k)} = \zeta_{11}^k + \zeta_{11}^{-k}$ for $k = 1, \dots, 5$). We found the maximum value of $|\alpha(\|M_{3,j}\|_F^2)|$ over all embeddings α and all j; this value proved to be 680.170 to six significant figures, so it is certainly bounded above by $\Omega = 680.18$.

Thus, in any complex embedding,

$$\begin{aligned}
\|M_3\|_2 &\leqslant \max_j \|M_{3,j}\|_2 && \text{by (4.1.3)}\\
&\leqslant \max_j \|M_{3,j}\|_F && \text{by (4.1.2)}\\
&\leqslant \sqrt{\Omega}.
\end{aligned}$$

So $\|M_4\|_2 \leqslant 11^3 \cdot 2 \cdot \sqrt{8} \cdot \sqrt{\Omega}$ by (4.1.1). Then by (4.1.2),

$$\|M_4\|_F \leqslant 11^3 \cdot 2 \cdot \sqrt{8} \cdot \sqrt{\Omega} \cdot \sqrt{50}.$$

If $\sigma_1, \dots, \sigma_{50}$ are the singular values of M_4 in a given complex embedding, then by (4.2.1) we know $\sum_{i=1}^{50} \sigma_i^2 = \|M_4\|_F^2$. Now for any k, the maximum of the function $x_1 x_2 \cdots x_k$ on the orthant $\{(x_1, \dots, x_k) \mid x_i \geqslant 0,\ \sum_{i=1}^k x_i^2 = r^2\}$ of the sphere of radius r is attained when $x_1 = \cdots = x_k = r/\sqrt{k}$, and the maximum has the value $\left(\frac{r}{\sqrt{k}}\right)^k$. So in any complex embedding, the product of the singular values of M_4 is bounded by

$$\begin{aligned}
\left(\frac{11^3 \cdot 2^{5/2} \cdot \sqrt{\Omega} \cdot \sqrt{50}}{\sqrt{50}}\right)^{50} &\approx 4.5 \cdot 10^{264}\\
&< 10^{265}.
\end{aligned}$$

The lemma now follows from the lemma in (4.3). □

(5.6) We now conclude the discussion of how we proved the main part of Theorem 2.4.

We chose a particular set $\{l_1, \dots, l_{64}\}$ of rational primes such that $\prod_{i=1}^{64} l_i > 10^{265}$. Our l_i satisfied $l_i \equiv 1 \pmod{77}$, so that each l_i splits completely

into prime ideals $L_{i,j}$ of $\mathscr{O}_{77}$—that is, $(l_i) = L_{i,1} \cdot \cdots \cdot L_{i,60}$ as an ideal in $\mathscr{O}_{77}$ (the 60 is the degree of $\mathbb{Q}(\zeta_{77})$ over $\mathbb{Q}$). Because the l_i split completely, $\mathscr{O}_{77}/L_{i,j} \cong \mathbb{F}_{l_i}$ for all i, j. For each $L_{i,j}$ we reduced M_4 mod $L_{i,j}$, obtaining a matrix over $\mathbb{F}_{l_i}$, and we computed its rank. The rank proved to be 49 for all $i = 1, \ldots, 64$ and all $j = 1, \ldots, 60$.

Assume M_4 had maximal rank. Then there would exist a maximal square submatrix of M_4 whose determinant $D \in \mathscr{O}_{77}$ was nonzero. By the preceding paragraph, D lies in the ideal $(l_1 \cdot \cdots \cdot l_{64})$ of $\mathscr{O}_{77}$. It follows that in some complex embedding α we have $|\alpha(D)| \geqslant |l_1 \cdot \cdots \cdot l_{64}| > 10^{265}$. This contradicts the lemma in (5.5). So M_4 must have rank $\leqslant 49$ after all. By the computations mod $L_{i,j}$, the rank is exactly 49.

This proves the main part of Theorem 2.4. □

(5.7) In this subsection and the next, we make a few remarks about how we checked our computations. In (5.9) we comment on the amount of computer time we used.

