ON THE REMAINDER OF GAUSSIAN QUADRATURE FORMULAS FOR BERNSTEIN-SZEGÖ WEIGHT FUNCTIONS

F. PEHERSTORFER

Abstract. We give an explicit expression for the kernel of the error functional for Gaussian quadrature formulas with respect to weight functions of Bernstein-Szegö type, i.e., weight functions of the form \( (1 - x)^\alpha (1 + x)^\beta /\rho(x) \), \( x \in (-1, 1) \), where \( \alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\} \) and \( \rho \) is a polynomial of arbitrary degree which is positive on \([-1, 1]\). With the help of this result the norm of the error functional can easily be calculated explicitly for a wide subclass of these weight functions.

1. Introduction and notation

We consider Gaussian quadrature formulas with respect to a nonnegative weight function \( w \) on the interval \([-1, 1]\),

\[
\int_{-1}^{1} f(x)w(x)\,dx = \sum_{j=1}^{n} \lambda_j f(x_j) + R_n(f, w),
\]

where \( x_j = x_{j,n} \) are the zeros of the \( n \)th-degree monic orthogonal polynomial \( P_n(\cdot, w) \) and \( \lambda_j = \lambda_{j,n} \) are the corresponding Christoffel numbers. If \( f \) is analytic in a domain \( D \) which contains in its interior the interval \([-1, 1]\) and a contour \( \Gamma \) surrounding \([-1, 1]\), the remainder term can be represented as a contour integral (see, e.g., [3])

\[
R_n(f, w) = \frac{1}{2\pi i} \int_{\Gamma} K_n(z, w)f(z)\,dz,
\]

where the kernel \( K_n(\cdot, w) \) is given by

\[
K_n(z, w) = R_n\left(\frac{1}{z - \cdot}, w\right)
\]

or, alternatively, by

\[
K_n(z, w) = \frac{Q_n(z, w)}{P_n(z, w)},
\]

where \( Q_n(\cdot, w) \) is the \( n \)th function of the second kind, i.e.,

\[
Q_n(z, w) = \int_{-1}^{1} \frac{P_n(x, w)}{z - x}w(x)\,dx \quad \text{for} \ z \in \mathbb{C}\setminus[-1, 1].
\]
Let us note that by (1.3), $K_n(\cdot, w)$ has the following series expansion:

\begin{equation}
K_n(z, w) = \sum_{k=2n}^{\infty} \frac{R_n(x^k, w)}{z^{k+1}} \quad \text{for } |z| > 1.
\end{equation}

From (1.2) the following well-known estimate of the remainder, based on contour integration, follows:

\begin{equation}
|R_n(f, w)| \leq \frac{l(\Gamma)}{2\pi} \max_{z \in \Gamma} |K_n(z, w)| \max_{z \in \Gamma} |f(z)|,
\end{equation}

where $l(\Gamma)$ denotes the length of $\Gamma$.

Another useful method to estimate the remainder for a function analytic in $C_r = \{ z \in \mathbb{C} : |z| < r \}$, $r > 1$, has been suggested by Hämmerlin [4], namely: For a function $f(z) = \sum_{k=0}^{\infty} a_k(f) z^k$ analytic in $C_r$ define

\[ |f|_r := \sup\{ |a_k(f)| r^k : k \in \mathbb{N}_0 \text{ and } R_n(x^k, w) \neq 0 \}. \]

Then, $| \cdot |_r$ in the space $\mathcal{X}_r := \{ f : f \text{ analytic in } C_r \text{ and } |f|_r < \infty \}$ is a seminorm. The error functional $R_n(f, w)$ is continuous in $(\mathcal{X}_r, | \cdot |_r)$, and we have

\[ |R_n(f, w)| \leq ||R_n|| |f|_r, \]

where $||R_n||$ can be estimated by

\begin{equation}
||R_n|| \leq \sum_{k=2n}^{\infty} \frac{|R_n(x^k, w)|}{r^k}.
\end{equation}

