EXCEPTIONAL GRAPHS WITH SMALLEST EIGENVALUE $-2$ AND RELATED PROBLEMS

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Abstract. This paper summarizes the known results on graphs with smallest eigenvalue around $-2$, and completes the theory by proving a number of new results, giving comprehensive tables of the finitely many exceptions, and posing some new problems. Then the theory is applied to characterize a class of distance-regular graphs of large diameter by their intersection array.

Introduction

This paper presents a theory of graphs with smallest eigenvalue around $-2$ (in §§1 and 2, with tables in the appendix of the microfiche section) and their application to a characterization problem for distance-regular graphs (§3).

Apart from the classical line graph theorem of Cameron, Goethals, Seidel, and Shult [10]—which is introduced here in a new way by means of root lattices—and consequences of it observed by Bussemaker, Cvetković, Doob, Kumar, Rao, Seidel, Simić, Singhi, and Vijayan [8, 16, 18, 20, 21, 30, 35, 44], we obtain a number of new results, namely

(i) a classification of graphs $\Gamma$ with smallest eigenvalue $-2$ such that $\Gamma$ or its complement are edge-regular (Theorem 1.2),
(ii) a complete list of minimal graphs with smallest eigenvalue $-2$ (Theorem 1.7 and Table 3),
(iii) a complete list of minimal forbidden subgraphs for the class of graphs with smallest eigenvalue $\geq -2$ (Table 4), and
(iv) the computation of the eigenvalue gap at $-2$ (Theorem 2.4).

The importance of the eigenvalue gap is demonstrated by the characterization of a class of distance-regular graphs (folded cubes, folded half-cubes, and folded Johnson graphs of large diameters) by their intersection arrays, in the spirit of earlier work of Terwilliger [41] and Neumaier [34].

The proofs for (i)–(iv) are based on extensive computer calculations which enumerate the finitely many exceptions arising from the exceptional root lattices (or root systems) $E_6$, $E_7$, and $E_8$. We challenge the reader at several places to provide conceptional proofs of some remarkable observations deduced here from lists of graphs generated by computer. We also point out a number of open questions.

Notation. If $\Gamma$ is a graph and $S$ a set of vertices of $\Gamma$, we denote by $\Gamma \setminus S$ the
Figure 1. The two minimal forbidden line graphs with five vertices.

graph obtained by deleting from $\Gamma$ the vertices in $S$ and all edges containing a vertex of $S$. A subgraph of $\Gamma$ always refers to an induced subgraph, i.e., a graph of the form $\Gamma \setminus S$. For a vertex $\gamma \in \Gamma$, $\Gamma(\gamma)$ denotes the subgraph induced on the set of neighbors of $\gamma$, and $\gamma^\perp$ denotes the subgraph induced on the set consisting of $\gamma$ and its neighbors. The relation $\equiv$ defined by $\gamma \equiv \delta$ if and only if $\gamma^\perp = \delta^\perp$ is an equivalence relation on the set of vertices, and if we identify equivalent vertices, we obtain a reduced graph $\overline{\Gamma}$. $\Gamma$ can be recovered from $\overline{\Gamma}$ as a clique extension, i.e., by replacing each vertex $\gamma$ of $\overline{\Gamma}$ by a suitable clique $C_\gamma$, and joining the vertices of $C_\gamma$ with the vertices of $C_\delta$ when $\gamma$ and $\delta$ are adjacent. We shall draw a clique extension of $\overline{\Gamma}$ by drawing vertices of $\overline{\Gamma}$ replaced by an $i$-clique as circles with label $i$ if $i > 1$, and as black dots if $i = 1$ (cf. Figure 1). The eigenvalues of a graph $\Gamma$ are the eigenvalues of its $(0,1)$-adjacency matrix; the spectrum of $\Gamma$ is the collection of its eigenvalues (together with their multiplicities). For a general discussion of graph spectra, see the book by Cvetković, Doob, and Sachs [17]. We denote the largest eigenvalue of $\Gamma$ by $\lambda_{\max}(\Gamma)$ and the smallest eigenvalue of $\Gamma$ by $\lambda_{\min}(\Gamma)$. By interlacing (cf. [17]), we have for a subgraph $\Gamma'$ of $\Gamma$ the relations

$$\lambda_{\min}(\Gamma) \leq \lambda_{\min}(\Gamma'), \quad \lambda_{\max}(\Gamma') \leq \lambda_{\max}(\Gamma).$$

The minimal valency of a graph $\Gamma$ is denoted by $k_{\min}(\Gamma)$. A graph $\Gamma$ is called regular if every vertex has the same valency $k$, edge-regular (coedge-regular) if $\Gamma$ is regular and any two adjacent (nonadjacent) vertices have the same number $\mu$ of common neighbors, amply regular if $\Gamma$ is edge-regular and any two vertices at distance 2 have the same number of common neighbors, and strongly regular if it is edge-regular and coedge-regular.

Since isomorphic graphs have the same spectrum, we do not distinguish between different isomorphic graphs.

1. Graphs with smallest eigenvalue $\geq -2$

The well-known fact that all line graphs have smallest eigenvalue $\geq -2$ prompted a great deal of interest in the characterization of certain classes of graphs $\Gamma$ with $\lambda_{\min}(\Gamma) \geq \lambda^*$ for $\lambda^*$ around $-2$. The work done on this problem culminated in a beautiful theory of Cameron, Goethals, Seidel, and Shult [10] who related the question to root systems. Together with computer calculations by Bussemaker, Cvetković, and Seidel [8], this theory implies a complete
classification of all regular graphs with smallest eigenvalue ≥ -2, and, as noted by Doob and Cvetković [21], a classification of all graphs (whether regular or not) with smallest eigenvalue > -2. In this section we summarize these results, and give some numerical information on the exceptional graphs.

A root lattice is an additive subgroup \( L \) of \( \mathbb{R}^n \) generated by a set \( X \) of vectors such that \( (x, x) = 2 \) and \( (x, y) \in \mathbb{Z} \) for all \( x, y \in X \); here, \( (x, y) = \sum x_i y_i \) is the standard inner product in \( \mathbb{R}^n \). The vectors in \( L \) of norm \( (x, x) = 2 \) are called the roots of \( L \). A root lattice is called irreducible if it is not a direct sum of proper sublattices. Every irreducible root lattice is isomorphic to one of the lattices

\[
A_n = \{ x \in \mathbb{Z}^{n+1} | \sum x_i = 0 \} \quad (n \geq 1),
\]
\[
D_n = \{ x \in \mathbb{Z}^n | \sum x_i \text{ even} \} \quad (n \geq 4),
\]
\[
E_8 = D_8 \cup (c + D_8), \quad \text{where } c = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1),
\]
\[
E_7 = \{ x \in E_8 | \sum x_i = 0 \},
\]
\[
E_6 = \{ x \in E_7 | x_7 + x_8 = 0 \}
\]

(cf. Witt [45], Cameron et al. [10], Neumaier [33]). In terms of basis vectors \( e_1, \ldots, e_n \) of \( \mathbb{Z}^n \), the roots of \( A_n \) are the \( n(n+1) \) vectors

\[
e_i - e_j \quad (1 \leq i < j \leq n+1),
\]

those of \( D_n \) are the \( 2n(n-1) \) vectors

\[
\pm e_i \pm e_j \quad (1 \leq i < j \leq n),
\]

and those of \( E_8 \) are the \( 240 = 112 + 128 \) vectors

\[
\pm e_i \pm e_j \quad (1 \leq i < j \leq 8)
\]

and, with an even number of + signs,

\[
\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8).
\]

From this, one finds that \( E_7 \) contains \( 126 = 56 + 70 \) roots and \( E_6 \) contains \( 72 = 32 + 40 \) roots.

