COUNTING POINTS ON ELLIPTIC CURVES OVER $F_{2^n}$

ALFRED J. MENEZES, SCOTT A. VANSTONE, AND ROBERT J. ZUCCHERATO

Abstract. In this paper we present an implementation of Schoof's algorithm for computing the number of $F_{2^n}$-points of an elliptic curve that is defined over the finite field $F_{2^n}$. We have implemented some heuristic improvements, and give running times for various problem instances.

1. INTRODUCTION

The use of elliptic curves in public key cryptography was first proposed by N. Koblitz [5] and V. Miller [10]. Since then, a significant amount of research has been done on the implementation of practical and secure cryptosystems based on elliptic curves. For a secure system, one should select a curve $E$ over a finite field $F_q$ such that the order, $#E(F_q)$, of the group of points has a large prime divisor. There are some families of curves whose orders are trivial to compute (see [7] for some examples). However, if a random curve is chosen, then it is necessary to have an efficient algorithm for computing its order.

In 1985, Schoof [13] gave a polynomial-time algorithm for computing $#E(F_q)$. The algorithm has a running time of $O(\log^8 q)$ bit operations, and is rather cumbersome in practice. In [3] the authors combined Schoof's algorithm with Shanks' baby-step giant-step algorithm, and were able to compute orders of curves over $F_p$, where $p$ is a 27-decimal-digit prime. The algorithm took 4.5 hours on a SUN-1 SPARC station.

The work mentioned above was all described for the case $q$ odd. From a practical point of view, however, curves over fields of characteristic 2 are more attractive, since the arithmetic in $F_{2^n}$ is easier to implement in hardware than the arithmetic in $F_q$, $q$ odd. In [6] Koblitz adapted Schoof's algorithm to curves over $F_{2^n}$ and studied the implementation and security of a random-curve cryptosystem. Special emphasis was placed on the underlying field $F_{2^{133}}$. Recently, Agnew, Mullin, and Vanstone [1] have developed a VLSI device to perform arithmetic in $F_{2^{155}}$ and to perform computations on a random elliptic curve over this field. Consequently, it is of interest to determine the order of random curves over $F_{2^{155}}$.

We have implemented Schoof's algorithm for counting the points on an arbitrary curve over $F_{2^m}$, and have employed some heuristics to improve the actual running time. We are able to compute $#E(F_{2^m})$ for $m = 155$ (and so $#E(F_{2^m}) \approx 10^{47}$) in about 61 hours on a SUN-2 SPARC station.

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The remainder of the paper is organized as follows. In §2, we mention the relevant properties of elliptic curves over finite fields of characteristic 2. In §3, we outline Schoof’s algorithm, and in §4 we present our heuristics for improving School’s algorithm. Section 5 discusses details of our implementation, and gives some running times for various problem instances. Finally, in §6, we survey the latest research on the problem of counting points on an elliptic curve.

2. Elliptic curves in characteristic 2

Let \( q = 2^m \), and let \( K = F_q \) be the finite field of \( q \) elements. We denote the algebraic closure of \( K \) by \( \overline{K} \). If \( S \) is a field or an additive group, then \( S^* \) will denote the nonzero elements of \( S \). There are two types of elliptic curves over \( K \). A supersingular curve \( E \) over \( K \) is the set of solutions \((x, y) \in K \times K\) to an equation of the form

\[
y^2 + a_3y = x^3 + a_4x + a_6,
\]

with \( a_3, a_4, a_6 \in K \), \( a_3 \neq 0 \), together with the “point at infinity” denoted \( \mathcal{O} \). A nonsupersingular curve \( E \) over \( K \) is the set of solutions \((x, y) \in K \times K\) to an equation of the form

\[
y^2 + xy = x^3 + a_2x^2 + a_6,
\]

with \( a_2, a_6 \in K \), \( a_6 \neq 0 \), together with the point \( \mathcal{O} \).

If \( L \) is any field with \( K \subseteq L \subseteq \overline{K} \), then let \( E(L) \) denote the set of points in \( E \) both of whose coordinates lie in \( L \), together with the point \( \mathcal{O} \).

There are precisely three isomorphism classes of supersingular elliptic curves over \( K \) if \( m \) is odd, and seven classes if \( m \) is even. The number of points on a curve in each class is known [9]. Given a supersingular curve (1), we can then compute \( \#E(K) \) by determining the isomorphism class that \( E \) belongs to. For the remainder of the paper we will thus be interested in computing \( \#E(K) \), where \( E \) is a nonsupersingular elliptic curve.

There are \( 2q - 2 \) isomorphism classes of nonsupersingular curves over \( K \). A set of representatives of these classes is

\[
\{ y^2 + xy = x^3 + a_2x^2 + a_6 | a_6 \in K^*, a_2 \in \{0, \gamma\} \},
\]

where \( \gamma \in K \) is a fixed element of trace 1. If \( E \) and \( \tilde{E} \) are the curves \( y^2 + xy = x^3 + a_6 \) and \( y^2 + xy = x^3 + \gamma x^2 + a_6 \), respectively, then it is easily verified that \( \#E(K) + \#\tilde{E}(K) = 2q + 2 \). Henceforth we will always assume that the equation for \( E \) is of the form

\[
y^2 + xy = x^3 + a_6, \quad a_6 \in K^*.
\]

It is well known that \( E \) has the structure of an abelian group, with the point \( \mathcal{O} \) serving as its identity element. The rules for adding points on the curve (3) are the following. Let \( P = (x_1, y_1) \in E^* \); then \( -P = (x_1, y_1 + x_1) \). Notice that \( P \) and \( -P \) have the same \( x \)-coordinates. If \( Q = (x_2, y_2) \in E^* \) and \( Q \neq -P \), then \( P + Q = (x_3, y_3) \), where

\[
x_3 = \begin{cases} 
\left( \frac{y_1 + y_2}{x_1 + x_2} \right)^2 + \frac{y_1 + y_2}{x_1 + x_2} + x_1 + x_2, & P \neq Q, \\
\frac{a_6^2}{x_1^2} + x_2^2, & P = Q,
\end{cases}
\]
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and

$$y_3 = \begin{cases} \left( \frac{y_1 + y_2}{x_1 + x_2} \right)(x_1 + x_3) + x_3 + y_1, & P \neq Q, \\ x_1^2 + \left( \frac{y_1}{x_1} \right)x_3 + x_3, & P = Q. \end{cases}$$

If $K \subseteq L \subseteq \overline{K}$, then $E(L)$ is a subgroup of $E$. If $L$ is finite with $\#L = q^r$, then Hasse's theorem states that

$$\#E(L) = q^r + 1 - t,$$

where $|t| \leq 2\sqrt{q^r}$. Thus, to compute $\#E(L)$, it suffices to compute $t$.