Equality holds (put $f(z) = 1/(r-z)$, resp. $1/(r+z)$) if for all $k \geq 2n$ the condition

\begin{equation}
R_n(x^k, w) \geq 0, \quad \text{resp. } (-1)^k R_n(x^k, w) \geq 0,
\end{equation}

is fulfilled. Since by [3, Theorem 2.1], proved in [2], and the proof of Theorem 3.1 in [3], the condition

\begin{equation}
w(x)/w(-x) \text{ nondecreasing on } (-1, 1),
\end{equation}

resp.

\begin{equation}
w(x)/w(-x) \text{ nonincreasing on } (-1, 1),
\end{equation}

implies that condition (1.9) holds for all $k \in \mathbb{N}_0$, it follows by (1.6) (see [3, Theorem 3.1]) that

\[
\max_{|z|=r} |K_n(z, w)| = \begin{cases} K_n(r, w) & \text{if } w \text{ satisfies (1.10)}, \\ |K_n(-r, w)| & \text{if } w \text{ satisfies (1.11)}, \end{cases}
\]

and

\[
||R_n|| = \begin{cases} rK_n(r, w) & \text{if } w \text{ satisfies (1.10)}, \\ -rK_n(-r, w) & \text{if } w \text{ satisfies (1.11)}. \end{cases}
\]

Thus, we see that for the estimation of the remainder it is very desirable to have an explicit expression for the kernel $K_n(z, w)$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Very recently, Notaris [8] computed $\|R_n\|$ explicitly for weight functions of the form

$$w(x) = (1 - x)^\alpha (1 + x)^\beta / \rho_2(x) \quad \text{for } x \in (-1, 1),$$

where $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $\rho_2$ is a polynomial of degree at most two which is positive on $[-1, 1]$ and satisfies condition (1.10) or (1.11). For the special case when $\rho_2$ is a polynomial of degree one or a particular even polynomial of degree two, this has been done before by Akrivis [1] (see also Kumar [5, 6]). Let us also mention that a detailed study of the kernel function for the four Chebyshev weight functions, i.e., $\rho_2(x) \equiv 1$ in (1.12), can be found in Gautschi and Varga [3]. In this note we derive an explicit expression for the kernel $K_n(z, w)(\|R_n\|)$ for all Bernstein-Szegö weight functions $w$ (which satisfy condition (1.10) or (1.11)), where a weight function is called a Bernstein-Szegö weight function if it is of the form

$$\pi w_{\alpha, \beta}(x, \rho_m) = (1 - x)^\alpha (1 + x)^\beta / \rho_m(x) \quad \text{for } x \in (-1, 1),$$

with $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $\rho_m$ a polynomial of degree $m$, $m$ arbitrary, which is positive on $[-1, 1]$.

2. MAIN RESULT

First let us recall the well-known fact that the so-called Joukowski transformation

$$y = \frac{1}{2}(z + z^{-1})$$

maps $\{z \in \mathbb{C}: |z| < 1\} \backslash \{0\}$ ($\{z \in \mathbb{C}: |z| > 1\}$) one-to-one onto $\mathbb{C} \backslash [-1, 1]$ and that the inverse transformation is given by

$$z = y (\frac{1}{i} \sqrt{y^2 - 1}),$$

where that branch of $\sqrt{}$ is chosen for which $\sqrt{y^2 - 1} > 0$ for $y \in (1, \infty)$. Note that the transformation (2.1) maps the circumference $|z| = 1$ onto the interval $[-1, 1]$.

The following version of the Fejér-Riesz Theorem on the representation of positive trigonometric polynomials (compare Theorems 1.2.1 and 1.2.2 in [10]) will be needed.

**Lemma.** Let $\rho_m$ be a real positive polynomial on $[-1, 1]$ of exact degree $m$. Then there exists a unique real polynomial

$$g_m(z) = \prod_{\nu=1}^{m} (z - z\nu) \quad \text{with } 0 < |z\nu| < 1 \text{ for } \nu = 1, \ldots, m$$

such that

$$\rho_m(\cos \phi) = c|g_m(e^{i\phi})|^2 \quad \text{for } \phi \in [0, 2\pi],$$

where $c \in \mathbb{R}^+$.