If \( \Gamma \) is a connected graph with \( \lambda_{\text{min}}(\Gamma) > -2 \) and adjacency matrix \( A \), then \( G = A + 2I \) is a symmetric positive semidefinite matrix. Thus, \( G \) is the Gram matrix of a set \( X \) of vectors of \( \mathbb{R}^n \), i.e., there is a bijection \( \gamma : \Gamma \rightarrow X \) such that

\[
(\gamma, \delta) = \begin{cases} 
2 & \text{if } \gamma = \delta, \\
1 & \text{if } \gamma, \delta \text{ are adjacent,} \\
0 & \text{otherwise.}
\end{cases}
\]

Such a mapping is called a spherical \((2, 1, 0)\)-representation, and in this paper simply a representation of \( \Gamma \). The additive subgroup \( L^+(\Gamma) \) generated by \( X \) is a root lattice (whose isomorphism type depends on \( \Gamma \) but not on \( X \)), and since \( \Gamma \) is connected, \( L^+(\Gamma) \) is irreducible. This implies the basic observation of Cameron et al. [10] that every connected graph \( \Gamma \) with \( \lambda_{\text{min}}(\Gamma) \geq -2 \) has a representation by roots of \( A_n \) \((n \geq 1)\), \( D_n \) \((n \geq 4)\), or \( E_n \) \((n = 6, 7, 8)\). Conversely, if \( \Gamma \) is represented by roots of \( A_n \), \( D_n \), or \( E_n \), then the Gram matrix \( A + 2I \) of the image of \( \Gamma \) is positive semidefinite, so that \( \lambda_{\text{min}}(\Gamma) \geq -2 \).
The line graph of a graph \( \Delta \) is the graph \( L(\Delta) \) whose vertices are the edges of \( \Delta \), two edges being adjacent if they intersect. Generalized line graphs \( L(\Delta, a_1, \ldots, a_m) \), introduced by Hoffman [25], are obtained from the line graph \( L(\Delta) \) of a graph \( \Delta \) with vertex set \( \{1, \ldots, m\} \) by adding the vertices \((i, \pm l) \) \((i = 1, \ldots, m; l = 1, \ldots, a_i)\) and joining \((i, l)\) with all vertices \(i, j\) of \( L(\Delta) \) and all \((i, l')\), \(l' \neq \pm l\). We get a representation of a generalized line graph by representing the vertices \(ij\) of \( L(\Delta) \) by \(e_i + e_j\) and the adjoined vertices \((i, \pm l)\) by \(e_i \pm e_i(l, l')\), \(l = 1, \ldots, a_i\), where the \(e_i, e_i(l, l')\) form a set of orthonormal vectors. Moreover, if \( \Delta \) is bipartite with bipartite parts \( A, B \), then \( L(\Delta) \) has also a representation by roots of \( A_m - 1 \) obtained by representing an edge \(ij\) with \(i \in A, j \in B\) by \(e_i - e_j\).

By analyzing the possible representations by roots of \( A_n \) and \( D_n \), Cameron, Goethals, Seidel, and Shult [10] arrived at the following result.

1.1. **Theorem.** Let \( \Gamma \) be a connected graph with smallest eigenvalue \( \geq -2 \). Then one of the following holds:

(i) \( \Gamma \) is a generalized line graph.

(ii) \( \Gamma \) has a representation by roots of \( E_8 \). The number \( v \) of vertices, and the average valency \( k \), are restricted by \( v \leq \min(36, 2k + 8) \). Moreover, every vertex has valency at most 28.

An as yet unsolved problem is the characterization of the graphs under (ii). Since every subgraph of such a graph is again represented by roots of \( E_8 \), it suffices to determine the (finitely many) maximal graphs under (ii) which are not generalized line graphs. It seems that most graphs under (ii) can be obtained by switching.

Switching a graph \( \Gamma \) with respect to a set \( S \) of vertices (or with respect to its complement \( S^c \)) is the operation of removing all edges of \( \Gamma \) between \( S \) and \( S^c \) and adding the new edges \( \gamma_1, \gamma_2 \) \((\gamma_1 \in S, \gamma_2 \in S^c, \gamma_1 \not\sim \gamma_2) \) (cf. Seidel [36]). If \( \Gamma^1 \) is obtained from \( \Gamma \) by switching with respect to \( S_1 \), and \( \Gamma^2 \) is obtained from \( \Gamma^1 \) by switching with respect to \( S_2 \), then \( \Gamma^2 \) can be directly obtained from \( \Gamma \) by switching with respect to the symmetric difference \((S_1 \cap S_2) \cup (S_1^c \cap S_2^c) \). Therefore, switching defines an equivalence relation on the set of graphs with a given vertex set. If \( \Gamma \) is the line graph of a graph \( \Delta \) with vertex set \( \{1, 2, \ldots, 8\} \), then the graph \( \Gamma' \) obtained from \( \Gamma \) by switching with respect to \( S \) can be represented in \( E_8 \) by the roots \(e_i + e_j\) (if \(ij\) is an edge \( \not\in S \)) and \(c - e_i - e_j\) (if \(ij\) is an edge \( \in S \); here, \(c = \frac{1}{2}(e_1 + \cdots + e_8)\)). Therefore, \( \Gamma' \) has smallest eigenvalue \( \geq -2 \). The maximal graphs obtainable from this construction are the graphs which are switching-equivalent to the triangular graph \( T(8) \), the line graph of the complete graph on eight vertices.

A graph with 36 vertices, maximal valency 28, and smallest eigenvalue \(-2\) which is not a generalized line graph can, e.g., be obtained by adding to \( K_8 + L(K_8) \) edges joining \(i \in K_8\) with \(jk \in L(K_8)\) whenever \(i \not\in \{j, k\}\); a \((2, 1, 0)\)-representation in \( E_8 \) is given by the vectors \(\frac{1}{2}(f_1 + \cdots + f_8) - f_i\) \((i \leq 8)\) and \(f_i + f_j\) \((i < j \leq 8)\), where \(f_1, \ldots, f_8\) are obtained from \(e_1, \ldots, e_8\) by reversing the sign of one \(e_i\). Thus, examples satisfying equality in (ii) of the theorem exist.

If we restrict ourselves to regular graphs, sharper results are possible. Bussemaker, Cvetković, and Seidel [8], supported by a computer, used this theorem to show that, up to isomorphism, there are precisely 187 regular graphs with
smallest eigenvalue $-2$ (and none with $\lambda_{\min}(\Gamma) > -2$) which are not generalized line graphs, namely

4 graphs generating $E_6$ (nos. 5, 185–187 in [8]),
24 graphs generating $E_7$ (nos. 19, 69, 164–184 in [8]),
159 graphs generating $E_8$ (the remaining ones).

Reference [8] also contains explicit adjacency matrices and representations by roots of $E_8$. To simplify the application of their results, we describe here the most important ones, and give (in Table 1) a list of relevant numerical invariants of these graphs. As shown in [8], all 187 graphs are subgraphs of the Gosset graph $E_7(1)$ with 56 vertices $e_i + e_j$, $c - e_i - e_j$ ($1 \leq i < j \leq 8$), two vertices being adjacent if their inner product is 1. (Note that this defines not a representation in our sense, since $(e_i + e_j, c - e_i - e_j) = -1$; indeed, the smallest eigenvalue of $E_7(1)$ is $-9$. However, all subgraphs not containing such an antipodal pair of vertices are switching-equivalent to a line graph and hence have smallest eigenvalue $\geq -2$.) The Gosset graph is the skeleton of the Gosset polytope 3_21 (cf. Coxeter [15]), and is related to the 28 bitangents of a quartic surface (cf. Dickson [32]). A modern description of the structure of $E_7(1)$ is given by Taylor [39] in terms of a regular two-graph with 28 points. We are interested in the graphs $E_n(1)$ ($n = 1, \ldots, 6$), the subgraphs of $E_7(1)$ induced on the set of common neighbors of $e_i + e_8$ ($i = n + 1, \ldots, 7$), and some other graphs.

(i) The Schläfli graph $E_6(1)$ (no. 184 in Table 1) has the 27 vertices $e_i + e_7$, $e_i + e_8$ ($i \leq 6$), $c - e_i - e_j$ ($i < j \leq 6$) and valency 16. The graph $E_6(1)$ is the complement of the point graph of the generalized quadrangle of order $(2, 4)$ with 27 points, and is related to the 27 lines on a cubic surface (see Baker [1]; one easily recognizes a double six in the description given).

(ii) The Clebsch graph $E_5(1)$ (no. 187 in Table 1) has the 16 vertices $e_6 + e_7$, $e_i + e_8$ ($i \leq 5$), $c - e_i - e_j$ ($1 < j \leq 5$) and valency 10. The complement is a triangle-free graph obtained by identifying antipodal points of the 5-dimensional cube. The graph $E_5(1)$ contains two regular proper subgraphs which are not line graphs (nos. 185, 186 in Table 1); namely a graph with the 12 vertices $e_i + e_8$ ($i = 2, 3, 4$), $c - e_i - e_j$ ($i < j \leq 5$, $(i, j) \neq (1, 5)$) and valency 7, and a graph with the eight vertices $e_6 + e_7$, $e_i + e_8$ ($i = 2, 3, 4$), $c - e_i - e_{i+1}$ ($i \leq 4$) and valency 4 (cf. Figure 2).

(iii) The graph $E_4(1)$ is isomorphic to the triangular graph $T(5)$ with ten vertices and valency 6.

(iv) The Petersen graph (no. 5 in Table 1) has ten vertices and valency 3. It is obtained from the triangular graph $T(5)$ with vertices $e_i + e_j$ ($i < j \leq 5$) by switching with respect to $\{e_i + e_j | i, j \leq 5, j \equiv i + 1 \ (\text{mod} \ 5)\}$. This graph is strongly regular with parameters $(\nu, k, \lambda, \mu) = (10, 3, 0, 1)$.

