

ON REAL QUADRATIC FIELDS OF CLASS NUMBER TWO

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ABSTRACT. It is the primary purpose of the paper to determine all real quadratic fields $Q(\sqrt{d})$ of class number $h(d) = 2$ when $k \leq 24$ (with one possible exception). Here, k is the period length of the continued fraction expansion of either $\omega = \sqrt{d}$, in the case $d \equiv 2$ or $3 \pmod{4}$, or of $\omega = (1 + \sqrt{d})/2$, in the case $d \equiv 1 \pmod{4}$.

1. INTRODUCTION

In [6] the authors found all real quadratic fields $Q(\sqrt{d})$ with $h(d) = 1$ and $k \leq 24$, k as above, with one possible exception remaining. The result of [6] allowed a solution of several conjectures in the literature (see [3-6] for details). The techniques used there provide a basis for examining the class number 2 problem herein. We are able to improve upon them, and as a result, to reduce the computational workload for this paper. The new results (Theorem 2.1 and Lemmas 2.1 and 2.2) are of interest in their own right. In fact, Theorem 2.1 is a very useful means of immediately getting an explicit lower bound on the class number $h(d)$ in terms of k . The complete listings of our findings for $k \leq 24$ and $h(d) = 2$ are in Tables 2.1 and 2.2 at the end of the paper. Moreover, our results vastly generalize the results of (and improve upon the techniques of) [9].

Throughout, d will be a positive square-free integer. For convenience sake we give the basic continued fraction notation which we will use in the paper. For ω as in the abstract let the continued fraction expansion of ω be denoted by $\omega = \langle a, \overline{a_1, \dots, a_k} \rangle$. Then $a_0 = a = [\omega]$ and $a_i = \lfloor (P_i + \sqrt{d})/Q_i \rfloor$ for $i \geq 1$ (here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x), where $(P_0, Q_0) = (\sigma - 1, \sigma)$, with $\sigma = 2$ if $d \equiv 1 \pmod{4}$ and $\sigma = 1$ otherwise. Also, $P_{i+1} = a_i Q_i - P_i$ and $Q_{i+1} Q_i = d - P_{i+1}^2$ for $i \geq 0$. For more detailed information and connections with other topics, such as reduced ideals, the reader is referred to [2, 10].

2. CLASS NUMBER 2 FOR $k < 25$

In order to find the real quadratic fields $Q(\sqrt{d})$ with $k \leq 24$ and $h(d) = 2$, we proceed in a fashion similar to that in Mollin and Williams [6]. However,

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the techniques here are different in that we can reduce the amount of work by first proving the following result. This result is in fact an improvement on the inequality $R < k \log \sqrt{\Delta}$ used in [6] (here, $R = \log \varepsilon$, where ε is the fundamental unit of $Q(\sqrt{d})$).

Theorem 2.1. *If R is the regulator of $Q(\sqrt{d})$, then $R < \lfloor 3(k + 1)/4 \rfloor \log \sqrt{\Delta}$.*

Proof. By results noted in Stephens and Williams [7] we have

$$(2.1) \quad \varepsilon = \prod_{i=1}^k \varphi_i,$$

where $\varphi_i = (P_i + \sqrt{d})/Q_i$ and $0 < P_i < \sqrt{d}$. We can write (2.1) as

$$(2.2) \quad \varepsilon = \lambda \prod_{i=1}^{\lfloor k/2 \rfloor} \chi_i,$$

where $\chi_i = \varphi_i \varphi_{k-i+1}$ and

$$\lambda = \begin{cases} 1 & \text{when } 2|k, \\ \varphi_{(k+1)/2} & \text{when } 2 \nmid k. \end{cases}$$

By [7, Theorem 2.1] we have $Q_{k-i} = Q_i$ and $P_{k-i} = P_{i+1}$; hence,

$$\chi_i = ((P_i + \sqrt{d})/Q_i)((P_i + \sqrt{d})/Q_{i-1}) = (\sqrt{d} + P_i)/(\sqrt{d} - P_i)$$

from (2.2) of [7]. Furthermore, when k is odd we have $Q_{(k-1)/2} = Q_{(k+1)/2}$, which means that $d = P_{(k+1)/2}^2 + Q_{(k+1)/2}^2$ and therefore

$$\lambda = \varphi_{(k+1)/2} < \sqrt{\Delta},$$

as $\sigma|Q_i$ for all $i \geq 0$.