**Proof.** Let $\rho_m(x) = c \prod_{\nu=1}^{m} (\alpha\nu - x)$, where $c \in \mathbb{R}$ and the $\alpha\nu$’s are either in $\mathbb{R} \backslash [-1, 1]$ or appear in pairs of complex conjugate numbers. Hence, if we set

$$z\nu = \alpha\nu - \sqrt{\alpha\nu^2 - 1} \quad \text{for } \nu = 1, \ldots, m,$$
then

\[ 0 < |z_\nu| < 1 \quad \text{and} \quad \alpha_\nu = \frac{1}{2}(z_\nu + z_\nu^{-1}) \quad \text{for} \quad \nu = 1, \ldots, m. \]

Thus,

\[ \rho_m(\cos \varphi) = \hat{c} \prod_{\nu=1}^{m} \left( \frac{1 + z_\nu^{-2}}{2z_\nu} - \cos \varphi \right) = c \prod_{\nu=1}^{m} |e^{i\varphi} - z_\nu|^2, \]

since

\[ (e^{i\varphi} - z_\nu)(e^{-i\varphi} - z_\nu) = 2z_\nu \left( \frac{1 + z_\nu^{-2}}{2z_\nu} - \cos \varphi \right) \]

and the \( z_\nu \)'s are real or appear in pairs of complex conjugate numbers.

Now the uniqueness remains to be shown. Suppose that

\[ \rho_m(\cos \varphi) = d \prod_{\nu=1}^{m} |e^{i\varphi} - v_\nu|^2, \]

where \( d \in \mathbb{R}^+, \quad v_\nu \in \{ z \in \mathbb{C} : |z| < 1 \}\{0\} \) for \( \nu = 1, \ldots, m \), and the \( v_\nu \)'s are real or complex conjugate. Then it follows as above that \( (v_\nu + v_\nu^{-1})/2 \), \( \nu = 1, \ldots, m \), is a zero of \( \rho_m(\cos \varphi) \) and thus, since \( 0 < |v_\nu| < 1 \) and since the Joukowski transformation is one-to-one, the uniqueness follows. \( \square \)

Let us note that other representations of \( \rho_m \) of the form (2.4), but with \( g_m \) having \( m - l \), resp. \( l \), \( l \in \{1, \ldots, m\} \), zeros inside, resp. outside, of the unit disk, can be obtained by replacing (2.5) by

\[ z_{\nu_j} = \alpha_{\nu_j} + \sqrt{\alpha_{\nu_j}^2 - 1} \quad \text{for} \quad j = 1, \ldots, l, \]

where \( \{\nu_1, \ldots, \nu_l\} \) is an arbitrary subset of \( \{1, \ldots, m\} \), and

\[ z_\nu = \alpha_\nu - \sqrt{\alpha_\nu^2 - 1} \quad \text{for} \quad \nu \in \{1, \ldots, m\}\{\nu_1, \ldots, \nu_l\}. \]

Now let us set

\[ \pi w(x, \rho_m) = \frac{1}{\sqrt{1 - x^2 \rho_m(x)}} \quad \text{for} \quad x \in (-1, 1) \]

and let \( g_m \) be the unique polynomial from the above lemma. Then it follows by well-known results of Bernstein and Szegö (see, e.g., [10, p. 31] and set \( h_m(z) = \sqrt{c} \pi \rho_m(x) \)) that, with \( z = e^{i\varphi} \) and \( x = \cos \varphi \),

\[ 2^{n-1} P_n(x, w(\cdot, \rho_m)) = \sum_{j=0}^{m} a_j T_{n-j}(x) \]

= \text{Re}\{z^{n-m}g_m(z)\} \quad \text{for} \quad 2n > m, 

\[ 2^{n-1} P_{n-1}(x, (1 - x^2)w(\cdot, \rho_m)) = \sum_{j=0}^{m} a_j U_{n-1-j}(x) \]

= \text{Im}\{z^{n-m}g_m(z)\}/\sin \varphi \quad \text{for} \quad 2n > m, 

\[ (2.7) \]

\[ 2^n P_n(x, (1 + x)w(\cdot, \rho_m)) = \sum_{j=0}^{m} a_j \frac{T_{n+1-j}(x) + T_{n-j}(x)}{x + 1} \]

= \text{Re}\{z^{n-m+1/2}g_m(z)\}/\cos(\varphi/2) \quad \text{for} \quad 2n + 1 > m, 

\[ 2^n P_n(x, (1 - x)w(\cdot, \rho_m)) = \sum_{j=0}^{m} a_j \frac{T_{n+1-j}(x) - T_{n-j}(x)}{x - 1} \]

= \text{Im}\{z^{n-m+1/2}g_m(z)\}/\sin(\varphi/2) \quad \text{for} \quad 2n + 1 > m, \]
where $g_m(z) = \sum_{j=0}^{m} a_j z^{m-j}$ and $T_j$, resp. $U_j$, denotes the Chebyshev polynomial of degree $j$ of the first, resp. second, kind on $[-1, 1]$.