**Figure 2.** A subgraph of the Clebsch graph.
(v) The Shrikhande graph $L_2^*(4)$ (no. 69 in Table 1) has 16 vertices and valency 6. It is obtained from the $(4 \times 4)$-grid $L_2(4)$ with vertices $e_i + e_j$ ($i \leq 4 < j$) by switching with respect to $\{e_i + e_{i+4} \mid i \leq 4\}$. This graph is strongly regular with the same parameters $(\nu, k, \lambda, \mu) = (16, 6, 2, 2)$ as $L_2(4)$ (cf. Shrikhande [37]). It is a quotient of the triangular lattice in $\mathbb{R}^2$.

(vi) The three Chang graphs $T'(8)$, $T''(8)$, $T'''(8)$ (nos. 161–163 in Table 1) have 28 vertices and valency 12. They are obtained from the triangular graph $T(8)$ with vertices $e_i + e_j$ ($i < j < 8$) by switching with respect to one of the sets

$$\{e_i + e_{i+4} \mid 1 \leq i \leq 4\} \text{ for } T'(8),$$

$$\{e_i + e_j \mid 1 \leq i \leq 8, \ j \equiv i + 1 \pmod{8}\} \text{ for } T''(8),$$

$$\{e_1 + e_2, e_2 + e_3, e_3 + e_1, e_4 + e_5, e_5 + e_6, e_6 + e_7, e_7 + e_8, e_8 + e_4\}$$

for $T'''(8)$.

These graphs are all strongly regular with the same parameters $(\nu, k, \lambda, \mu) = (28, 12, 6, 4)$ as $T(8)$ (cf. Chang [11, 12], and Seidel [36] for equivalence under switching).

The information collected by Bussemaker et al. [8] about the 187 regular exceptional graphs can be summarized together with the analysis of regular generalized line graphs in Cameron et al. [10] in the following theorem.

1.2. Theorem. Let $\Gamma$ be a connected regular graph with $v$ points, valency $k$, and smallest eigenvalue $\lambda \geq -2$. Then one of the following holds:

(i) $\Gamma$ is the line graph of a regular or a bipartite semiregular connected graph $\Delta$.

(ii) $v = 2(k + 2) \leq 28$, and $\Gamma$ is a subgraph of the graph $E_7(1)$, switching-equivalent to the line graph of a graph $\Delta$ on eight vertices, where all valencies of $\Delta$ have the same parity (graphs nos. 1–163 in Table 1).

(iii) $v = \frac{3}{2}(k + 2) \leq 27$, and $\Gamma$ is a subgraph of the Schlafli graph (graphs nos. 164–184 in Table 1).

(iv) $v = \frac{4}{3}(k + 2) \leq 16$, and $\Gamma$ is a subgraph of the Clebsch graph (graphs nos. 185–187 in Table 1).

(v) $v = k + 2$, and $\Gamma \cong K_{m \times 2}$ for some $m \geq 3$.

Moreover, $L^+(\Gamma) \cong A_n$ if and only if (i) holds with a bipartite graph $\Delta$ with $n + 1$ vertices, and $L^+(\Gamma) \cong D_n$ if and only if either (i) holds with a graph $\Delta$ with $n$ vertices with is not bipartite or (v) holds with $m = n − 1$.

New and computer-free proofs of Theorems 1.1 and 1.2 are contained in Brouwer, Cohen and Neumaier [6].

A glance through Table 1, together with a straightforward analysis of line graphs, leads to the following application of the preceding result, which generalizes the characterization of strongly regular graphs with smallest eigenvalue $-2$ by Seidel [36].

1.3. Theorem. Let $\Gamma$ be a connected regular graph with smallest eigenvalue $-2$.

(i) If $\Gamma$ is strongly regular, then $\Gamma$ is a triangular graph $T(n)$, a square grid $n \times n$ (also called a lattice graph $L_2^2(n)$), a complete multipartite graph $K_{n \times 2}$, or one of the graphs of Petersen, Clebsch, Schlafli, Shrikhande, or Chang.
(ii) If $\Gamma$ is edge-regular, then $\Gamma$ is strongly regular or the line graph of a regular triangle-free graph.

(iii) If $\Gamma$ is amply regular, then $\Gamma$ is strongly regular or the line graph of a regular graph of girth $\geq 5$.

(iv) If $\Gamma$ is coedge-regular, then $\Gamma$ is strongly regular, an $(m \times n)$-grid, or one of the two regular subgraphs of the Clebsch graph with eight and 12 vertices, respectively.

The multiplicity of the eigenvalue $-2$ of a graph with $\lambda_{\text{min}}(\Gamma) \geq -2$ can be found quite easily from the following results of Doob [19] (case (i)) and Cvetković, Doob, and Simić [18] (case (ii)).

1.4. Theorem. Let $\Gamma$ be a connected graph with $\lambda_{\text{min}}(\Gamma) \geq -2$.

(i) If $\Gamma$ is the line graph of a graph $\Delta$ with $n$ vertices and $e$ edges, then $-2$ is an eigenvalue of $\Gamma$ with multiplicity $e - n + 1$ if $\Delta$ is bipartite, and $e - n$ otherwise.

(ii) If $\Gamma = L(\Delta; a_1, \ldots, a_n)$ ($\sum a_i > 0$) is a generalized line graph of a graph $\Delta$ with $n$ vertices and $e$ edges, then $-2$ is an eigenvalue of multiplicity $e - n + \sum a_i$.

(iii) If $\mathbb{L}^+(\Gamma) \cong E_n$ ($n = 6, 7, 8$), then $-2$ is an eigenvalue of multiplicity $|\Gamma| - n$.

The case when the multiplicity of $-2$ is zero corresponds to the graphs $\Gamma$ with $\lambda_{\text{min}}(\Gamma) > -2$. Since $\lambda_{\text{min}}(\Gamma) > -2$ implies that $A + 2I$ is positive definite, so that $\Gamma$ is represented by a linearly independent set of roots, we get the following results of Doob and Cvetković [21].

1.5. Theorem. Let $\Gamma$ be a connected graph with $\lambda_{\text{min}}(\Gamma) > -2$. Then $\Gamma$ is one of the following cases:

(i) The line graph of a connected graph without cycles of even length and with at most one cycle of odd length.

(ii) The generalized line graph $L(\Delta; 1, 0, \ldots, 0)$ obtained from the line graph of a tree $\Delta$ by adding two nonadjacent vertices $\infty^+, \infty^-$ which are adjacent with all edges of $\Delta$ containing a fixed vertex $\infty$ of $\Delta$.

(iii) A graph represented by a set of $n \in \{6, 7, 8\}$ linearly independent roots generating $E_n$.

1.6. Corollary. A connected regular graph with smallest eigenvalue $> -2$ is a complete graph or a polygon with an odd number of vertices.

Calculations of the first author (quoted in [21]) imply that, up to isomorphism, there are precisely 573 graphs of the form (iii), namely

- 20 graphs with six vertices generating $E_6$,
- 110 graphs with seven vertices generating $E_7$,
- 443 graphs with eight vertices generating $E_8$.

Their adjacency matrices and smallest eigenvalues are listed in Table 2. The 20 graphs with six vertices generating $E_6$ are drawn in Figure 3 (see next page). It is a useful fact that every graph with $n$ vertices generating $E_n$ ($n = 7, 8$) contains a subgraph with $n - 1$ vertices generating $E_{n-1}$. It would be interesting to have a simple explanation of this fact.
Figure 3. The graphs with six vertices generating $E_6$.
($2.i$ is graph number $i$ in Table 2; $G_i$ is the notation of [18].) $G_{12} - G_{17}$ are the minimal forbidden line graphs with six vertices.

As another consequence of Theorem 1.4 we determine the minimal graphs with smallest eigenvalue $-2$. The proof is straightforward and left to the reader.

1.7. **Theorem.** Let $\Gamma$ be a connected graph with $\lambda_{\text{min}}(\Gamma) = -2$ such that $\lambda_{\text{min}}(\Gamma') > -2$ for all proper subgraphs $\Gamma'$ of $\Gamma$. If $\Gamma$ is a generalized line
Figure 4. The minimal generalized line graphs with smallest eigenvalue $-2$ and associated eigenvectors ($C_i$ is a cycle with $i \geq 3$ vertices, $P_i$ a path with $i \geq 1$ vertices).

There are precisely 777 minimal graphs $\Gamma$ with smallest eigenvalue $-2$ which are not generalized line graphs, namely

12 graphs with seven vertices generating $E_6$,
79 graphs with eight vertices generating $E_7$,
686 graphs with nine vertices generating $E_8$.