Let $n$ be a positive integer, and let $\mathbb{Z}_n$ denote the cyclic group of $n$ elements. The group $E(K)$ has rank either 1 or 2; we can write $E(K) \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$, where $n_2|n_1$ and $n_2|q - 1$. A point $P \in E$ is called an $n$-torsion point if $nP = \mathcal{O}$.

Let $E[n]$ denote the group of $n$-torsion points in $E$. If $\text{gcd}(n, q) = 1$, then $E[n] \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$. If $n = 2^e$, then either $E[n] \cong \{\mathcal{O}\}$ if $E$ is supersingular, or else $E[n] \cong \mathbb{Z}_n$ if $E$ is nonsupersingular.

We introduce the division polynomials $f_n \in \mathbb{K}[x]$ associated with the nonsupersingular curve $E$ given by the equation (2) (see [6]):

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = x, \quad f_3 = x^4 + x^3 + a_6, \quad f_4 = x^6 + a_6x^2,$$

(4)

$$f_{2n+1} = f_n^3f_{n+2} + f_{n-1}f_{n+1}^3, \quad n \geq 2,$$

(5)

$$xf_{2n} = f_{n-1}f_nf_{n+2} + f_{n-2}f_nf_{n+1}^2, \quad n \geq 3.$$

The polynomials $f_n$ are monic in $x$, and if $n$ is odd, then the degree of $f_n$ is $(n^2 - 1)/2$. The division polynomials have the following useful properties which will enable us to perform computations in $E[n]$. Theorem 1 is from [8], while Theorem 2 is from [6].

**Theorem 1.** Let $P = (x, y) \in E^*$ and let $n \geq 0$. Then $P \in E[n]$ if and only if $f_n(x) = 0$.

**Theorem 2.** Let $n \geq 2$, and let $P = (x, y) \in E^*$ with $nP \neq \mathcal{O}$. Then

$$nP = \left( x + \frac{f_{n-1}f_{n+1}}{f_n^2}, x + y + \frac{f_{n-1}f_{n+1}}{f_n^2} + \frac{f_{n-2}f_{n+1}^2}{xf_n^3} + (x^2 + y^2)\frac{f_{n-1}f_{n+1}}{xf_n^2} \right).$$

The ring of endomorphisms of $E$ that are defined over $K$ is denoted by $\text{End}_K E$. The map $\phi \in \text{End}_K E$ sending $(x, y)$ to $(x^q, y^q)$ and fixing $\mathcal{O}$ is called the Frobenius endomorphism of $E$. In $\text{End}_K E$, $\phi$ satisfies the relation

$$\phi^2 - t\phi + q = 0$$

for a unique $t \in \mathbb{Z}$. In fact, $t = q + 1 - \#E(K)$. If $l$ is an odd prime, then $E[l]$ can be viewed as a vector space over $F_l$; the vector space has dimension 2. The map $\phi$ restricted to $E[l]$ is a linear transformation on $E[l]$ with characteristic equation $\phi^2 - t\phi + q = 0$.

3. **Outline of Schoof's algorithm**

We give an outline of Schoof's algorithm for computing $\#E(K)$, where $K = F_q \cdot q = 2^m$, and $E$ is given by equation (3). The method in [13] is described.
for fields of odd characteristic. More details for the case \( q \) even will be given in §4.

Let \( \#E(F_q) = q + 1 - t \). Choose a prime \( L' \) such that \( \prod l > 4\sqrt{q} \), where the product ranges over all primes \( l \) from 3 to \( L' \). We proceed to compute \( t \pmod{l} \) for each odd prime \( l \leq L' \); since \( |l| \leq 2\sqrt{q} \), we can then recover \( t \) by the Chinese Remainder Theorem.

Let \( P = (\bar{x}, \bar{y}) \in E[l]^* \), and let \( k \equiv q \pmod{l}, 0 \leq k \leq l - 1 \). We search for an integer \( \tau, 0 \leq \tau \leq l - 1 \), such that

\[
\phi^2(P) + kP = \tau \phi(P).
\]

Since \( \phi^2(P) + kP = \tau \phi(P) \), we deduce that \( (t - \tau)\phi(P) = \emptyset \), and hence \( t \equiv \tau \pmod{l} \). The problem with implementing this idea is that the coordinates of \( P \), which are in \( \bar{K} \), may not lie in any small extension of \( K \), and thus cannot be efficiently found in general. We overcome this problem by observing that \( \bar{x} \) is a root of the division polynomial \( f_1(x) \in K[x] \). Moreover, we can use Theorem 2 to obtain an expression for \( kP \) and \( \tau \phi(P) \), where the coordinates of the expressions are rational functions in \( x \) and \( y \). We may then use the addition rules to sum \( \phi^2(P) \) and \( kP \).

To test whether there exists some \( P = (x, y) \in E[l]^* \) satisfying (6), we equate the \( x \)-coordinates of the expressions for \( \phi^2(P) + kP \) and \( \tau \phi(P) \), and eliminate denominators and the variable \( y \) to obtain an equation \( h_1(x) = 0 \). We then compute \( H_1(x) = \gcd(h_1(x), f_1(x)) \). If \( H_1(x) = 1 \), then there is no \( P \in E[l]^* \) satisfying (6). If \( H_1(x) \neq 1 \), then there exists \( P \in E[l]^* \) with \( \phi^2(P) + kP = \pm \tau \phi(P) \). To determine the sign, we equate the \( y \)-coordinates of the expressions for \( \phi^2(P) + kP \) and \( \tau \phi(P) \), eliminate denominators and the variable \( y \) to obtain an equation \( h_2(x) = 0 \), and then compute \( H_2(x) = \gcd(h_2(x), f_1(x)) \). If \( H_2(x) \neq 1 \), then \( P \) satisfies (6), otherwise \( P \) satisfies \( \phi^2(P) + kP = -\tau \phi(P) \). Note that all computations now take place in the ring \( K[x] \).