We mention in passing that if $g_m$ in (2.7) is replaced by a polynomial $\tilde{g}_m$ which also satisfies (2.4) but does not have all zeros in the open unit disk, then the polynomials on the right-hand side in (2.7) are not orthogonal with respect to $w_{a, b} (\cdot, \rho_m)$, $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$. In fact (see [9, Corollary 5], corresponding results hold also for $\alpha = -\beta = \pm \frac{1}{2}$), they are orthogonal with respect to a functional $\Psi$ of the form

$$\Psi(p) = \int_{-1}^{1} p(x) w_{a, b} (x, \rho_m) \, dx + L(p) \text{ for } p \in \mathbb{P},$$

where $L$ is a functional given by

$$L(p) = \sum_{j} \sum_{\kappa} \mu_{\kappa, j} p^{(\kappa-1)} (\alpha_{\nu_j}),$$

and the $\alpha_{\nu_j}$'s are those zeros of $\rho_m$ which correspond to the zeros of $\tilde{g}_m$ lying outside of the unit disk by (2.6), $l_j$ is the multiplicity of the zero $\alpha_{\nu_j}$, and the $\mu_{\kappa, j}$'s are certain real numbers.

We now give the announced explicit expression for the kernel function $K_n(z, w_{a, b})$, $|\alpha| = |\beta| = \frac{1}{2}$.

**Theorem.** Let $\rho_m$ be given by (2.4). Then we have for $y \in \mathbb{C} \setminus [-1, 1]$, on writing $y = \frac{1}{2} (z + z^{-1})$ with $|z| < 1$, i.e., $z = y - \sqrt{y^2 - 1}$, that

$$cK_n(y, w(\cdot, \rho_m)) = \frac{4 z^{2n+1}}{(1 - z^2) g_m^* (z) \left\{ z^{2n-m} g_m(z) + g_m^*(z) \right\} } \text{ for } 2n > m,$$

$$cK_n(y, (1 + x) w(\cdot, \rho_m)) = \frac{2 z^{2n+1} (z + 1)}{(1 - z) g_m^* (z) \left\{ z^{2n+1-m} g_m(z) + g_m^*(z) \right\} } \text{ for } 2n + 1 > m,$$

$$cK_n(y, (1 - x^2) w(\cdot, \rho_m)) = \frac{z^{2n+1} (z^2 - 1)}{g_m^* (z) \left\{ z^{2n+2-m} g_m(z) - g_m^*(z) \right\} } \text{ for } 2n + 2 > m,$$

where $g_m^* (z) = z^m g_m (\frac{1}{z})$.

**Proof.** Let $R$ and $S$ be monic polynomials of degree at most two such that

$$R(y) S(y) = y^2 - 1,$$

and let us put for abbreviation

$$P_n(x) := P_n(x, R w(\cdot, \rho_m)) \text{ and } \tilde{P}_n(x) := P_n(x, S w(\cdot, \rho_m)).$$

Using the simple fact that for $k \in \mathbb{Z}$ and $\varphi \in [0, 2\pi]$

$$(2.8) \quad \left[ \text{Re} \{ e^{ik\varphi} g_m(e^{i\varphi}) \} \right]^2 + \left[ \text{Im} \{ e^{ik\varphi} g_m(e^{i\varphi}) \} \right]^2 = |g_m(e^{i\varphi})|^2,$$

we get, using (2.7) and (2.4), that with $l = n + \partial R - 1$, where $\partial R$ denotes the exact degree of $R$,