Their adjacency matrices are listed in Table 3 together with an eigenvector belonging to the eigenvalue $-2$, normalized such that its absolutely smallest entries have the value $\pm 1$. It is a useful fact that the normalized eigenvectors (belonging to $\lambda = -2$) of all minimal graphs with smallest eigenvalue $-2$ are integral, and it implies that one can delete a vertex whose normalized eigenvector coefficient is $\pm 1$ without changing the lattice generated. It would be interesting to have a simple explanation of this fact.

A reader who wants to check the information given for the exceptional graphs in Theorem 1.5 and Theorem 1.7 can use the fact that a quadrangle has smallest eigenvalue $-2$; thus it is sufficient to check all graphs $\Gamma$ with $v \leq 9$ vertices and without quadrangles for their smallest eigenvalue, and if $\lambda_{\text{min}}(\Gamma) = -2$ to determine the multiplicity of $-2$ ($\Gamma$ is minimal if and only if $-2$ is a simple eigenvalue and the corresponding eigenvector contains no zero entry).
2. Minimal forbidden subgraphs

Let $\mathcal{F}$ be a class of graphs such that if $\Gamma$ is in $\mathcal{F}$ then every subgraph of $\Gamma$ is also in $\mathcal{F}$. A minimal forbidden subgraph for $\mathcal{F}$ is a graph $\Gamma \notin \mathcal{F}$ all of whose proper subgraphs are in $\mathcal{F}$. A complete list of minimal forbidden subgraphs is a list $\mathcal{F}^*$ of pairwise nonisomorphic minimal forbidden subgraphs for $\mathcal{F}$ such that every graph $\Gamma \notin \mathcal{F}$ contains a subgraph isomorphic to some graph of $\mathcal{F}^*$. Thus, $\Gamma \in \mathcal{F}$ if and only if $\Gamma$ contains no subgraph isomorphic to some graph of $\mathcal{F}^*$; in particular, if $\mathcal{F}^*$ is a known finite list, then we have an obvious finite algorithm for deciding whether a given graph is in $\mathcal{F}$ or not.

A complete list of minimal forbidden subgraphs for the class $\mathcal{L}$ of line graphs has been found by Beineke [3] (who also gives credit to unpublished work by N. Robertson). The list $\mathcal{L}^*$ consists of nine graphs, the 3-claw $K_{1,3}$ (with four vertices), the two graphs drawn in Figure 1 (with five vertices), and the graphs $G_1^2$-$G_1^7$ in Figure 3 (with six vertices).

A complete list of minimal forbidden subgraphs for the class $\mathcal{L}_0$ of generalized line graphs has been found independently by Rao, Singhi, and Vijayan [35] and Cvetković, Doob, and Simić [18]. The list $\mathcal{L}_0^*$ consists of 31 graphs, namely the 20 graphs drawn in Figure 3 and the 11 graphs drawn in Figure 6. Their adjacency matrices and smallest eigenvalues are given as the first 20 entries of Table 2 and the first 11 entries of Table 4.

We discuss some properties of the list $\mathcal{L}_0^*$.

1. The minimal forbidden subgraphs $\Gamma$ for $\mathcal{L}_0$ with $\lambda_{\min}(\Gamma) > -2$ are precisely the graphs represented by a set of linearly independent generators for the lattice $E_6$. Indeed, $L^+ (\Gamma) \not\subseteq A_n$ or $D_n$, since $\Gamma$ is not a generalized line graph. Moreover, $L^+ (\Gamma) \not\subseteq E_7$ or $E_8$, since (as observed above) any set of linearly independent generators for $E_7$ and $E_8$ contains a subset generating $E_6$. 

**Figure 5.** The minimal graphs with smallest eigenvalue $-2$ and $v \leq 6$ vertices, and associated eigenvectors.
2. There is no minimal forbidden subgraph for $\mathcal{L}_0$ with $\lambda_{\text{min}}(\Gamma) = -2$. Indeed, as observed above, the minimal graphs $\Gamma$ with smallest eigenvalue $-2$ have a proper subgraph generating the same lattice as $\Gamma$. (Note that the argument given in Cvetković et al. [18, Corollary 4.2] to prove $\lambda_{\text{min}}(\Gamma) \neq -2$ for $\Gamma \in \mathcal{L}_0^*$ is incorrect, since it does not cover the case where some $\Gamma \setminus \{y\}$ is disconnected or generates $A_n$; however, their argument can be replaced by the simple fact that such a $\Gamma$ would have at most nine vertices and thus is ruled out by McKay’s computer search mentioned in Proposition 4.5 of [18].)

3. The set of minimal forbidden subgraphs for $\mathcal{L}_0$ with $\lambda_{\text{min}}(\Gamma) < -2$ coincides with the set of minimal graphs with smallest eigenvalue $< -2$ and at
most six vertices. Indeed, if $\Gamma$ is a graph with $\lambda_{\min}(\Gamma) < -2$ and at most six vertices, then $\Gamma \setminus \{y\}$ generates a lattice of dimension $\leq 5$, hence $A_n$ or $D_n$, so that each $\Gamma \setminus \{y\}$ is a generalized line graph and $\Gamma \in \mathcal{L}_0^*$. However, the remarkable fact that there are no graphs in $\mathcal{L}_0$ with $\lambda_{\min}(\Gamma) < -2$ and more than six vertices has not yet found a simple explanation.

4. Every minimal forbidden subgraph for $\mathcal{L}_0$ with $\lambda_{\min}(\Gamma) < -2$ contains one of the minimal graphs with smallest eigenvalue $-2$ and $\leq 5$ vertices (cf. Figure 5). Again, a simple explanation is missing.

Rao et al. [35] observed that the complete list of minimal forbidden subgraphs for the class $\mathcal{G}_{-2}$ of graphs with smallest eigenvalue $\geq -2$ is finite, since $\mathcal{L}_0^*$ is finite and there are only finitely many graphs in $\mathcal{L}_{-2} \setminus \mathcal{L}_0^*$ (by Theorem 1.1). In particular, $\mathcal{G}_{-2}^* \setminus \mathcal{L}_0^*$ consists of graphs with at most 37 vertices. Kumar, Rao, and Singhi [30] improved this estimate by showing that the maximal number of vertices of a graph in $\mathcal{G}_{-2}^*$ is ten. (Note, however, that the graph with ten vertices they give is not in $\mathcal{G}_{-2}^*$.) They also determine the graphs in $\mathcal{G}_{-2}^*$ with at most seven vertices (see Figures 6 and 7), but incorrectly state that $\mathcal{G}_{-2}^*$ contains more than 100 graphs with eight vertices. Their complicated arguments were simplified in Vijayakumar [44]. We shall give a new proof of the results in [35] and [30], together with a complete list $\mathcal{G}_{-2}^*$, based on the following variation of Lemma 4.3 of Cvetković, Doob, and Simić [18].

2.1. **Proposition.** Let $\Gamma$ be a minimal forbidden subgraph for the class $\mathcal{F}_{-m}$ of graphs with smallest eigenvalue $\geq -m$ ($m \geq 1$). Then for any two distinct vertices $\gamma, \delta \in \Gamma$, the graph $\Gamma \setminus \{\gamma, \delta\}$ has smallest eigenvalue $> -m$.

**Proof.** Let $v$ be the number of vertices of $\Gamma$, and denote by $p(x)$ and $p_\alpha(x)$ ($\alpha \in \Gamma$) the characteristic polynomials of $\Gamma$ and $\Gamma \setminus \{\alpha\}$, respectively. By Clarke [14], the derivative $p'(x)$ can be expressed as $p'(x) = \sum_{\alpha \in \Gamma} p_\alpha(x)$. Since $\Gamma \in \mathcal{G}_{-m}$, all proper subgraphs of $\Gamma$ have smallest eigenvalue $\geq -m$; in particular, $(-1)^{v-1} p_\alpha(x)$ is positive for all $x < -m$. Hence, $(-1)^{v-1} p(x)$ is strictly increasing for $x < -m$, and since $\lambda_{\min}(\Gamma) < -m$, it follows that $\lambda_{\min}(\Gamma)$ is a simple eigenvalue of $\Gamma$ and all other eigenvalues are $> -m$.