The running time of \( O(\log^8 q) \) bit operations is obtained as follows. We have that \( L' = O(\log q) \). For each \( l \), the search for \( \tau \) satisfying (6) is dominated by the computations of the residues of \( x^{q^2} \) and \( y^{q^2} \) modulo \( f_1(x) \) (note that \( \phi^2(P) = (x^{q^2}, y^{q^2}) \)). Since the degree of \( f_1(x) \) is \( O(\log^2 q) \), these residues can be computed in \( O(\log^5 q) \) field operations, or \( O(\log^7 q) \) bit operations. If fast multiplication techniques are used for multiplication in \( K[x] \) and in \( F_q \), then the total running time reduces to \( O(\log^{5+\varepsilon} q) \), for any \( \varepsilon > 0 \). However, since the fast multiplication techniques are only practical for very large \( q \), we will henceforth only use classical multiplication algorithms.

4. Some heuristics

Again, we assume that \( K = F_q \), where \( q = 2^m \), and that the curve \( E \) has equation (3). Let \( \#E(F_q) = q + 1 - t \), where \( |t| \leq 2\sqrt{q} \). From the expression for the division polynomial \( f_4 \) we can deduce that \( \#E(F_q) \equiv 0 \pmod{4} \), so we can easily determine \( t \pmod{4} \).

In §§4.1 and 4.2 we describe how to find \( t \pmod{l} \), where \( l \) is an odd prime.

4.1. Finding an eigenvalue of \( \phi \), if one exists. Recall from §2 that when viewing \( \phi \) as a linear transformation on \( E[l] \), the characteristic equation of \( \phi \)
is \( \phi^2 - t\phi + q = 0 \). Thus, \( \phi \) has eigenvalues in \( F_l \) if and only if either \( t^2 - 4q \) is a quadratic residue mod \( l \), or \( t^2 - 4q \) is 0 mod \( l \). Assume that \( s, r \) are eigenvalues of \( \phi \) in \( F_l \). The following two observations are useful.

- Since \( s^2 - ts + q = 0 \), we have \( t \equiv s + q/s \) (mod \( l \)).
- If \( s \neq r \), then let \( S \) denote the set of \( x \)-coordinates of nonzero points in the one-dimensional eigenspace corresponding to \( s \). Observe that if \( \alpha \in S \), then \( \alpha^q \in S \); it follows that \( f(x) = \prod_{\alpha \in S}(x - \alpha) \) is a degree \((l - 1)/2\) factor of \( f_l(x) \) in \( K[x] \).

Let \( w \) be an integer, \( 1 \leq w \leq (l-1)/2 \). To test whether \( \pm w \) is an eigenvalue of \( \phi \), we have to check if there exists \( P = (x, y) \in E[l]^* \) with \( \phi(P) = \pm wP \). Explicitly, we equate the \( x \)-coordinates of \( \phi(P) \) and \( \pm wP \) to obtain

\[
x^q = x + \frac{f_w-1f_{w+1}}{f_w^2}.
\]

Thus, the search is successful if and only if

\[
g_1(x) = \gcd((x^q + x)f_w^2 + f_w-1f_{w+1}, f_l) \neq 1.
\]

The dominant step in these calculations is the computation of \( x^q \) modulo \( f_l(x) \).

If \( g_1(x) \neq 1 \), then we need to test if \( \phi(P) = wP \) or \( \phi(P) = -wP \). The roots of \( g_1(x) \) are the \( x \)-coordinates of points \( P \in E[l]^* \) satisfying \( \phi(P) = \pm wP \). If the eigenvalues of \( \phi \) are \( w \) and \(-w \), then \( t \equiv 0 \) (mod \( l \)), and this will be detected since the degree of \( g_1(x) \) will be \( l - 1 \). If the eigenvalues of \( \phi \) are the same, then \( g_1(x) = f_l(x) \) or the degree of \( f_l(x) \) is \((l - 1)/2\). Otherwise, if either \( w \) or \(-w \) (but not both) is one of the two eigenvalues of \( \phi \) in \( F_l \), then the degree of \( g_1(x) \) is \((l - 1)/2 \). In the following computations, all polynomials in \( x \) are reduced modulo \( g_1(x) \). Equating \( y \)-coordinates of \( \phi(P) \) and \( \pm wP \), and clearing denominators, we obtain the equation

\[
h(x, y) = xf_w^2(y + y^q) + f_w-2f_{w+1}^2 + (x^2 + y)f_{w-1}f_wf_{w+1} = 0.
\]

Since \( y^2 = x^3 + a_6 + xy \), we can compute \( y^q \) by repeatedly squaring \( y^2 \). After \( m - 1 \) squarings, we obtain

\[
y^q = a(x) + b(x)y,
\]

with \( a(x) \) and \( b(x) \) both reduced modulo \( g_1(x) \). Equation (8) then reduces to

\[
\overline{a(x)} + \overline{b(x)}y = 0.
\]

Substituting \( y = \overline{a(x)}/\overline{b(x)} \) into the equation of the curve (3) yields the following equation of the curve:

\[
h(x) = \overline{a(x)}x + \overline{b(x)}x + (x^3 + a_6)\overline{b(x)} = 0.
\]

Finally, if \( \gcd(h(x), g_1(x)) = 1 \), then \( t \equiv w + q/w \) (mod \( l \)), otherwise \( t \equiv -w - q/w \) (mod \( l \)).

We comment that this method of searching for eigenvalues of \( \phi \) easily extends to the case \( q \) an odd prime power.

4.2. Schoof's algorithm. If there is no eigenvalue of \( \phi \) in \( F_l \), i.e., if \( t^2 - 4q \) is a quadratic nonresidue mod \( l \), then we apply Schoof's test to determine the \( \tau \) satisfying (6).
We first check if there is a \( P = (x, y) \in E[l]^* \) with \( \phi^2(P) = \pm kP \), where \( k \) is \( q \) modulo \( l \). This is the case if and only if
\[
\gcd((x^q + x) f_k^2 + f_{k-1} f_{k+1}, f_l) \neq 1.
\]
Observe that if \( t \equiv 0 \) (mod \( l \)), then \( \phi^2(P) = -kP \). Now, if \( \phi^2(P) = kP \), then \( \phi(P) = (2k/t)P \), whence \( \phi \) has an eigenvalue in \( F_l \). But \( t^2 - 4q \) is a quadratic nonresidue \( \text{mod} \ l \), so we conclude that \( \phi^2(P) = -kP \). Thus, \( t\phi(P) = \emptyset \) and \( t \equiv 0 \) (mod \( l \)).