$$R P_n^2 - S \tilde{P}_l^2 = k_n \rho_m,$$
where
\[ k_n = \begin{cases} 
2^{-2n+2}/c & \text{for } R(x) \equiv 1, \\
-2^{-2n}/c & \text{for } R(x) \equiv x^2 - 1, \\
(-2)^{-2n+1}/c & \text{for } R(x) \equiv x + 1 (x - 1).
\end{cases} \]
Furthermore, it follows from Theorem 3(a) of our paper [9] that for \( 2n \geq m + 1 - \partial R \)
\[ R P_n^2 - S(Y P_n + \rho_m P_{n-1}^{(1)})^2 = d_n \rho_m, \]
where \( P_{n-1}^{(1)} \) denotes the associated polynomial of \( P_n \), i.e.,
\[ P_{n-1}^{(1)}(y) = \int_{-1}^{1} \frac{P_n(y) - P_n(x)}{y - x} R(x) w(x, \rho_m) \, dx, \]
and \( Y \in \mathbb{P}_\mu, \mu = \max\{m - 1, \partial R - 1\} \), is uniquely determined by the conditions that at each zero \( \alpha\nu \) of \( \rho_m(x) = c \prod_{\nu=1}^{m^*_\nu} (\alpha\nu - x)^{m\nu} \), where \( c \in \mathbb{R} \) and \( m\nu \) is the multiplicity of the zero \( \alpha\nu \),
\[ Y^{(j)}(\alpha\nu) = \left( R/\sqrt{y^2 - 1} \right)^{(j)}(\alpha\nu) \quad \text{for } j = 0, \ldots, m\nu - 1, \]
and that for \( y \to \infty \)
\[ \frac{\left( R/\sqrt{y^2 - 1} - Y \right)(y)}{\rho_m(y)} = O(y^{-1}); \]
furthermore,
\[ d_n = 2 \int_{-1}^{1} P_n^2(x) R(x) w(x, \rho_m) \, dx. \]
(We note that in the definition of \( 1/h \) in [9, p. 461] \((-1)^{l-k}/\sqrt{-H}\) is to be replaced by \((-1)^{l-k}/\pi\sqrt{-H}\). It now follows from [10, (2.6.5)] that the leading coefficient of the orthonormal polynomial of degree \( n \) with respect to \( Rw(\cdot, \rho_m) \) is equal to \( \sqrt{2/k_n} \) for \( 2n \geq m + 1 - \partial R \), which implies that \( d_n = k_n \) and thus, in view of (2.9) and (2.10),
\[ \pm P_n = Y P_n + \rho_m P_{n-1}^{(1)} \quad \text{for } 2n \geq m + 1 - \partial R. \]
For a function \( f \) defined on \( C\backslash[-1, 1] \) and for \( x \in (-1, 1) \) we write, provided the limits involved exist,
\[ f^{(-)}(x) := \lim_{z \to x^-} f(z), \]
where \( C^{(-)} := \{ z \in C: \text{Im } z \geq 0 \} \). Observing that by (2.11) and (2.12)
\[ \Phi(y) := \frac{(R/\sqrt{y^2 - 1} - Y)(y)}{\rho_m(y)} \]
is analytic on \( C\backslash[-1, 1] \) and vanishes at infinity, and that the boundary values \( \Phi^\pm(x), \ x \in (-1, 1), \) from the upper (lower) half plane satisfy the relation
\[ \Phi^+(x) - \Phi^-(x) = \frac{2}{i} \frac{R(x)}{\rho_m(x) \sqrt{1 - x^2}} \quad \text{for } x \in (-1, 1), \]
where we have used the fact that
\[(\sqrt{y^2 - 1})^+(x) = i\sqrt{1 - x^2} = -(\sqrt{y^2 - 1})^-(x),\]
we get by the Sokhotski-Plemelj formula (see, e.g., [7])
\[
\Phi(y) = \frac{1}{\pi} \int_{-1}^{+1} \frac{R(x)}{y - x - (\lambda + \mu)} dx = Q_0(y, \rho(y, \rho_m)) \quad \text{for } y \in \mathbb{C}\{[-1, 1]\}.
\]
Recalling the well-known fact (see, e.g., [10, §3.5]) that for sufficiently large \(|y| > 7\),
\[
\frac{P_n^{(1)}(y)}{P_n(y)} = Q_0(y, \rho(y, \rho_m)) + O(y^{-n+1}),
\]
we get, using (2.16) and (2.15), that
\[
(Y P_n + \rho_m P_n^{(1)})(y) = P_n(y) R(y) / \sqrt{y^2 - 1} + O(y^{-n+1+m})
\]
and thus, since for \(2n > m + 1 - \partial R\)
\[
\lim_{y \to \infty} y^{-n+1}\{P_n(y) R(y) / \sqrt{y^2 - 1} + O(y^{-n+1+m})\} = 1,
\]
the polynomial \(Y P_n + \rho_m P_n^{(1)}\), which by (2.14) is of exact degree \(n + \partial R - 1\),
has leading coefficient one. Hence, the plus sign holds in (2.14). Thus, the \(n\)th function of the second kind is of the form
\[
Q_n(y, \rho(y, \rho_m)) = -P_n^{(1)}(y) + P_n(y) Q_0(y)
\]
(2.17)
\[
= \frac{\sqrt{R(y) P_n(y) - \sqrt{S(y) \mathcal{P}_n(y)}}}{\sqrt{S(y) \rho_m(y)}} = \frac{k_n}{(\sqrt{y^2 - 1 P_n + \mathcal{S} \mathcal{P}_n)(y)}},
\]
where the second equality follows with the help of (2.16), (2.15), and (2.14),
and the third equality with the help of (2.9). Now the following equalities hold on the circumference \(|z| = 1\):
\[
2^n P_n \left(\frac{1}{2}(z + z^{-1}), w(\cdot, \rho_m)\right) = z^{-n}(z^{2n-m} g_m(z) + g_m^*(z)),
\]
(2.18)
\[
2^{n-1} P_{n-1} \left(\frac{1}{2}(z + z^{-1}), (1 - x^2)w(\cdot, \rho_m)\right)
\]
\[
= \frac{z^{-n}(z^{2n-m} g_m(z) - g_m^*(z))}{z - z^{-1}},
\]
\[
2^n P_n \left(\frac{1}{2}(z + z^{-1}), (1 + x)w(\cdot, \rho_m)\right)
\]
\[
= \frac{z^{-n}(z^{2n+1-m} g_m(z) + g_m^*(z))}{z + z^{-1}}.
\]
Since all functions appearing in (2.18) are analytic in the domain \(\mathbb{C}\{0\}\),
it follows that in (2.18) equality holds also on \(\mathbb{C}\{0\}\). Hence, we get from (2.17)
and (2.18) that for \( y \in C[-1,1] \), on writing \( y = \frac{1}{2} (z + z^{-1}) \) with \(|z| < 1\),