Now suppose that $\Gamma = \Gamma \setminus \{\gamma, \delta\}$ has smallest eigenvalue $\leq -m$ (and hence equal to $-m$). Let $z = (z_\alpha | \alpha \in \Gamma')$ be a corresponding eigenvector, and denote by $x_{c,d}$ the vector $x = (x_\alpha | \alpha \in \Gamma)$ with $x_\gamma = c$, $x_\delta = d$, $x_{\alpha} = z_\alpha$ for $\alpha \in \Gamma'$. Writing $A$ for the adjacency matrix of $\Gamma$, we have $x_{c,d}^T (A + mI)x_{c,d} = 0$, and therefore

$$x_{c,0}^T (A + mI)x_{c,0} = c^2 m + 2c \sum_{\alpha \in \Gamma \setminus \{\gamma, \delta\}} x_\alpha.$$ 

Since $\Gamma \setminus \{\gamma\}$ has smallest eigenvalue $\geq -m$, this expression must be nonnegative for all $c \in \mathbb{R}$, and this is possible only if $\sum_{\alpha \in \Gamma \setminus \{\gamma, \delta\}} x_\alpha = 0$. By the same reasoning we find that $\sum_{\alpha \in \Gamma \setminus \{\gamma, \delta\}, \alpha \neq \gamma} x_\alpha = 0$. Now, by construction of $z$, we find the relation $(A + mI)x_{c,0} = 0$, which is impossible, since $-m$ is not an eigenvalue of $\Gamma$. Therefore, the smallest eigenvalue of $\Gamma \setminus \{\gamma, \delta\}$ is $> -m$. □

In the special case $-m = -2$, this result can be combined with the results of §1 and yields the following restrictions on graphs in $\mathcal{G}_{-2}^*$. 

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2.2. **Theorem.** Let $\Gamma$ be a graph with $v$ vertices, and suppose that $\Gamma$ is a minimal forbidden subgraph for the class $F_{-2}$ of graphs with smallest eigenvalue $\geq -2$. Then every subgraph of $\Gamma$ with smallest eigenvalue $-2$ has $v-1$ vertices, and one of the following holds:

**Figure 7.** The minimal graphs with smallest eigenvalue $<-2$ and seven vertices, and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue $-2$. (4.1 is graph number 1 in Table 4.)
(i) \( v \leq 10 \), and there exists a vertex \( \gamma \in \Gamma \) such that \( \Gamma \setminus \{\gamma\} \) is a minimal graph with smallest eigenvalue \(-2\).

(ii) \( v \in \{7, 8, 9\} \), \( \lambda_{\min}(\Gamma \setminus \{\gamma\}) > -2 \) for every vertex \( \gamma \in \Gamma \), and for some \( \gamma \in \Gamma \), the graph \( \Gamma \setminus \{\gamma\} \) has a representation by \( n = v - 1 \) linearly independent roots generating \( E_n \).

**Proof.** We showed already that subgraphs with smallest eigenvalue \(-2\) have \( v - 1 \) vertices. To prove the remainder, we distinguish two cases.

**Case 1.** All proper subgraphs of \( \Gamma \) with \( v - 1 \) vertices are generalized line graphs. Then \( \Gamma \) is a minimal forbidden subgraph for \( \mathcal{L}_0 \) with smallest eigenvalue \(< -2\), and hence one of the graphs of Figure 6. This implies that \( \Gamma \) satisfies (i).

**Case 2.** The graph \( \Gamma \) contains a subgraph \( \Gamma \setminus \{\delta\} \) which is not a generalized line graph. Then \( L^+(\Gamma \setminus \{\delta\}) \cong E_{v-1} \); in particular, \( \Gamma \setminus \{\gamma, \delta\} \), having smallest eigenvalue \( > -2 \), is represented by \( v - 2 \) linearly independent roots of \( E_n \subseteq E_8 \), so that \( v \leq 10 \). If \( \Gamma \) contains a vertex \( \gamma \in \Gamma \) such that \( \lambda_{\min}(\Gamma \setminus \{\gamma\}) = -2 \), then \( \Gamma \setminus \{\gamma\} \) is a minimal graph with this property and (i) holds. Otherwise, \( \Gamma \setminus \{\delta\} \) is represented by \( v - 1 \) linearly independent roots generating \( E_{v-1} \), and \( \Gamma \) satisfies (ii). \( \square \)

In particular, a comparison with Figure 5 yields:

**2.3. Corollary.** Let \( \Gamma \) be a graph in \( \mathcal{G}_{-2}^* \) with \( v \) vertices. If \( v \geq 5 \), then \( \Gamma \) contains no quadrangle; if \( v \geq 6 \), then \( \Gamma \) contains no subgraph of the form \( M_i \) \((i \leq 3)\); and if \( v \geq 7 \), then \( \Gamma \) contains no subgraph of the form \( M_i \) \((i \leq 7)\).

Theorem 2.2 and the corollary now allow a reasonably fast determination of a complete list of forbidden subgraphs for \( \mathcal{G}_{-2} \) by computer. We have already seen that a graph \( \Gamma \in \mathcal{G}_{-2}^* \) with at most six vertices is a minimal forbidden subgraph and hence one of the 11 graphs in Figure 6. For graphs with more than six vertices, the fact that \( \Gamma \) contains no quadrangle drastically restricts the possibilities for extending subgraphs of \( \Gamma \) so that a systematic extension process together with checks on the smallest eigenvalues of \( \Gamma \) and the \( \Gamma \setminus \{\gamma\} \) yields a complete list in a reasonable time. (Several earlier trials to get a complete list turned out to be much too time consuming. The breakthrough was when Aart Blokhuis noticed that no minimal forbidden subgraph with six or seven vertices contained a quadrangle. After further experiments, this finally led to the corollary and then to the above theorem.)

The result of the computer search was that a complete list of minimal forbidden subgraphs for the class \( \mathcal{G}_{-2} \) of graphs with smallest eigenvalue \( \geq -2 \) consists of 1812 graphs; cf. the following statistics \((\# = \text{number of graphs in} \mathcal{G}_{-2}^* \text{with} \ v \text{vertices})\).

\[
\begin{array}{cccccccc}
\hline
v & 5 & 6 & 7 & 8 & 9 & 10 & \text{total} \\
\hline
\# & 3 & 8 & 14 & 67 & 315 & 1405 & 1812 \\
\hline
\end{array}
\]

The adjacency matrices and smallest eigenvalues of the 1812 graphs in \( \mathcal{G}_{-2}^* \) are listed in Table 4. The graphs in \( \mathcal{G}_{-2}^* \) with up to seven vertices are drawn in Figures 6 and 7.
Figure 8. Minimal triangle-free graphs with smallest eigenvalue $< -2$ and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue $-2$. (4.₁ is graph number $i$ in Table 4.)

An inspection of Table 4 shows that there are only 14 graphs in $G_{-2}$ without triangles; they are drawn in Figure 8. The completeness of the list of triangle-free graphs in $G_{-2}$ can be established easily by hand on the basis of Theorem 2.2 and its corollary.

2.4. **Theorem** (Doob [20]). Let $\Gamma$ be a graph with $\lambda_{\min}(\Gamma) < -2$. Then $\Gamma$ contains a minimal graph with smallest eigenvalue $-2$ and at most nine vertices.
• ••       < (n> 1 vertices)

♦••       • (n > 4 vertices)

Figure 9. The graphs with largest eigenvalue < 2.

Proof. The graph Γ contains a subgraph isomorphic to a graph in \( \mathcal{G}_2 \). Inspection of the list of Table 4 shows that each such graph contains a proper subgraph with smallest eigenvalue \(-2\). □

Note that Doob [20] proved the theorem in a slightly different way, relying on computer calculations of Brendan McKay. It would be very interesting to have a computer-free proof of this result. By Theorems 2.2 and 2.4, the graphs in \( \mathcal{G}_2 \) can be characterized as the graphs obtained by adding to a minimal graph with smallest eigenvalue \(-2\) and \( \leq 9 \) vertices a new vertex \( \infty \) and edges containing \( \infty \) in such a way that the eigenvector coefficients (with respect to the eigenvalue \(-2\)) of the neighbors of \( \infty \) do not sum up to zero. This follows from a similar argument as in the second part of the proof of Proposition 2.1.

Let us digress for a moment and consider some related work on the largest eigenvalue of a graph. Denote by \( \mathcal{G}_m \) \( (m \geq 1) \) the class of graphs \( Y \) with largest eigenvalue \( \lambda_{\max}(Y) \leq m \). The graphs in \( \mathcal{G}_2 \), listed in Figures 9 and 10, have been determined by Smith [38] (cf. also Lemmens and Seidel [31]); they are precisely the spherical and affine Dynkin diagrams for so-called simply-laced root systems (cf. Hiller [24]).

A complete list of minimal forbidden subgraphs for \( \mathcal{G}_2 \) is easily established and can be deduced from the list of minimal hyperbolic Dynkin diagrams given in Chein [13] and Koszul [29]. The list \( \mathcal{G}_2^\# \) consists of 18 graphs, namely the 13 bipartite graphs of Figure 8 (4.410 is not bipartite; \( \lambda_{\min}(\Gamma) = -\lambda_{\max}(\Gamma) \) if \( \Gamma \) is bipartite) and five further graphs drawn in Figure 11 which are not bipartite. The maximal number of vertices of graphs in \( \mathcal{G}_2^\# \) is ten. One can read off from Figures 8–11 that every graph not containing a graph with largest eigenvalue \( 2 \) is contained in such a graph (cf. Doob [20]).