If we define

$$\nu = \begin{cases} 0 & \text{when } 2|k, \\ 1 & \text{when } 2 \nmid k, \end{cases}$$

then it is an easy matter to show that

$$(2.3) \quad \lfloor k/2 \rfloor + \lfloor (k + 2)/4 \rfloor + \nu \leq \lfloor 3(k + 1)/4 \rfloor.$$

If $\sigma = 2$ and $2|\lfloor \sqrt{d} \rfloor$, then, since $P_i \equiv 1 \pmod{2}$ for all $i \geq 0$, we cannot have $P_i = \lfloor \sqrt{d} \rfloor$. Thus, in this case, $\chi_i < 2\sqrt{d}$ and $\varepsilon < \lambda(2\sqrt{\Delta})^{\lfloor k/2 \rfloor}$, so

$$(2.4) \quad \varepsilon < 2^{\lfloor k/2 \rfloor} (\sqrt{\Delta})^{\lfloor k/2 \rfloor + \nu}.$$

When $\Delta > 16$, we have

$$2^{\lfloor k/2 \rfloor} < (\sqrt{\Delta})^{\lfloor (k+2)/4 \rfloor},$$

hence by (2.3) we have our result in this case. For the remaining values of $\Delta < 16$ we see that $\sigma = 2$ and $\lfloor \sqrt{d} \rfloor$ even forces $d = 5$, for which the theorem is easily verified.

If $P_i \neq \lfloor \sqrt{d} \rfloor$ and $\lfloor \sqrt{d} \rfloor \equiv 1 \pmod{\sigma}$, then we must have $P_i \leq \lfloor \sqrt{d} \rfloor - \sigma$ and $\chi_i < (2/\sigma)\sqrt{d} = \sqrt{\Delta}$. Furthermore, if j is the least positive integer such that $P_j = P_{j+1}$, we must have $k = 2j$. Therefore, the case in which we have the largest possible number of P_i -values equal to $\lfloor \sqrt{d} \rfloor$ can only occur when

$$P_1 = P_2 = \dots = P_{\lfloor (k+2)/4 \rfloor} = \lfloor \sqrt{d} \rfloor.$$

Thus, if n is the number of values of P_i with $P_i = \lfloor \sqrt{d} \rfloor$ for $i \leq \lfloor k/2 \rfloor$, then

$$(2.5) \quad \varepsilon < \lambda(\sqrt{\Delta})^{\lfloor k/2 \rfloor - n} (\sqrt{d} + \lfloor \sqrt{d} \rfloor) / (\sqrt{d} - \lfloor \sqrt{d} \rfloor)$$

and $n \leq \lfloor (k + 2)/4 \rfloor$.

Since $d - \lfloor \sqrt{d} \rfloor^2 \geq \sigma^2$, we have $(\sqrt{d} + \lfloor \sqrt{d} \rfloor) / (\sqrt{d} - \lfloor \sqrt{d} \rfloor) < (2\sqrt{d}/\sigma)^2 = \Delta$. Hence,

$$\varepsilon < \lambda(\sqrt{\Delta})^{\lfloor k/2 \rfloor - n} \Delta^n = (\sqrt{\Delta})^{\lfloor k/2 \rfloor + n + \nu}.$$

By (2.3) the result follows. \square

We are now able to prove

Lemma 2.1. *If $k \leq 24$ and $\Delta > 6 \times 10^9$, then with at most one possible exception we must have $h(d) > 2$.*

Proof. By Tatzuzaawa [8], we have (with at most one exception) $L(1, \chi) > 0.655\eta\Delta^{-\eta}$ for $0 < \eta < \frac{1}{2}$ and $\Delta \geq \max(e^{1/\eta}, e^{11 \cdot 2})$ (where $L(1, \chi) = \sum_{n=1}^{\infty} (\Delta/n)/n$ and (\cdot/n) is the Kronecker symbol). Also, since $2Rh(d) = \sqrt{\Delta}L(1, \chi)$, by Theorem 2.1 we must have

$$2h(d) > 2\Delta^{1/2-\eta}(0.655\eta)/(\lfloor 3(k+1)/4 \rfloor \log \Delta) > 4$$

when $\Delta > 6 \times 10^9$, $\eta = 0.04442$, and $k \leq 24$. \square

In the case $\Delta = d \equiv 1 \pmod{4}$ we can improve Lemma 2.1 somewhat.