Many parts of the computer programs could be debugged by comparing them with mathematics done by hand. We worked out M_2, $t(h)$, and even some blocks of M_3 on paper, using the relatively simple formulas for t with respect to the basis $\mathscr{R}$, and we compared the results with the programs' output. We checked the programming of Helversen-Pasotto's complicated formula for $T^{\Lambda}(w)$ against examples done by hand. We verified this formula, both by checking her derivation of it [13, §2.5] and by writing special programs to compute $T^{\Lambda}(w)$ from first principles.

One can show directly that the solution of the GL(2) problem always has dimension $\frac{1}{24}N$, where $N = (p+1)(p-1)^2$ is as in (2.1). More precisely, we have the following result.

Proposition. *Let V be a cuspidal representation of* $\mathrm{GL}(3, \mathbb{Z})$ *for $p \geqslant 5$, and consider its restriction to* $\mathrm{GL}(2, \mathbb{Z})$ *(embedded in* $\mathrm{GL}(3)$ *in the usual way). Then*

$$\dim H_1(\mathrm{GL}(2, \mathbb{Z}), V) = \frac{1}{24}N.$$

The proposition is proved by working out a spectral sequence [7, Chapter VII] for the homology of $\mathrm{GL}(2, \mathbb{Z})$, using the well-rounded retract for $\mathrm{GL}(2, \mathbb{Z})$ described in (3.1, Remark 2).

The dimension of the computer's solution of the GL(2) problem always agreed with the dimension predicted by the proposition. Each of the $\frac{1}{2}(p-1)$ blocks of M_3 had $\frac{1}{24}N \cdot \frac{2}{p-1}$ independent columns, as predicted by the proof of the proposition. These experimental results held for all $p \leqslant 19$ and all the λ for which we ran the programs, even if the cuspidal cohomology vanished for these p and λ.

The very fact that the reduction of M_4 mod $L_{i,j}$ had less than maximal rank for $64 \cdot 60 = 3840$ different L's is evidence for the existence of the cusp form, since a generic matrix has maximal rank, and bugs tend to produce random, generic matrices. Compare the results (6.3) of the programs for other p and λ.

(5.8) In the early stages of our work we used programs which computed M_2, M_3, and M_4 directly over $\mathbb{C}$ in one particular complex embedding. In finding

M_3, the kernel was computed using a complete-pivoting Gauss-Jordan elimination algorithm [11, Algorithm 4.4-1]. We had confidence that the algorithm gave accurate results, because it is regarded as a stable algorithm and because the results we obtained with it always agreed with the Proposition in (5.7). The rank of M_4 was found using LINPACK's singular value decomposition algorithm. The computations were carried out on the Cray XMP of the Ohio Supercomputer Center. The Cray's 64-bit words provide accuracy to about 20 significant figures. To three significant figures, the fifty singular values of M_4 were found to be (in descending order, and omitting the factor of p^3)

$$7.22,\ 5.60,\ \ldots,\ 0.178,\ 0.116,\ 8.46 \cdot 10^{-13}.$$

The drop of more than 10^{-11} between the 49th and 50th singular value is overwhelming numerical evidence that M_4 has rank 49.

(5.9) The computations described in §5 took a great deal of time, almost all of which was for (5.6): computing the reductions of M_4 modulo the $64 \cdot 60 = 3840$ prime ideals $L_{i,j}$ and finding their ranks. One program had to be run 64 times, once for each l_i; we ran it on several machines simultaneously, with different l_i as input on each. Over a period of months, off and on, we did runs on seven Sun's at Harvard, on two Sun's at Ohio State, and on one Sun and a VAX 8350 at Oklahoma State. The Sun's ranged from 3/60's to a 4/260 and a SparcStation 1. A sample run on the Sparc 1, for one value l_i and about twenty of the sixty associated $L_{i,j}$, showed that the computation for a single $L_{i,j}$ required about 1212 seconds of CPU time on this machine. Thus, the whole computation of (5.6) would have taken about $3840 \cdot 1212$ CPU-sec $= 53.86$ CPU-days on the Sparc 1. It took much longer in real life, partly because most of the computers available were the slower 3/60's.