Assume now that there is no \( P \in E[l]^* \) with \( \phi^2(P) = \pm kP \). In order to determine \( t \) (mod \( l \)), we check for each \( \tau, 1 \leq \tau \leq l - 1 \), if there exists \( P \in E[l]^* \) satisfying (6). Since \( \phi^2(P) \neq \pm kP \), we can use the rule for adding distinct points to compute an expression for \( \phi^2(P) + kP \). Explicitly, let \( (P)_x \) denote the \( x \)-coordinate of point \( P \). Then for \( k \geq 2 \)
\[
(\pm t\phi(P))_x = x^q + \frac{f_q}{f_2^q} t\tau_{\frac{1}{2}} f_{\frac{1}{2}+1} \quad \text{\(\text{and}\)}
\]
\[
(\phi^2(P) + kP)_x = x^q + \frac{f_{k-1} f_{k+1}}{f^2_k} + \lambda^2 + \lambda,
\]
where
\[
\lambda = \frac{(y^q + y + x)f^3_k + f_{k-2} f_{k+1}^2 + (x^2 + x + y)(f_{k-1} f_k f_{k+1})}{x f_k^2 (x + x^q) + x f_{k-1} f_{k+1}}.
\]
Similar equations can be obtained for the case \( k = 1 \). Equate the \( x \)-coordinates of \( \phi^2(P) + kP \) and \( \pm t\phi(P) \), and eliminate denominators and the variable \( y \), to get an identity \( h_3(x) = 0 \). Then there exists a \( P \in E[l]^* \) with \( \phi^2(P) + kP = \pm t\phi(P) \) if and only if \( h_3(x) = \gcd(h_3(x), f_l(x)) \neq 1 \). This is repeated for each \( \tau, 1 \leq \tau \leq (l - 1)/2 \), for which \( \tau^2 - 4q \) is a nonresidue (mod \( l \)). If the gcd is nontrivial, then we can determine the correct sign by first equating the \( y \)-coordinates of \( \phi^2(P) + kP \) and \( t\phi(P) \). Explicitly, for \( \tau \geq 2 \),
\[
(t\phi(P))_y = x^q + y^q + \frac{f_q}{f_2^q} t\tau_{\frac{1}{2}} + \frac{f_q}{x f_2^q} + \frac{f^2_q}{x^q f_3^q} + (x^2 q + y^q) \frac{f^2_q}{x f_2 q},
\]
and
\[
(\phi^2(P) + kP)_y = \lambda(x^q + x_3) + x_3 + y^q,
\]
where \( x_3 = (\phi^2(P) + kP)_x \) and \( \lambda \) is as in (10) (similar equations can be obtained for the case \( \tau = 1 \)). As was done above, we then proceed to eliminate the denominator and the variable \( y \) to get an identity \( h_5(x) = 0 \). Then, if \( \gcd(f_l(x), h_5(x)) \neq 1 \), we have \( t = \tau \); otherwise \( t = -\tau \). The dominant step in these calculations is the computation of \( x^q \) and \( y^q \) modulo \( f_l(x) \).

To determine \( t \) (mod \( l \)) in practice, one would first search for an eigenvalue of \( \phi \) in \( F_l \), and if this fails, then Schoof’s algorithm is applied. The first method is faster since it only requires the residue of \( x^q \) modulo \( f_l(x) \), while the second method requires the residues \( x^q, x^q, y^q, \) and \( y^q \) modulo \( f_l(x) \). Heuristically, for a random curve, we would expect \( \phi \) to have an eigenvalue in \( F_l \) (i.e., \( t^2 - 4q \) is a quadratic residue in \( F_l \)) for half of all \( l \)’s. Moreover, if \( \phi \) does have eigenvalues in \( F_l \), then in most cases the eigenvalues will be distinct, and so the test if \( \phi(P) = wP \) or \( \phi(P) = -wP \) in (4.1) takes negligible time (since \( \deg g_1(x) = (l - 1)/2 \) or \( l - 1 \)).
4.3. Determining $t \bmod I = 2^c$. If $l = 2^c$, then the following lemma proves that $f_l(x)$ has a factor of small degree.

**Lemma 3.** If $l = 2^c$, then $f_l(x)$ has a factor $f(x)$ of degree $1/4$ in $K[x]$.

*Proof.* Since $E[l] \cong Z_l$, $f_l(x)$ has only $l/2$ distinct roots. Of these, only $l/4$ are $x$-coordinates of points of order $l$. Thus, $f_l(x)$ has a factor $f(x)$ of degree $1/4$ in $F_q[x]$, whose roots are precisely the $x$-coordinates of points of order $l$. \(\Box\)

The next lemma shows how the factor $f(x)$ may be easily constructed.

**Lemma 4.** Let $l = 2^c$. Define the sequence of polynomials $\{g_i(x)\}$ in $K[x]$ as follows:

- $g_0 = x$,
- $g_1 = b_1 + x$, where $a_6 = b_1^4$,
- $g_i = g_{i-1}^2 + b_i x \prod_{j=1}^{i-2} g_j^2$, where $a_6 = b_i^{2^{i+1}}$, for $i \geq 2$.

Then $f(x) = g_{c-1}(x)$ is a degree $1/4$ factor of $f_l(x)$ in $K[x]$. Moreover, the roots of $f(x)$ are precisely the $x$-coordinates of points of order $l$.

*Proof.* Define the sequence of polynomials $\{h_i(x)\}$ in $K[x]$ by

- $h_0 = 1$,
- $h_1 = x$,
- $h_i = x \prod_{j=1}^{i} g_j^2$ for $i \geq 2$.

Let $P = (x, y) \in E^*$, and let $(2^n P)_x = G_n/H_n$ for $n \geq 0$. From the formula for doubling a point, we see that $G_n$ and $H_n$ are polynomials in $K[x]$. We prove by induction that $G_n = (g_n)^{2^n+1}$ and $H_n = (h_n)^{2^n}$ for $n \geq 1$.