Lemma 2.2. *If $d \equiv 1 \pmod{4}$, $\Delta > 4.75 \times 10^9$, and $k \leq 24$, then with at most one possible exception we must have $h(d) > 2$.*

Proof. This result can be easily verified by using the methods of Theorem 2.1 and Lemma 2.1 in the case where $\lfloor \sqrt{d} \rfloor$ is even. Thus, we will assume that $\lfloor \sqrt{d} \rfloor$ is odd and write (2.5) as

$$\varepsilon < (\sqrt{\Delta})^{\lfloor (k+1)/2 \rfloor - n} (\sqrt{d} + \lfloor \sqrt{d} \rfloor)^n \gamma^{-n},$$

where $\gamma = \sqrt{d} - \lfloor \sqrt{d} \rfloor$. We also note that the value of n (the number of values of $P_i = \lfloor \sqrt{d} \rfloor$ for $i \leq \lfloor k/2 \rfloor$) cannot exceed the number of divisors of $(d - \lfloor \sqrt{d} \rfloor^2)/4$. This is a fact because each Q_i associated with one of the P_i -values must be distinct from any other, must be even, and must be a divisor of $(d - \lfloor \sqrt{d} \rfloor^2)/2$.

Now,

$$\varepsilon < (\sqrt{\Delta})^{\lfloor (k+1)/2 \rfloor - n} (2\sqrt{\Delta}/\gamma)^n = (\sqrt{\Delta})^{(k+1)/2} (2/\gamma)^n$$

and

$$R < \lfloor (k+1)/2 \rfloor \log \sqrt{\Delta} + n \log(2/\gamma).$$

Hence, if $\Delta > 4.75 \times 10^9$, $n \log(2/\gamma) < 52.6$, $\eta = 0.045$, and $k \leq 24$, then

$$2h(d) > \Delta^{1/2-\eta}(0.655\eta)/(\lfloor (k+1)/2 \rfloor \log \sqrt{\Delta} + n \log(2/\gamma)) > 4.$$

If $n \log(2/\gamma) \geq 52.6$, then, since $k \leq \lfloor (k+2)/4 \rfloor \leq 6$, we have $-\log \gamma > 8.0$, $\gamma < 0.000335$, and $d - \lfloor \sqrt{d} \rfloor^2 < 2\sqrt{d}\gamma < 46.2$. It follows that in this case $(d - \lfloor \sqrt{d} \rfloor^2)/4 < 11$. However, the maximum value of the number of divisors of l for $1 \leq l \leq 11$ is 4, thus we must have $n \geq 4$. This now means that $-\log \gamma > 12.4$ and $d - \lfloor \sqrt{d} \rfloor^2 < 1$, which is impossible. \square

Thus, to find all real quadratic fields with $k \leq 24$ and $h(d) = 2$ (with at most one more value remaining), we need only examine those with $d < 1.5 \times 10^9$ when $d \not\equiv 1 \pmod{4}$ and those with $d < 4.75 \times 10^9$ when $d \equiv 1 \pmod{4}$.

A computer search was run on all numbers of these forms up to the bounds given above to find all values of d such that $k \leq 24$. Once this had been done, we used the method in Mollin and Williams [1] to eliminate most of the values of d for which the corresponding field has $h(d) > 2$. The value of $h(d)$ was actually determined for those fields which remained and those for which $h(d) > 2$ were also eliminated, leaving only those for which $h(d) = 2$. Our results are summarized in Tables 2.1 and 2.2. There were a surprisingly large number of them, 1958 to be exact.