For \( \tilde{m} = \sqrt{2 + \sqrt{5}} \approx 2.058171 \) \( = \tau^{3/2} = \tau^{1/2} + \tau^{-1/2} \), where \( \tau = (1 + \sqrt{5})/2 \), \( \mathcal{G}_\tilde{m} \) has been determined by Cvetković, Doob, and Gutman [16] and Brouwer and Neumaier [7]. \( \mathcal{G}_\tilde{m} \) consists of the paths, polygons, the trees \( Y_{ij1}, \ Y_{i22}, \ Y_{332} \) (where \( Y_{ijk} \) is the \( Y \)-shaped tree with a unique vertex of valency 3, the deletion of which leaves three disjoint paths with \( i, j, \) and \( k \) vertices), and the trees \( \Pi_{ijk} \) (\( \pi \)-shaped, consisting of a path with \( i+j+k-1 \) vertices and two further vertices of valency 1 adjacent to the \( i \) th and \( (i+j) \)th vertex of the path), where \( \tau^j \geq (\tau^i-2)(\tau^k-2) \) (i.e., \( j \geq i+k-\epsilon_{ik} \), where \( \epsilon_{23} = \epsilon_{32} = 4 \).
GRAPHS WITH SMALLEST EIGENVALUE −2

(n ≥ 3 vertices)

(n ≥ 5 vertices)

Figure 10. The graphs with largest eigenvalue 2, with a corresponding eigenvector.

Figure 11. The minimal graphs with largest eigenvalue > 2 which are not bipartite, and their largest eigenvalue. A vertex is starred when its deletion leaves a graph with largest eigenvalue −2.

\[ e_{2i} = e_{i2} = 3 \text{ for } i > 3, \quad e_{33} = e_{34} = e_{43} = 2, \quad e_{3i} = e_{i3} = e_{44} = e_{45} = e_{54} = 1, \]
and \( e_{ik} = 0 \) otherwise. It is remarkable that \( \hat{m} = \sup \{ \lambda_{\text{max}}(\Gamma) | \Gamma \in \mathcal{G}_m \} \) although no graph with \( \lambda_{\text{max}}(\Gamma) = \hat{m} \) exists; in particular, this shows that the set of maximal eigenvalues of graphs is not closed. As observed by Hoffman [27], \( \mathcal{G}_m^\# \) is infinite, since it contains all subgraphs obtained by adding a vertex of valency 1 to the vertices of a polygon (and \( \hat{m} \) is maximal with this property). It would be interesting to know the set of numbers \( m, -m \) such that \( \mathcal{G}_m^\# \) or \( \mathcal{G}_{-m}^\# \) are finite; however, these seem to be very difficult problems.
The fact that $F^*_2$ and $F^*_2$ are finite implies the existence of "eigenvalue gaps" at ±2 in the following sense (cf. [23] for $\lambda_{max}$).

2.5. **Theorem.** Let $\rho \approx 2.006594$ be the largest solution of the equation

$$(\rho^3 - \rho)^2(\rho^2 - 3)(\rho^2 - 4) = 1.$$ 

Then there is no graph $\Gamma$ such that $-\rho < \lambda_{min}(\Gamma) < -2$ or $2 < \lambda_{max}(\Gamma) < \rho$. Moreover, every graph with $\lambda_{min}(\Gamma) = -\rho$ or $\lambda_{max}(\Gamma) = \rho$ is isomorphic to the graph $Y_{621}$.

**Proof.** Inspection of Table 5 shows that the only graph in $F^*_2$ with $\lambda_{min}(\Gamma) \geq -\rho$ is $Y_{621}$, which has $\lambda_{min}(\Gamma) = -\rho$. Now every graph with $\lambda_{min}(\Gamma) < -2$ not in $F^*_2$ contains a proper subgraph $\Gamma' \in F^*_2$; hence, $\lambda_{min}(\Gamma) \leq \lambda_{min}(\Gamma') \leq -\rho$.

If equality holds, then $\Gamma' = Y_{621}$, and all subgraphs of $\Gamma$ strictly containing $\Gamma'$ have $-\rho$ as a multiple eigenvalue. In particular, the subgraph obtained by adjoining to $\Gamma'$ one further vertex of $\Gamma$ and deleting one of the vertices of $\Gamma'$ again has $-\rho$ as smallest eigenvalue, and hence must be isomorphic to $Y_{621}$. But no graph with 11 vertices has this property. Hence, $\lambda_{min}(\Gamma) = -\rho$ implies $\Gamma \cong Y_{621}$. The statement about $\lambda_{max}$ follows immediately, since $Y_{621}$ is bipartite and the nonbipartite minimal graphs with largest eigenvalue > 2 (Figure 11) have largest eigenvalue > $\rho$. □

For graphs with large minimum valency, the eigenvalue gap at $-2$ is considerably larger. The following highly nontrivial result was proved by Hoffman [26] using Ramsey's theorem.

2.6. **Theorem.** Let $\hat{\lambda}_k = \sup\{\lambda_{min}(\Gamma) \mid k_{min}(\Gamma) \geq k, \lambda_{min}(\Gamma) < -2\}$. Then $\hat{\lambda}_k$ is a monotonic decreasing sequence with limit $-1 - \sqrt{2} \approx 2.414214$.

Theorem 2.5 implies the value $\hat{\lambda}_1 = -\rho \approx -2.006594$. Lower bounds for the values of $\hat{\lambda}_i$ can be obtained from particular graphs with minimal valency $k$. In particular, we get

$$\hat{\lambda}_2 \geq \hat{\lambda}_2 = \frac{1 - 2\sqrt{13}}{3} \approx -2.070368,$$

since the clique extension of an $n$-cycle, where a single vertex is replaced by a 2-clique, provides a sequence of graphs $\Gamma_i$ with minimal valency 2 and $\lambda_{min}(\Gamma_i) \to \hat{\lambda}_2$ for $i \to \infty$. The graph of Figure 13 gives the lower bound

$$\hat{\lambda}_k \geq \hat{\lambda}_k \approx (1 + \sqrt{2})(1 - k^{-1} + O(k^{-2})) \text{ for } k \geq 3,$$

where $\hat{\lambda}_k$ is the smallest solution of the equation

$$(x + 1)^2(x + 2)(x + 3) + k(2x^3 + 9x^2 + 10x + 1) + k^2(x^2 + 2x - 1) = 0.$$

The reader is challenged to provide more extreme examples or to prove that the examples given are extremal. An explicit upper bound for $\hat{\lambda}_k$ which tends to $-1 - \sqrt{2}$ for $k \to \infty$ would also be of considerable interest. In particular, for

**Figure 13.** A graph with minimal valency $k$. 

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applications to distance-regular graphs (see §3), we would like to know whether
\[
\hat{\lambda}_k < -2.4 \quad \text{if } k \geq 64;
\]
note that \(\hat{\lambda}_k < -2.4\) if \(k > 29.4\).

If we require that \(\Gamma\) is regular of large valency, then \(-1 - \sqrt{2}\) seems no longer to be the right limit. Among the regular graphs of valency \(k\), the largest value of \(\lambda_{\min}(\Gamma)\) observed in a limited number of test cases occurred for the graph in Figure 14, where \(\lambda_{\min}(\Gamma) = -1 - \alpha_k\) with the largest zero \(\alpha_k\) of the equation \(x^3 + 2x^2 + x - 3 - k(x^2 + 2x - 2) = 0\), and \(\lim_{k \to \infty}(-1 - \alpha_k) = -1 - \sqrt{3}\).

The result corresponding to Theorem 2.5 for the largest eigenvalue is trivial for minimal valency \(k > 2\), since (by Perron-Frobenius theory) the largest eigenvalue of a graph \(\Gamma\) with minimal valency \(k\) is at least \(k\), with equality if and only if the graph is regular of valency \(k\). The more relevant value
\[
\inf \{\lambda_{\max}(\Gamma) \mid k_{\min}(\Gamma) > k, \lambda_{\max}(\Gamma) > k\}
\]
\[
= \inf \{\lambda_{\max}(\Gamma) \mid \Gamma \text{ not regular, } k_{\min}(\Gamma) > k\}
\]
is not known, not even for \(k = 2\).