For $n = 1$, we have

$$\frac{G_1}{H_1} = \frac{g_1^4}{h_1^2} = \frac{(b_1 + x)^4}{x^2} = \frac{a_6}{x^2} + x^2,$$

which indeed is $(2P)_x$. Assuming that the statement is true for $n = i$, we have

$$\frac{(2^{i+1} P)_x}{G_{i+1}} = \frac{(b_{i+1} H_i + G_i)^4}{(G_i H_i)^2} = \frac{(b_{i+1} H_i + g_{i+1}^2)^{2^{i+2}}}{(g_{i+1}^2)^{2^{i+1}}} = \frac{(g_{i+1})^{2^{i+2}}}{(h_{i+1})^{2^{i+1}}}.$$

It is also easily proved by induction that $\deg g_n = 2^{n-1}$ for $n \geq 1$, and $\gcd(g_n, h_n) = 1$ for $n \geq 0$.

Now, let $P = (\overline{x}, \overline{y}) \in E^*$. Since $(2^{c-1} P)_x = (g_{c-1})^{2^c}/(h_{c-1})^{2^{c-1}}$, we have $\text{ord}(P) = 2^c$ if and only if $g_{c-1}(\overline{x}) = 0$ and $g_i(\overline{x}) \neq 0$ for $0 \leq i \leq c-2$.

But, since $h_{c-1} = g_0 \prod_{j=1}^{c-2} g_j^2$ and $\gcd(g_{c-1}, h_{c-1}) = 1$, we have $\text{ord}(P) = 2^c$ if and only if $g_{c-1}(\overline{x}) = 0$. Finally, since $\deg g_{c-1} = l/4$, the desired factor $f(x)$ must in fact be $g_{c-1}(x)$. \(\Box\)

For $l = 2^c$ that divides $q$, we have $q \equiv 0 \pmod l$. Hence, for $P \in E[l]^*$, we know that $\phi^2(P) - \tau\phi(P) = \mathcal{O}$. Since $\phi$ is the Frobenius endomorphism,
\( \phi(P) \neq \emptyset \) for \( P \neq \emptyset \). Therefore, \( \phi(P) - \tau P = \emptyset \) and \( \tau \) is an eigenvalue of \( \phi \) in \( \mathbb{Z}_l \).

Since we know that \( \#E(F_q) \equiv 0 \pmod{4} \), we have that \( t \equiv 1 \pmod{4} \) and \( \tau \equiv 1 \pmod{4} \). This gives us only two choices for \( \tau \) modulo 8. We can easily obtain this eigenvalue using a factor of \( f_\ell(x) \) obtained as above, and using our heuristic for finding eigenvalues. This procedure can then similarly be applied to finding eigenvalues for \( l = 16, 32, 64, \ldots \). The method is efficient for \( l \) being a small power of 2, since the polynomial arithmetic is performed modulo a degree \( l/4 \) factor of \( f_\ell(x) \).

4.4. Baby-step giant-step algorithm. The calculation of \( t \) modulo \( l \) using Schoof’s algorithm for small primes \( l \) is very simple. However, since \( \deg(f_\ell(x)) = (l^2 - 1)/2 \), the calculation quickly becomes infeasible as the value of \( l \) increases. In [3], the authors combined Schoof’s algorithm with Shanks’ baby-step giant-step method. In this method, one first computes \( \#E(F_q) \) modulo \( l_0 \cdot l_1 \cdots l_r \), where \( l_1, \ldots, l_r \) are small primes and \( l_0 \) is a small power of 2. We then use the baby-step giant-step algorithm to determine \( \#E(F_q) \).

We describe Shanks’ algorithm with suitable modifications for use with Schoof’s algorithm.

Step 1. Choose a random point \( P \) in \( E(F_q) \) and set

\[
k = \min\{k' | k' \geq \lceil \sqrt{L \cdot 4 \cdot \sqrt{q}} \rceil, \ k' \equiv 0 \pmod{L} \}\.
\]

Step 2. Compute \( iP \) for \( i \equiv (|q + 1 - 2\sqrt{q}| - \#E(F_q)) \pmod{L} \) for \( 0 \leq i \leq k - 1 \). If for some \( i \) we have \( iP = \emptyset \), then return to Step 1. Otherwise, store \( i \) and the first 32 bits of the \( x \)-coordinate of \( iP \) in a table sorted by the entry \( iP \).

Step 3. Set \( Q = kP \).

Step 4. Compute

\[
H_j = \lfloor q + 1 - 2\sqrt{q} \rfloor P + jQ
\]

for \( j = 1, 2, \ldots, k/L \) and check (by a binary search) whether the first 32 bits of the \( x \)-coordinate of \( H_j \) correspond to the first 32 bits of the \( x \)-coordinate of \( iP \) for some \( i \). If it does, we then check if \( H_j = iP \) (by recalculating \( iP \)). If we have only one pair \((i, j)\) with \( H_j = iP \), then

\[
\#E(F_q) = \lfloor q + 1 - 2\sqrt{q} \rfloor + k j - i,
\]

and the algorithm terminates. If not, then return to Step 1.

We sketch the correctness and running time of the algorithm.

Since \( P \in E(F_q) \), then \( \text{ord}(P) \) divides \( \#E(F_q) \). Thus, if there exists a unique integer \( r \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \) such that \( rP = \emptyset \), then

\[
r = \#E(F_q) ; \text{if not, then } \text{ord}(P) \leq 4\sqrt{q}.
\]

Either case is detected in Step 4. Thus in Step 1 we hope that \( \text{ord}(P) > 4\sqrt{q} \).