TABLE 2.1. $h(d) = 2$ for $k \leq 24$

k	d
1	10, 26, 85, 122, 362, 365, 533, 629, 965, 1685, 1853, 2813
2	15, 30, 35, 39, 42, 51, 66, 87, 102, 110, 123, 143, 146, 165, 182, 203, 221, 230, 258, 285, 327, 357, 402, 447, 635, 645, 678, 741, 843, 902, 957, 1085, 1245, 1298, 1517, 1533, 2037, 2045, 2085, 2397, 2613, 4245, 4277, 4773, 5645, 5957, 6573, 8333
3	65, 185, 458, 485, 1157, 2117, 2285, 3077, 3293, 3365, 12365
4	34, 55, 78, 95, 119, 138, 155, 174, 194, 205, 215, 222, 266, 287, 299, 305, 318, 335, 377, 395, 429, 482, 527, 623, 755, 782, 861, 885, 1022, 1055, 1205, 1405, 1469, 1965, 2013, 2093, 2222, 2301, 2373, 2877, 3005, 3237, 3597, 3813, 4893, 5117, 5397, 5757, 5885, 6005, 6285, 6293, 7157, 7733, 7973, 8357, 9005, 9077
5	74, 218, 493, 565, 1037, 1565, 1781, 2138, 2165, 2173, 3869, 5165, 5213, 5837, 6485, 8021, 10397, 14213
6	70, 105, 111, 114, 178, 183, 187, 267, 273, 303, 371, 374, 407, 418, 470, 498, 518, 545, 551, 590, 602, 618, 642, 803, 805, 822, 923, 1005, 1007, 1034, 1118, 1167, 1173, 1178, 1202, 1581, 1605, 1623, 1653, 1707, 1749, 1790, 2103, 2109, 2147, 2245, 2261, 2445, 2717, 2723, 2765, 2845, 3405, 3605, 3638, 3737, 3893, 4085, 4301, 4445, 4605, 5133, 5453, 7805, 10237, 10317, 10653, 11837, 12845, 13253, 13277, 13445, 14405, 14573, 15197, 19445, 21677, 23693, 25437
7	58, 202, 314, 538, 685, 949, 1165, 1261, 2885, 3133, 3277, 3653, 5429, 5765, 6437, 7373, 9197, 9509, 12557, 16757, 17141, 17261, 18317, 22301
8	91, 238, 282, 638, 695, 707, 710, 854, 866, 942, 1247, 1403, 1643, 1655, 1869, 1883, 1943, 2238, 2390, 2483, 2685, 2978, 3205, 3333, 3765, 4247, 4565, 5069, 5141, 5829, 6341, 6365, 6693, 6773, 6837, 6965, 7405, 7469, 8165, 8853, 9141, 9453, 9485, 10013, 10293, 10373, 10517, 10797, 10805, 11357, 11501, 15677, 16805, 17357, 17853, 19493, 31533, 37373, 38213
9	106, 698, 1073, 1189, 1285, 1385, 1418, 1865, 2581, 3233, 4469, 4553, 4709, 5597, 8885, 9365, 9773, 9893, 10229, 10685, 12053, 12077, 13565, 14285, 16733, 23285, 28757, 29957
10	115, 154, 159, 186, 246, 259, 286, 339, 345, 354, 403, 411, 451, 465, 494, 515, 534, 543, 561, 583, 591, 598, 665, 671, 682, 687, 703, 705, 762, 779, 830, 938, 978, 1047, 1102, 1203, 1263, 1265, 1363, 1383, 1645, 1671, 1727, 1742, 2098, 2123, 2127, 2485, 2651, 2658, 2701, 2747, 2802, 2829, 2867, 2882, 3157, 3165, 3218, 3587, 3685, 3741, 3743, 3827, 3867, 4103, 4619, 4667, 5057, 5061, 5205, 5253, 5285, 5405, 5522, 6149, 6613, 6789, 7005, 7845, 8045, 8445, 8517, 8533, 8621, 9085, 9093, 9581, 9701, 9821, 10365, 10645, 10877, 11373, 11557, 11973, 12117, 12165, 12837, 14773, 14861, 16037, 16077, 16205, 17045, 17741, 17877, 18093, 18357, 18717, 19253, 21405, 21749, 21885, 22413, 22517, 22781, 23933, 23997, 24213, 24845, 25077, 25133, 26333, 26477, 27173, 28005, 28853, 30245, 30693, 33677, 37565, 39245, 41477, 47195