The time for the rest of the computations was miniscule by comparison. The calculations over $\mathscr{O}_{11}$, as described in (5.2–5.3), took 74.2 CPU-sec on the Sparc 1. We did not time precisely the CRAY programs described in (5.8); but for $p = 11$ and a given value of λ, the entire computation over $\mathbb{C}$ took only a minute or two in real time.

All the programming was done in Fortran (the language of the Cray).

6. Conclusion of the proof of the main theorem

We now prove and/or discuss the parts of Theorem 2.4 not covered in §5.

(6.1) When $p \neq 2$ and λ is odd, it is easy to prove the theorem. For if λ is odd, then $\Lambda(-1) = -1$, and (3.1.1) applied to $T^{\Lambda}(-I) = -I$ implies that $H_3(\mathrm{GL}(3, \mathbb{Z}), V)$ vanishes.

(6.2) Let $p = 11$ and $\lambda = 38$, and use the other notation of §2. We have obtained very convincing numerical evidence that

$$\dim H_3(\mathrm{GL}(3, \mathbb{Z}), V) = 1 \tag{6.2.1}$$

for this p and λ. We used the techniques outlined in (5.8), working over $\mathbb{C}$ in a particular complex embedding on the Cray. To three significant figures, the

fifty singular values of M_4 were found to be

$$7.62,\ 5.79,\ \ldots,\ 0.194,\ 0.158,\ 1.51 \cdot 10^{-13}.$$

As in (5.8), the drop of 10^{-12} between the 49th and 50th values is overwhelming evidence that M_4 has rank 49.

We expect that the techniques of (5.2–5.6) would prove (6.2.1) as a theorem. We have not carried out this proof, though, since it would involve even more computer time than the case $p = 11$, $\lambda = 190$ did.

(6.3) We now discuss how we proved Theorem 2.4 for the primes p other than 11 and for the nonasterisked even λ.

First, let p be one of 5, 7, 13, 17, or 19, and let λ have one of the nonasterisked even values in the statement of the theorem. We considered the cyclotomic field $K = \mathbb{Q}(\zeta_p, \zeta_{\frac{p^3-1}{\lambda}})$ over which the representation T^Λ is defined, and chose a rational prime l which splits completely in this field. We found a prime L of K over l, and we reduced M_4 (defined in (5.2–5.4)) mod L in the manner of (5.6), obtaining a matrix over $\mathbb{F}_l$. In each case, the matrix mod L proved to have maximal rank. Hence, M_4 has maximal rank, and by Theorem 3.1 the cohomology vanishes for this representation T^Λ.

The case $p = 2$ was done on paper. The case $p = 3$ was handled as in the previous paragraph, though our programs needed minor modifications because the solution to the GL(2) problem is a little different when $p = 3$. (For example, the proposition of (5.7) is false for $p = 3$.)

This proves Theorem 2.4. □

Remarks. 1. For the λ's with asterisks, we could not run our programs as we had originally written them. A prime l splits completely if and only if it is congruent to 1 modulo $p \cdot \frac{p^3-1}{\lambda}$, and in the asterisked cases the smallest such l exceeds 65536. Finding the rank of M_4 mod L would have required recoding our routines to use double-precision integers.

2. For all primes p with $5 \leqslant p \leqslant 19$ and $p \neq 11$, we computed the cohomology over $\mathbb{C}$ on the Cray by the techniques of (5.8). This confirmed Theorem 2.4, since for all such p and all even λ, the smallest singular value of M_4 was always greater than 10^{-3} and was of the same order of magnitude as the neighboring singular values. The computations over $\mathbb{C}$ worked for the asterisked λ's just as well as for the others, giving strong numerical evidence that the cuspidal cohomology vanishes for the asterisked λ's which occur for $p = 17$ or 19.

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