Recall that \( E(F_q) \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \), where \( n_1 \mid n_2 \) and \( n_2 \mid (q - 1) \). For a random elliptic curve, we would expect \( n_1 \gg n_2 \) and so \( n_1 \gg 4\sqrt{q} \). Thus, with very high probability, \( \text{ord}(P) > 4\sqrt{q} \). Since \( \#E(F_q) \geq (\sqrt{q} - 1)^2 \), we have \( n_1 \geq \sqrt{q} - 1 \). Moreover, since \( 4 \mid \#E(F_q) \) and \( n_2 \) is odd, we have \( n_1 \geq 2(\sqrt{q} - 1) \). If in fact \( n_1 \leq 4\sqrt{q} \), then there is no point in \( E(F_q) \) of order greater than \( 4\sqrt{q} \). This will be detected since the algorithm will fail in Step 4.
each time. If this happens, then we determine \( \text{ord}(P) \) and repeat the algorithm until \( \text{ord}(P) \geq 2(\sqrt{q} - 1) \). We then search for a point \( P' \) which has order \( \geq 3 \) in the quotient group \( E(F_q)/(P) \). For more details, consult [3].

The table in Step 2 has about \( S = 2q^{1/4}/\sqrt{L} \) entries, which are computed with \( O(S) \) field operations. The table is then sorted using \( O(S \log S) \) comparisons. Computing \( H_j \) for \( j = 1, 2, \ldots, k/L \) takes \( O(S) \) field operations, while each binary search takes \( O(\log S) \) comparisons. Thus the whole algorithm takes \( O(q^{1/4}(\log q)^2/\sqrt{L}) \) bit operations, and requires \( O(q^{1/4}(\log q)/\sqrt{L}) \) bits of storage.

4.5. Checking results. Let \( \#E(F_q) = q + 1 - t \), where \( t \) is unknown, and suppose that our algorithm outputs \( \#E(F_q) = q + 1 - t' \). We may verify that \( t = t' \) as follows.

Let \( P \) be the point in the baby-step giant-step algorithm. Since the algorithm terminated, we believe that \( \text{ord}(P) > 4\sqrt{q} \). We first verify that \( (q + 1 - t')P = \mathcal{O} \); if this does not hold, then \( t \neq t' \). We then proceed to factor \( q + 1 - t' \), which is an easy task since \( q + 1 - t' \leq 10^{50} \) for the \( q's \) we are concerned with. Given the prime factorization of \( q + 1 - t' \), we can easily determine \( \text{ord}(P) \), and we then check that \( \text{ord}(P) > 4\sqrt{q} \). Now, since \( (q + 1 - t)P = \mathcal{O} \) and \( (q + 1 - t')P = \mathcal{O} \), we deduce that \( (t - t')P = \mathcal{O} \). Finally, since \( \text{ord}(P) > 4\sqrt{q} \) and \( |t - t'| \leq 4\sqrt{q} \), we conclude that \( t = t' \).

Of course, this check is only successful if \( n_1 > 4\sqrt{q} \), which, as pointed out in §4.4, is true for most curves.

5. Implementation and results

The algorithm described in §4 was implemented in the C programming language on a SUN-2 SPARC station with 64 Mbytes of main memory. We make some comments on our implementation.

(i) The elements of \( F_q = F_{2^m} \) were represented with respect to a normal basis. This has the advantage that squaring a field element involves only a cyclic shift of the vector representation. Explicitly, if \( \beta \) is a normal basis generator and \( \alpha = \sum_{i=0}^{m-1} \lambda_i \beta^{2^i} \), where \( \lambda_i \in F_2 \), then \( \alpha^2 = \sum_{i=0}^{m-1} \lambda_i \beta^{2^{i+1}} \) (with subscripts reduced modulo \( m \)). For computational efficiency in multiplying field elements, we use the special class of normal bases known as optimal normal bases [11]; these bases only exist for certain values of \( m \) but are perhaps the most important for practical purposes.

(ii) Let \( n = \deg f_i(x) \). To compute \( \gcd(A(x), f_i(x)) \) for some \( A(x) \in K[x] \), we first reduce \( A(x) \) modulo \( f_i(x) \), and then compute the gcd of the resulting polynomial with \( f_i(x) \). In order to compute \( x^q \left( \mod f_i(x) \right) \), which is needed, for example, in (7), we precompute the residues \( x^{2j} \) modulo \( f_i(x) \), for \( 0 \leq j \leq n - 1 \). Then \( x^q \left( \mod f_i(x) \right) \) is obtained by repeatedly squaring \( x \). Explicitly,

\[
x^{2j} \left( \mod f_i(x) \right) = (x^{2^{j-1} \left( \mod f_i(x) \right)})^2 \left( \mod f_i(x) \right) = \left( \sum_{j=0}^{n-1} a_j x^j \right)^2 \left( \mod f_i(x) \right) = \sum_{j=0}^{n-1} a_j^2 \left( x^{2j} \left( \mod f_i(x) \right) \right).
\]

The residues of \( x^{q^2}, y^q \), and \( y^{q^2} \) modulo \( f_i(x) \) are obtained in a similar manner.
(iii) In calculating (9) and (11), we need to compute \( f^q_i \pmod{f_i(x)} \), for \( 0 \leq \tau \leq (l - 1)/2 + 1 \). Since we already know \( x^q \pmod{f_i(x)} \), we can easily compute \( f^q_i \pmod{f_i(x)} \) recursively:
\[
\begin{align*}
    f^q_0 &= 0 \pmod{f_i(x)}, \\
    f^q_1 &= 1 \pmod{f_i(x)}, \\
    f^q_2 &= x^q \pmod{f_i(x)}, \\
    f^q_3 &= x^{4q} + x^{3q} + a_6 \pmod{f_i(x)}, \\
    f^q_4 &= x^{6q} + a_6x^{2q} \pmod{f_i(x)}, \\
    f^q_{2i+1} &= f_i^{3q}f^q_{i+2} + f_i^q f_{i+1}^{3q} \pmod{f_i(x)}, \quad i \geq 2, \\
    f^q_{2i} &= s(x)f^q_{i-1} + f^q_{i+2} \pmod{f_i(x)}, \quad i \geq 3,
\end{align*}
\]
where \( s(x) \in K[x] \) satisfies \( s(x)x^q \equiv 1 \pmod{f_i(x)} \). Note that indeed \( \gcd(x^q, f_i(x)) = 1 \)
when \( l \) is odd, since the only points with \( x \)-coordinates equal to 0 have order 2.

(iv) We chose \( l \)'s up to 31 in order to keep manageable the size of the space searched in the baby-step giant-step part of the method. If more memory is available, then the cases \( l = 29 \) and \( l = 31 \) may be excluded, at the expense of an increase in the time for the baby-step giant-step part.