- 11 265, 298, 554, 794, 1322, 1658, 2218, 2509, 3242, 4181, 4682, 4685, 11413, 11773, 13085, 14453, 15685, 16085, 18485, 20285, 20765, 25565, 28013, 28685, 31037, 39797, 40157, 43733, 46637, 51917, 56117
- 12 247, 295, 355, 366, 385, 386, 426, 535, 609, 767, 802, 815, 851, 969, 995, 1027, 1113, 1162, 1207, 1343, 1353, 1355, 1358, 1535, 1538, 1703, 1717, 1799, 1910, 1946, 2018, 2047, 2054, 2105, 2231, 2318, 2327, 2334, 2365, 2438, 2507, 2735, 2855, 2987, 3002, 3263, 3302, 3563, 3695, 4322, 4382, 4415, 4453, 4542, 4717, 4917, 5447, 6605, 6853, 6905, 7365, 7413, 7797, 9429, 10262, 11077, 12341, 12453, 12485, 12605, 12669, 13837, 15333, 16365, 18557, 18805, 22893, 23253, 24293, 24485, 25397, 25413, 25517, 27053, 27389, 27605, 29141, 29405, 29861, 30173, 32357, 36533, 40533, 44117, 44693, 45485, 45573, 47157, 52037, 59213, 59573, 75677
- 13 746, 778, 1082, 1241, 1514, 1649, 2042, 2426, 3085, 3338, 3349, 4058, 4573, 4589, 4885, 5389, 7418, 7421, 8765, 9389, 9965, 10085, 12965, 14837, 16277, 17533, 19357, 21053, 22373, 25877, 30733, 31373, 31853, 36965, 38597, 39437, 40757, 53477, 69893, 81413
- 14 190, 319, 406, 430, 471, 474, 611, 667, 670, 699, 742, 745, 806, 807, 1001, 1043, 1070, 1115, 1119, 1309, 1315, 1338, 1347, 1398, 1542, 1545, 1562, 1634, 1670, 1691, 1826, 1839, 1874, 2282, 2294, 2315, 2323, 2337, 2345, 2427, 2435, 2463, 2630, 2714, 2782, 2821, 3297, 3378, 3478, 3621, 3878, 4115, 4154, 4178, 4307, 4331, 4381, 4499, 4506, 4646, 4835, 5222, 5246, 5282, 5442, 5673, 5781, 5917, 6098, 6213, 6357, 6443, 6461, 6477, 6611, 6645, 7145, 7285, 7445, 7619, 7885, 8205, 8393, 8437, 8483, 8565, 8733, 8805, 8877, 8965, 9285, 9645, 9717, 9877, 10149, 10573, 11051, 11805, 12578, 12621, 12733, 12869, 12885, 13197, 13213, 13973, 14181, 15861, 15965, 16541, 17013, 17805, 18845, 18941, 19205, 19277, 19365, 19397, 19677, 21197, 21245, 21549, 21909, 21917, 22557, 22965, 23493, 24069, 24893, 25109, 25597, 26285, 26373, 26885, 27677, 27845, 28397, 28605, 28797, 28805, 31349, 31413, 32973, 34013, 34133, 35045, 35477, 35765, 35789, 36917, 38253, 40445, 42413, 42933, 43493, 44957, 45453, 47253, 51653, 52013, 54557, 54677, 55893, 64181, 64253, 71357, 85973, 98045
- 15 481, 1417, 1466, 2858, 3065, 3589, 3785, 3977, 4538, 5317, 5941, 6641, 6749, 7082, 11861, 12701, 12833, 13793, 14909, 16589, 17153, 18185, 18365, 18581, 20885, 24221, 27989, 29069, 32885, 33365, 44813, 47165, 51173, 66197, 67973, 70493, 78917