3. Applications to distance-regular graphs

In this section we apply the preceding results to a characterization problem in the theory of distance-regular graphs. A connected graph \(\Gamma\) is called distance-regular if for any two vertices \(\gamma\) and \(\delta\) at distance \(i = d(\gamma, \delta)\), there are precisely \(c_i\) neighbors of \(\delta\) at distance \(i - 1\) from \(\gamma\), and \(b_i\) neighbors of \(\delta\) at distance \(i + 1\) from \(\gamma\) (see Biggs [4], Bannai and Ito [2]). The sequence
\[
\tau(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\},
\]
where \(d\) is the diameter of \(\Gamma\), is called the intersection array of \(\Gamma\). A fundamental problem in the theory of distance-regular graphs is the characterization of known graphs by their intersection array. Recently, Paul Terwilliger and the second author achieved a breakthrough in this direction by utilizing the classification of graphs with smallest eigenvalue \(-2\) to settle this problem for a large class of intersection arrays containing those for the Hamming graphs, the Johnson graphs, and the half-cubes (Terwilliger [41], Neumaier [34]). Here we show that knowledge of the eigenvalue gap in Theorem 2.4 allows the characterization of distance-regular graphs for another class of intersection arrays, at least for large diameter. We begin by summarizing the background needed.
The adjacency matrix $A$ of a distance-regular graph $\Gamma$ of diameter $d$ has precisely $d + 1$ distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$; the largest eigenvalue $\theta_0$ is the valency $k = b_0$ of $\Gamma$ and has multiplicity 1. To each eigenvalue $\theta$ of $\Gamma$ there corresponds a unique idempotent matrix $E$ in the algebra of polynomials in $A$ satisfying the equation $AE = \theta E$, and the rank $f$ of $E$ agrees with the multiplicity of $\theta$. The $(\gamma, \delta)$-entries $E_{\gamma \delta}$ of $E$ depend only on the distance of $\gamma$ and $\delta$.

$$E_{\gamma \delta} = u_i \quad \text{if} \quad d(\gamma, \delta) = i,$$

and the $u_i$ satisfy the recurrence relations

$$
\begin{align*}
u_0 &= 1, & u_1 &= \theta/k, \\
c_i u_{i-1} + a_i u_i + b_i u_{i+1} &= \theta u_i \quad (i = 1, \ldots, d - 1), \\
c_d u_{d-1} + a_d u_d &= \theta u_d,
\end{align*}
$$

where $a_i = k - b_i - c_i$. Conversely, if (3) holds, then $\theta$ is an eigenvalue of $A$ and (2) defines the corresponding idempotents. These facts can be found, e.g., in [2, 4, 6]. Since an idempotent symmetric matrix is positive semidefinite, $E$ can be considered as the Gram matrix of a set of vectors in $\mathbb{R}^f$; hence, there is a mapping $- : \Gamma \to \mathbb{R}^f$ such that the images $\bar{\gamma}, \bar{\delta}$ of two vertices $\gamma, \delta \in \Gamma$ have inner product

$$(\bar{\gamma}, \bar{\delta}) = u_i \quad \text{if} \quad d(\gamma, \delta) = i.$$ 

From this graph representation in $\mathbb{R}^f$ it is possible to deduce the following result concerning the smallest eigenvalues of the local subgraphs $\Gamma(\gamma)$:

3.1. **Proposition** (Terwilliger [43]). Let $\Gamma$ be a distance-regular graph with intersection array (1), and suppose that $\theta$ is an eigenvalue of $\Gamma$ with multiplicity $f$. If $-1 < \theta < k$, then

$$\lambda_{\text{min}}(\Gamma(\gamma)) \geq -b_1/(\theta + 1) \quad \text{for all } \gamma \in \Gamma.$$ 

Moreover, if $f < k$, then (4) holds with equality, $\theta$ is the second-largest eigenvalue of $\Gamma$, and either $\theta + 1$ is an integer dividing $b_1$, or $\theta + 1$ and $b_1/(\theta + 1)$ are irrational quadratic algebraic integers.

**Proof.** Inequality (4) is essentially Theorem 1(2) in [43]. The second assertion is part of Theorem 5 in [43], apart from the statement about equality in (4), which derives from the proof of that theorem. \(\square\)

The following result of Terwilliger is also relevant in the present context.

3.2. **Proposition** (Terwilliger [40]). Let $\Gamma$ be a distance-regular graph with intersection array (1).

(i) If $\Gamma$ contains a quadrangle, then

$$c_i - b_i \geq c_{i-1} - b_{i-1} + a_i + 2 \quad (i = 1, \ldots, d).$$

(ii) If $c_2 - b_2 = c_1 - b_1 + a_1 + 1$, then every 2-claw of $\Gamma$ is in at most one quadrangle.

**Proof.** (i) is in Terwilliger [40], and (ii) is a simple consequence of the simplified proof of (i) in Terwilliger [42]. \(\square\)
Terwilliger classified in [42] the distance-regular graphs satisfying (5) with equality for all $i$. Here we consider a class of intersection arrays which have equality in (5) for all $i \neq d$, namely the arrays (1) defined by

\begin{align*}
  b_i &= \mu \binom{m-i}{2} - (m-i)(m-i-1) \quad (i = 0, \ldots, d-1), \\
  c_i &= \mu \binom{i}{2} - i(i-2) \quad (i = 1, \ldots, d-1), \\
  c_d &= \gamma \binom{d}{2} - d(d-2),
\end{align*}

where

\[(d, \gamma) \in \left\{ \left( \frac{m}{2}, 2 \right), \left( \frac{m-1}{2}, 1 \right) \right\}, \quad d > 1.\]

Note that $\mu = c_2$ and $m \in \{2d, 2d+1\}$ are positive integers, $m \geq 4$. There are three families of distance-regular graphs realizing these arrays:

(i) The folded $m$-cube with $v = \frac{1}{2}2^m$ vertices is the graph obtained by identifying antipodal vertices in the $m$-cube, and realizes (6) with $\mu = 2$. It can be described as the graph whose vertices are the partitions of an $m$-set into two sets, where two such partitions are adjacent whenever their common refinement contains two singletons.

(ii) The folded Johnson graph with $v = \frac{1}{2}(2^m)$ vertices is the graph whose vertices are the partitions of a $2m$-set into two $m$-sets, with adjacency defined as before. Its intersection array realizes (6) with $\mu = 4$.

(iii) The folded half $2m$-cube with $v = \frac{1}{2}2^{2m-1}$ vertices is the graph whose vertices are the partitions of a $2m$-set into two sets of even size, where two such partitions are adjacent whenever their common refinement contains two sets of size 2. Its intersection array realizes (6) with $\mu = 6$.

In view of these examples, we call a distance-regular graph with intersection array (6) a pseudopartition graph. We shall prove the following characterization theorem.

3.3. **Theorem.** Let $\Gamma$ be a pseudopartition graph with diameter $d$.

(i) If $\mu = 2$, then either $\Gamma$ is a folded cube, or $d = 3$ and $\Gamma$ is the incidence graph of a $(16, 6, 2)$-biplane.

(ii) If $d \geq 3$, then $\mu \in \{2, 4, 6\}$.

(iii) If $d \geq 154$, then $\Gamma$ is a folded cube, a folded Johnson graph, or a folded half-cube.

**Proof.** We proceed in several steps.

**Step 1.** $\Gamma$ has an eigenvalue $\theta = m - 4 + (\mu - 2)\binom{m-2}{2}$ with multiplicity

\[ f = \frac{m(m-1)(2 + (\mu - 2)(m-1))(4 + (\mu - 2)(2m-5))}{(4 + (\mu - 2)(m-2))(4 + (\mu - 2)(m-3))}. \]

To show this, we note that the intersection array belongs to the family of $Q$-polynomial intersection arrays of type II discussed in Bannai and Ito [2], with
parameters (in the notation of [2])
\[ r_1 = \frac{-m+1}{2} , \quad r_2 = \frac{-m+2}{2} , \quad r_3 = -m - \frac{2}{\mu-2} , \]
\[ h = 2\mu - 4 , \quad s = -m - \frac{1}{2} - \frac{2}{\mu-2} , \quad s^* = -m - 1. \]
Therefore, the eigenvalues of \( \Gamma \) are given by
\[ \theta_i = k - 4i + (\mu - 2)i(2i - 1 - 2m) \quad (i = 0, \ldots, d), \]
and their multiplicity is
\[ f_i = \prod_{j=1}^{i} q_j , \tag{8} \]
where
\[ q_j = \frac{b_{j-1}^*}{c_j^*} = \frac{(j+s)(2j+1+s)}{j(2j+1+s)} \cdot \frac{(j+r_1)(j+r_2)(j+r_3)}{(j+s-r_1)(j+s-r_2)(j+s-r_3)}. \]
In particular, since \( k = b_0 = \mu \left( \frac{m}{2} \right) - m(m-2) = \frac{m}{2}(2 + (\mu - 2)(m-1)) \), we get for \( i = 1 \) by simplification the above values \( \theta \) for \( \theta_1 \) and \( f \) for \( f_1 \).