Using the method of §4.3, we also computed \( t \) modulo 64. If \( (t \pmod{64}) < 31 \), then we compute \( t \pmod{128} \) (for this we only need the division polynomials \( f_i(x), 1 \leq i \leq 31 \), modulo the degree-32 factor of \( f_{28}(x) \)). Similarly, if \( (t \pmod{128}) < 31 \), we compute \( t \pmod{256} \). In this way we may compute \( t \pmod{1024} \).

(v) In the baby-step giant-step algorithm we need to select points uniformly at random from \( E(F_q) \). This is accomplished as follows. First pick a random element \( \overline{x} \in F_q \). The probability that \( \overline{x} \) is the \( x \)-coordinate of some \( P \in E(F_q) \) is roughly \( \frac{1}{2} \); this follows from Hasse's theorem. We then attempt to solve the equation
\[
y^2 + \overline{x}y = \overline{x}^3 + a_6
\]
for \( y \). There is a solution if and only if there is a solution to \( y^2 + y = b \), where \( b = \overline{x}^{-2}(\overline{x}^3 + a_6) \). Compute \( b \), and let
\[
b = \sum_{i=0}^{m-1} b_i\beta^{2i} \quad \text{and} \quad \overline{y} = \sum_{i=0}^{m-1} y_i\beta^{2i}.
\]
Then
\[
\overline{y}^2 + \overline{y} = \sum_{i=0}^{m-1} (y_{i-1} + y_i)\beta^{2i} = \sum_{i=0}^{m-1} b_i\beta^{2i}.
\]
Select \( y_0 = 0 \) or \( y_0 = 1 \) at random. Since \( y_0 + y_1 = b_1 \), this determines \( y_1 \). Similarly, \( y_2, y_3, \ldots, y_{m-1} \) are determined. Finally, if \( y_{m-1} + y_0 = b_0 \), then \((\overline{x}, \overline{x}\overline{y})\) is a random point in \( E(F_q) \). Otherwise, \( \overline{x} \) is not the \( x \)-coordinate of a point in \( E(F_q) \).

In Table 1, we list the time taken for the major steps in (4.1), (4.2), and (4.3) of our algorithm for counting points on a single randomly chosen curve over
Table 1. Times (in seconds) for the major steps in (4.1), (4.2), and (4.3) of the algorithm for counting points on a single randomly chosen curve over $F_{q}$, $q = 2^{155}$

| Time to compute $f(x)$, $0 \leq i < 31$ | 245.3 |
| Time to compute $t$ modulo 128 | 162.7 |
| $l$ | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
| (4.1) (a) | 1.7 | 9.4 | 35.6 | 278.1 | 469.8 | 1231.3 | 24.9 | 4612.9 | 11939.1 | 14170.2 |
| (b) | 0.1 | 0.7 | 1.1 | 31.5 | 69.8 | 0.9 | 0.7 | 1243.2 | 778.2 | 5252.0 |
| (c) | - | - | 13.1 | - | - | 88.3 | - | - | 72.3 | - |
| (4.2) (d) | 1.7 | 9.7 | 247.7 | 488.9 | - | 2268.1 | 4890.6 | - | 15188.2 |
| (e) | 11.5 | - | - | 552.6 | 1026.8 | - | 4539.4 | 9525.4 | - | 28869.2 |
| (f) | 3.4 | - | - | 495.4 | 977.7 | - | 4536.3 | 9805.2 | - | 30141.0 |
| (g) | 0.1 | - | - | 87.2 | 299.7 | - | 2036.9 | 6072.8 | - | 22463.9 |
| (h) | 0.7 | - | - | 173.2 | 177.3 | - | 2018.3 | 786.3 | - | 6298.5 |
| (i) | 0.9 | - | - | 213.0 | 348.8 | - | 1831.9 | 3444.4 | - | 9971.7 |

Legend

(a) Compute $x^q$ (mod $f_j(x)$).
(b) Search for an eigenvalue.
(c) Determine the sign of the eigenvalue.
(d) Compute $x^{q^2}$ (mod $f_j(x)$).
(e) Compute $y^q$ (mod $f_j(x)$).
(f) Compute $y^{q^2}$ (mod $f_j(x)$).
(g) Compute $f_j^q$ (mod $f_j(x)$), $0 \leq i \leq (l-1)/2+1$.
(h) Search for $t$, $1 \leq t \leq (l-1)/2$.
(i) Determine the sign of $t$.

As was expected, the computation of $x^q$ (mod $f_j$) dominated the time to search for an eigenvalue, while the computation of $x^{q^2}$, $y^q$, and $y^{q^2}$ modulo $f_j$ is the dominant step in the Schoof part of the algorithm. If an eigenvalue exists, then determining its sign takes negligible time. Observe that searching for an eigenvalue is a useful heuristic, and results in a big time savings should one exist. Lastly, note that the time taken to compute the division polynomials, and to compute $t$ modulo 128, is also negligible.

In Table 2, we list the time for the baby-step giant-step method (step (4.4)) for various problem instances. The size of the space searched is $4\sqrt{q}/L$, where $L$ is the product of those $l$'s for which $t$ modulo $l$ is known.