- 16 310, 391, 415, 654, 655, 679, 955, 1038, 1146, 1166, 1267, 1282, 1346, 1391, 1578, 1662, 1739, 1833, 1858, 1895, 1902, 2183, 2195, 2198, 2407, 2526, 2553, 2615, 3227, 3278, 3374, 3497, 3565, 3611, 3755, 3818, 3918, 4043, 4069, 4087, 4142, 4233, 4298, 4405, 4955, 4958, 5123, 5198, 5267, 5543, 5558, 5579, 5726, 5855, 6062, 6167, 6254, 6383, 6501, 6527, 7322, 7337, 7355, 7898, 8029, 8078, 8207, 8378, 8421, 8493, 8718, 9107, 9309, 9373, 10509, 11303, 11405, 11517, 11917, 12878, 12957, 13802, 13943, 15213, 15365, 15573, 16685, 17673, 17997, 19293, 19389, 21965, 22029, 22173, 24101, 25885, 26133, 29685, 30413, 30581, 31493, 34989, 35861, 38309, 38405, 38517, 40685, 41741, 43053, 43253, 43805, 45773, 48965, 49565, 49685, 50973, 55013, 55173, 57293, 58253, 58373, 58973, 59237, 61277, 63557, 67133, 67205, 67997, 69621, 75413
- 17 1018, 1994, 2965, 4285, 5354, 5498, 5585, 8917, 9242, 9665, 10265, 12085, 13061, 13957, 14677, 15242, 15613, 16109, 16565, 16613, 17173, 17285, 17429, 17861, 18037, 18737, 18965, 34037, 34957, 35285, 36413, 37949, 40085, 40501, 41165, 47813, 48149, 50357, 53285, 57797, 58853, 61133, 62957, 63653, 86957, 146453
- 18 519, 562, 831, 879, 951, 1185, 1199, 1209, 1281, 1310, 1362, 1379, 1505, 1506, 1526, 1606, 1630, 1686, 1698, 1842, 1903, 1919, 1923, 1983, 1991, 2202, 2219, 2283, 2363, 2631, 2697, 2771, 2985, 3183, 3282, 3414, 3470, 3642, 3702, 3707, 3830, 3839, 4029, 4287, 4343, 4430, 4562, 4697, 4791, 4803, 5027, 5363, 5705, 5797, 5845, 5870, 6177, 6182, 6278, 6407, 6470, 6758, 6767, 6830, 6842, 7358, 7485, 7802, 7869, 7958, 8582, 8589, 8697, 8843, 9119, 9269, 9381, 9383, 9445, 9470, 10245, 10461, 10502, 10643, 11397, 12093, 12162, 12278, 12549, 12722, 12749, 12909, 13019, 13237, 13593, 13605, 13682, 13821, 14053, 14309, 14605, 15485, 15933, 16845, 17197, 17381, 17733, 18173, 19229, 19245, 20253, 20541, 20645, 20677, 20917, 21533, 22085, 22181, 22245, 23309, 24357, 24365, 24477, 24837, 25445, 25485, 25781, 26781, 27485, 28461, 28589, 29309, 30317, 30957, 31733, 32021, 32669, 32773, 33789, 33845, 34405, 34685, 34853, 35189, 36669, 36813, 39893, 40613, 41789, 41837, 41973, 43581, 43589, 43797, 44645, 47333, 47549, 48245, 49805, 50133, 51557, 51845, 52853, 54197, 54845, 57845, 59637, 61477, 62405, 64277, 66549, 70133, 71213, 73253, 74973, 76037, 80013, 80117, 92477, 96701, 128117, 138773, 139277, 146333, 151373, 168773, 171797
- 19 922, 1706, 2186, 2257, 3386, 8522, 8714, 9997, 16781, 17177, 20513, 20813, 21509, 24341, 26165, 28453, 29597, 30365, 31085, 35333, 35885, 36173, 37685, 37757, 38765, 41765, 43469, 46157, 50453, 52637, 53765, 57965, 59765, 62285, 65501, 70733, 75197, 79085, 82277, 84773, 107333, 109757, 139037, 144317