**Step 2.** If \( d \geq 3 \), then \( \mu \neq 1, 3, 5 \).

\( c_3 > 0 \) excludes \( \mu = 1 \). For \( \mu = 3, \) (7) reduces to
\[ f = \frac{m(m-1)(2m-1)}{(m+2)} , \]
so that \( m + 2 \mid 30 \). For \( d \geq 3 \) (\( m \geq 6 \)) this leaves the cases \( m = 8, 13, 28 \), and (8) yields a nonintegral \( f_3, f_4, f_5 \) in the respective cases. And for \( \mu = 5 \), (7) becomes
\[ f = \frac{m(m-1)(3m-1)(6m-11)}{(3m-2)(3m-5)} , \]
which is nonintegral for all \( m > 3 \), hence for \( d > 2 \).

**Step 3.** If \( d \geq 3 \), then \( \mu \in \{2, 4, 6\} \).

To get this, we apply Proposition 3.1; note that \( \theta < k \) and \( \theta + 1 = \frac{m-3}{m-1} b_1 > 0 \). Since \( m \geq 2d \geq 6 \), \( \theta + 1 \) is no divisor of \( b_2 \), and since \( \theta \) is rational, we must have \( f \geq k = \frac{m}{2}(2 + (\mu - 2)(m-1)) \). This implies
\[ 2(m-1)(4 + (\mu - 2)(2m - 3)) \leq (4 + (\mu - 2)(m-2))(4 + (\mu - 2)(m-3)) , \]
which simplifies to \( (\mu - 6)(m-3)(2 + (\mu - 2)(m-2)) \leq 0 \). Therefore, \( \mu \leq 6 \) and thus \( \mu \in \{2, 4, 6\} \) by Step 2.

**Step 4.** If \( \mu = 2 \), then the conclusion of (i) holds.

For \( m \geq 7 \) this follows from Egawa [22]. For \( m = 6 \), \( \Gamma \) has \( v = 32 \) vertices and intersection array \( \{6, 5, 4; 1, 2, 6\} \); hence, \( \Gamma \) is bipartite of diameter 3 and must be the incidence graph of a \( 2 - (\frac{m}{2}, k, \mu) \)-design, i.e., of a \( (16, 6, 2) \)-biplane. For \( m = 4, 5 \), the graph \( \Gamma \) is easily seen to be \( K_4 \) and \( K_{4,4} \), respectively, and hence a folded \( m \)-cube.

**Step 5.** For any two nonadjacent vertices \( \alpha, \beta \in \Gamma(\gamma) \), the number \( \mu(\alpha, \beta) \) of common neighbors of \( \alpha \) and \( \beta \) in \( \Gamma(\gamma) \) is \( \mu - 1 \) or \( \mu - 2 \).
For, if $\mu(\alpha, \beta) \leq \mu - 3$, then there are two distinct vertices $\delta, \delta' \in \Gamma_2(\gamma)$ adjacent with $\alpha$ and $\beta$ so that the 2-claw $\alpha\gamma\beta$ is in two distinct quadrangles, contradicting Proposition 3.2(ii). Since $\mu(\alpha, \beta) \leq \mu - 1$, this forces $\mu(\alpha, \beta) \in \{\mu - 1, \mu - 2\}$.

**Step 6.** Each neighborhood $\Gamma(\gamma)$ has smallest eigenvalue $\geq -2 - \frac{2}{m-3}$.

This follows from Proposition 3.1 since

$$\frac{b_1}{\theta + 1} = \frac{m - 1}{m - 3} = 1 + \frac{2}{m - 3}.$$ 

**Step 7.** If $d \geq 154$, then each neighborhood $\Gamma(\gamma)$ is a line graph.

In this case, $m \geq 308$ so that $\lambda_{\min}(\Gamma(\gamma)) \geq -2 - \frac{2}{305} > -2.00656 > -\rho$, and by Theorem 2.5, $\lambda_{\min}(\Gamma(\gamma)) \geq -2$. Now $\Gamma(\gamma)$ is a regular graph with $k = \frac{m}{2}(2 + (\mu - 2)(m - 1)) \geq m^2 > 28$ (since we may assume $\mu \geq 4$ by Steps 3 and 4) vertices and valency $a_1 = k - 1 - b_1 < k - 2$, and by Theorem 1.2, each $\Gamma(\gamma)$ must be a line graph.

**Step 8.** If $\mu = 4$, then each neighborhood $\Gamma(\gamma)$ which is a line graph is in fact an $(m \times m)$-grid.

By Step 5, the hypothesis of Proposition 5 of Neumaier [34] is satisfied with $c = 2$ for $\Gamma(\gamma) = L(\Delta)$, and part (iii) of that proposition, together with the fact that $\Gamma(\gamma)$ contains $k = m^2$ vertices and is regular of valency $a_1 = k - 1 - b_1 = 2(m - 1)$, only leaves the case $\Delta = K_{m,m}$. Therefore, $\Gamma(\gamma) = L(K_{m,m})$ is an $(m \times m)$-grid.

**Step 9.** If $d \geq 154$ and $\mu = 4$, then $\Gamma$ is a folded Johnson graph.

For $\Gamma$ is locally an $(m \times m)$-grid by Steps 7 and 8, and has the same intersection array, hence the same number $\frac{1}{2}(2m)$ of vertices as a folded Johnson graph. By Blokhuis and Brouwer [5], $\Gamma$ is a quotient of a Johnson graph $J(2m, m)$, and distance regularity forces that antipodal vertices (corresponding to complementary $m$-sets) must be identified, so that $\Gamma$ is a folded Johnson graph.

**Step 10.** If $\mu = 6$, then each neighborhood $\Gamma(\gamma)$ which is a line graph is in fact a triangular graph $T(2m)$.

By Step 5, the hypothesis of Proposition 5 of Neumaier [34] is satisfied with $c = 4$ for $\Gamma(\gamma) \cong L(\Delta)$, and part (i) of that proposition, together with the fact that $\Gamma(\gamma)$ contains $k = m(2m - 1)$ vertices, implies $\Delta = K_{2m}$ and $\Gamma(\gamma) \cong T(2m)$.

**Step 11.** If $d \geq 154$ and $\mu = 6$, then $\Gamma$ is a folded half-cube.

For $\Gamma$ is locally $T(2m)$ by Steps 7 and 10, and has the same intersection array, hence the same number $\frac{1}{2}2^{2m-1}$ of vertices as the folded half $m$-cube. Since $d \geq 3$ and $m \geq 6$, the vertices and $m$-cliques of $\Gamma$ form a semiplane, i.e., distinct vertices are in precisely zero or two blocks ($m$-cliques), and distinct blocks intersect in zero or two vertices. The incidence graph $\Gamma^*$ of this semiplane is an amply regular graph with $\lambda = 0$ and $\mu = 2$. Application of Egawa [22] shows that $\Gamma^*$ is a folded $2m$-cube, so that $\Gamma$ is a folded half-cube. □

Together with the results of Terwilliger [41] (which inspired the first three steps of the preceding proof), this implies that for large diameters ($d \geq 154$), all $Q$-polynomial distance-regular graphs of type II are known.
The diameter bound seems much too pessimistic, and there should be no exceptions for \( d \geq 4 \). In order to obtain assertion 3.3(iii) for \( d \geq 4 \) in place of \( d \geq 154 \), the argument of Step 7 has to be improved; since \( m \geq 8 \) for \( d \geq 4 \), we would need a result like
\[
\Gamma \text{ regular of valency } k > 64 \Rightarrow \lambda_{\min}(\Gamma) < -2.4 \text{ or } \lambda_{\min}(\Gamma) \geq -2.
\]
For \( d \leq 3 \), there are many exceptions: Husain [28] shows that there are precisely three \((16, 6, 2)\)-bipartite graphs, and since each of them is self-dual, their incidence graphs give three nonisomorphic distance-regular graphs with intersection array \(\{6, 5, 4; 1, 2, 6\}\), one of which is the folded 5-cube. Bussemaker et al. [9] show that there are at least 1853 strongly regular graphs with the same intersection array \(\{16, 9; 1, 4\}\) as the folded Johnson graph with \( v = \frac{1}{2}(\frac{9}{2}) = 35 \) vertices. Any pair of orthogonal Latin squares of order 8 gives a Latin square graph \(LS_4(8)\) with the same intersection array \(\{28, 25; 1, 6\}\) as the folded half 8-cube. Any Latin square of order 16 gives a Latin square graph \(LS_3(16)\) with the same intersection array \(\{45, 28; 1, 6\}\) as the folded half 10-cube.

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