Table 2. Times for the baby-step giant-step part (step (4.4)) for a curve over $F_{2^{m}}$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$l$'s used in steps 4.1, 4.2, and 4.3</th>
<th>Size of space searched</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>3, 5, 64</td>
<td>$3.9 \cdot 10^2$</td>
<td>0.2 sec</td>
</tr>
<tr>
<td>52</td>
<td>3, 5, 7, 11, 128</td>
<td>$1.8 \cdot 10^3$</td>
<td>0.5 sec</td>
</tr>
<tr>
<td>65</td>
<td>3, 5, 7, 11, 13, 64</td>
<td>$2.5 \cdot 10^4$</td>
<td>1 sec</td>
</tr>
<tr>
<td>82</td>
<td>3, 5, 7, 11, 13, 17, 64</td>
<td>$5.4 \cdot 10^5$</td>
<td>4 sec</td>
</tr>
<tr>
<td>100</td>
<td>3, 5, 7, 11, 13, 17, 64</td>
<td>$2.8 \cdot 10^8$</td>
<td>1 min 43 sec</td>
</tr>
<tr>
<td>113</td>
<td>3, 5, 7, 11, 13, 17, 64</td>
<td>$2.5 \cdot 10^{10}$</td>
<td>18 min 31 sec</td>
</tr>
<tr>
<td>135</td>
<td>3, 5, 7, 11, 13, 17, 19, 23, 64</td>
<td>$1.2 \cdot 10^{11}$</td>
<td>51 min 22 sec</td>
</tr>
<tr>
<td>148</td>
<td>3, 5, 7, 11, 13, 17, 19, 23, 29, 64</td>
<td>$3.6 \cdot 10^{11}$</td>
<td>100 min 42 sec</td>
</tr>
<tr>
<td>155</td>
<td>3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 128</td>
<td>$6.7 \cdot 10^{10}$</td>
<td>44 min 11 sec</td>
</tr>
</tbody>
</table>
Table 3. Total time for counting points on randomly chosen curves over $F_{2^m}$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$l$'s for which an eigenvalue of $\phi$ was found in $F_l$</th>
<th>Total running time (steps (4.1), (4.2), (4.3) and (4.4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>3</td>
<td>1 min 6 sec</td>
</tr>
<tr>
<td>52</td>
<td>3, 5, 7</td>
<td>4 min 51 sec</td>
</tr>
<tr>
<td>65</td>
<td>5</td>
<td>22 min 29 sec</td>
</tr>
<tr>
<td>82</td>
<td>3, 7, 11, 13</td>
<td>57 min 46 sec</td>
</tr>
<tr>
<td>100</td>
<td>5, 7, 11, 17</td>
<td>46 min 21 sec</td>
</tr>
<tr>
<td>113</td>
<td>3, 7, 17</td>
<td>1 hr 8 min 7 sec</td>
</tr>
<tr>
<td>135</td>
<td>3, 7, 13, 19, 23</td>
<td>5 hr 43 min 47 sec</td>
</tr>
<tr>
<td>148</td>
<td>5, 7, 11, 13, 17, 19, 29</td>
<td>16 hr 7 min 26 sec</td>
</tr>
<tr>
<td>155</td>
<td>7, 17, 29</td>
<td>60 hr 29 min 33 sec</td>
</tr>
</tbody>
</table>

Finally, Table 3 presents the total running time of our method for evaluating $#E(F_{2^m})$ for single randomly chosen curves and several values of $m$. For a fixed $m$, the running time for counting $#E(F_{2^m})$ has a large variance; the longest running times happen when no eigenvalue of $\phi$ exists in $F_l$ for the largest prime $l$'s used.

6. A SURVEY OF RECENT WORK

Let $K = F_q$. As observed in §4, there is a degree $(l - 1)/2$ factor $f(x)$ of $f_l(x)$ in $K[x]$ for those primes $l$ for which $\phi$ has distinct eigenvalues in $F_l$. If this factor exists and is known, then it may be used instead of $f_l(x)$ in Schoof's algorithm for a considerable savings in time. In unpublished work, Elkies and Miller independently showed how to construct the factor $f(x)$ without having to first construct $f_l(x)$. In [4], Elkies' work is modified, whereby $f(x)$ can be easily computed after some one-time work. These modifications reduce the work for determining $#E(K)$ from $O(\log^8 q)$ to $O(\log^6 q)$ bit operations. The running of $O(\log^6 q)$ is not rigorously proved since, for example, it is assumed that $t^2 - 4q$ is a quadratic residue modulo $l$ for roughly half of all odd primes $l$. The method is described only for the case $q$ an odd prime, and the generalization to the case $q = 2^m$ does not appear to be straightforward. We are unaware of any implementations of this method.

Recently Atkin [2] described a new algorithm for computing $#E(K)$ which uses modular equations. For each odd prime $l$, the algorithm performs operations in $K[x]$ modulo a polynomial of degree $l + 1$ instead of the polynomial $f_l(x)$ of degree $(l^2 - 1)/2$. Each iteration determines that $t (\mod l) \in S_l$, where $S_l$ is a subset of $\{0, 1, 2, \ldots, l\}$, and where $|S_l| < l/2$ but usually $|S_l| \ll l/2$. This partial information for various $l$'s is then combined to reveal $t$. Again, the algorithm has been described only for the case $q$ an odd prime. The algorithm has not been rigorously analyzed but performs remarkably well in practice. It is almost certain to work when $q \approx 10^{30}$, and Atkin has recently computed $#E(K)$, where $q$ is an odd prime and $q \approx 10^{68}$.

7. Concluding remarks

We have implemented Schoof's algorithm along with some heuristics, and we are able to compute $#E(F_{2^m})$, where $E$ is any elliptic curve over $F_{2^m}$ and $m \leq 155$. For the Schoof part, we were able to compute $t$ modulo
COUNTING POINTS ON ELLIPTIC CURVES OVER $F_{2^m}$

$l$ for $l = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \text{ and } 64 \text{ (and sometimes } l = 128, 256, 512, 1024).$

Computing $#E(F_{2^l})$ takes roughly 61 hours on a SUN-2 SPARC station. (The algorithm takes 61 hours or less provided that $\phi$ has an eigenvalue in either $F_{29}$ or $F_{31}$. Heuristically, one would expect this to occur about 75% of the time for random curves.) On the SPARC station, we can multiply field elements in $F_{2^l}$ at the rate of 900 multiplications per second. There exists a special purpose chip which does the field arithmetic in $F_{2^l}$ and can perform 250,000 multiplications per second [1]. Since roughly 90% of all time of the algorithm is spent in multiplying field elements in $F_{2^m}$, the use of this chip should reduce the time for computing $#E(F_{2^l})$ to about 6 hours.

Possible improvements which we did not implement are the computation of $t$ modulo 27, and using Pollard's Lambda method for catching kangaroos [12] instead of the baby-step giant-step algorithm. Pollard's method has the same expected running time as the latter method, but requires very little storage.

Finally, as pointed out by Atkin [2], we mention that the information obtained from Schoof's algorithm and the heuristics presented here can be combined with the information from Atkin's method to compute $#E(F_{2^m})$ for even larger values of $m$.

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Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

E-mail address: ajmeneze@math.waterloo.edu
E-mail address: savansto@math.waterloo.edu
E-mail address: rjzucche@math.waterloo.edu