TABLE 2.1. (continued)

20	511,559,606,790,1002,1065,1079,1182,1195,1374,1415,1510,1513,1537,1603,1687, 1961,2193,2215,2455,2471,2627,2863,3155,3239,3295,3383,3647,3746,3857,4163,4295, 4458,4535,4595,4727,4782,4847,4922,5038,5143,5195,5258,5678,5709,5759,5803,5822, 5962,6107,6141,6338,6415,6467,6702,6914,6943,7189,7295,7343,7367,7787,7813,7895, 8238,8258,8507,8567,8903,9527,9861,9885,10622,11949,13413,13461,14541,14565, 15029,15203,16237,16463,16989,18821,19085,19221,19337,19509,19533,20517,20642, 21093,21765,21962,23069,23133,25197,25557,26637,26697,28205,28965,29261,29365, 32053,32997,34773,35085,36093,36165,36237,36573,38685,39093,40557,41285,42605, 42749,43085,44765,45933,46277,48237,48453,49181,50213,52917,54893,57189,58493, 58533,60205,61053,61773,61973,62237,62933,63213,65285,65813,67373,69293,70213, 71405,74261,76677,77957,80693,80717,81317,81357,81437,82869,88413,93197,96773, 106685,109997,113693,115037,240077
21	394,865,1769,1985,2561,2762,3098,4385,5465,5485,5965,6122,7141,7265,10565,11101, 11485,11581,13285,13466,14381,14765,16442,21365,22565,28373,34493,35197,36221, 44861,47477,48485,48941,51365,54317,57317,58133,58589,69365,75917,78053,78557, 78773,80165,84173,85277,85949,89333,91013,92165,94877,97877,104837,120893, 127613,130037,156917,167477,212357
22	466,763,771,834,1059,1194,1266,1334,1558,1563,1798,1835,1843,1905,1963,1986, 2001,2082,2270,2274,2279,2406,2514,2519,2546,2585,2643,2778,2823,2859,2931,2937, 2947,3063,3107,3131,3147,3182,3207,3310,3417,3506,3635,3657,3687,3938,4119,4145, 4187,4202,4433,4630,4645,4814,4863,4883,4938,4965,5111,5163,5315,5345,5367,5603, 5703,5718,5747,5862,5989,6023,6061,6378,6387,6403,6431,6585,6635,6738,6743,7026, 7122,7143,7257,7334,7545,7553,7622,7701,7842,7982,8018,8027,8270,8365,8531,8630, 8749,8897,8998,9113,9138,9158,9205,9263,9687,9709,10190,10298,10307,11507, 11747,12257,12261,12305,12405,12662,12827,13067,13265,13405,13817,14205,14510, 14845,15085,15113,15617,15765,16149,16797,17445,17549,18002,20301,21837,23777, 24141,24405,25149,25653,25682,26197,27101,28673,29805,30165,31965,32469,33045, 33429,33549,33645,33989,34005,34933,36581,37077,37317,37445,37677,39693,40605, 42117,44373,45237,45605,46949,48045,48813,48885,49629,49677,50045,50861,51861, 52085,52421,54645,56069,56357,56397,57749,57893,58197,58205,59285,60277,62733, 63237,63437,63453,64085,65189,66557,67469,68933,71021,71885,72597,76053,76773, 77357,77693,78869,80597,83765,85917,86933,89957,92573,94085,94317,95933,98813, 99413,101213,101909,104357,106757,107645,112013,116045,116261,116597,120245, 121733,126245,130613,135453,136037,136253,138437,145277,145613,148133,148973, 149357,149957,150293,156053,157493,165557,168797,204245,248093
23	586,634,1585,2474,3578,4121,4141,5114,6074,6109,6362,6506,6602,7261,8042,8249 10673,12349,20557,22837,24869,26773,26869,33017,34165,34541,37661,37837,43693, 51757,55565,56285,56381,58277,59293,63677,64349,67253,74693,76565,77453,86213, 87485,90485,90557,90653,94973,99557,107117,107573,107957,113237,119477,139157, 154853,160277,172133,176837,247397
24	826,871,1147,1255,1614,1711,1795,2051,2119,2154,2409,2414,2534,2594,2698,2743, 2759,3009,3018,3110,3206,3633,3806,3854,3882,4031,4118,4310,4638,4665,4826,5322, 5466,6155,6302,6455,6618,7091,7222,7278,7763,8302,8489,8927,8939,9002,9287,9347, 9393,9469,9741,9785,10058,10142,10415,10442,10562,10823,11042,11262,11546, 11714,11843,12311,12741,13118,13502,13958,14198,14987,15815,16502,16765,16859, 17063,17447,17501,17531,18381,18501,18885,19685,20013,21047,21287,21565,21669, 22523,23037,23927,25053,25322,26277,27069,27669,27789,27853,29877,30093,31485, 34669,34709,35277,36453,37157,37293,37805,39957,40893,41109,42285,43301,43727, 44133,44429,44717,45069,47613,48005,48765,49101,50165,51837,54389,54797,54885, 56885,57813,59717,59853,60477,61269,61365,62493,63885,68285,70565,70685,71133, 71165,72197,72917,76973,78837,83613,84413,85533,87989,88877,89549,91077,92405, 92597,97277,102605,106413,109205,109805,112517,113813,118685,122405,125333, 126965,132117,136205,141797,151205,154013,158933,165413,169133,172493,175637, 197333,205805

TABLE 2.2

k	The number of $h(d) = 2$
1	12
2	48
3	11
4	58
5	18
6	79
7	24
8	59
9	28
10	135
11	31
12	102
13	40
14	169
15	37
16	130
17	46
18	187
19	44
20	161
21	59
22	245
23	59
24	176

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