\[ Q_n(y, w(\cdot, \rho_m)) = \frac{2^{-n+2}}{c} \frac{z^{n+1}}{(1 - z^2) g_m^*(z)}, \]

(2.19) \[ Q_n(y, (1 - x^2) w(\cdot, \rho_m)) = \frac{2^{-n}}{c} \frac{z^{n+1}}{g_m^*(z)}, \]

\[ Q_n(y, (1 + \epsilon) x w(\cdot, \rho_m)) = \frac{2^{-n+1}}{c} \frac{z^{n+1}}{(1 + \epsilon) g_m^*(z)}, \]

where we have used the fact that \( \sqrt{y^2 - 1} = (z^{-1} - z)/2 \). Relation (2.19) in conjunction with (2.18) and (1.4) gives the assertion. \( \square \)

In the remark below we state sufficient conditions on the weight function \( w_\alpha, \beta(x, \rho_m) \), defined in (1.13), such that (1.10), resp. (1.11), is fulfilled. Since the product \( w_1(x) w_2(x) \) of two weight functions \( w_1, w_2 \) satisfies condition (1.10), resp. (1.11) if \( w_1 \) and \( w_2 \) satisfy (1.10), resp. (1.11), we consider the behavior of \( w(x, \rho_m)/w(-x, \rho_m) \) for \( m \in \{1, 2\} \) only.

**Remark.** The ratio \( w(x, \rho)/w(-x, \rho) \) is nondecreasing (nonincreasing) on \((-1, 1)\) if

\[ \rho(x) = \left\{ \begin{array}{ll}
(\alpha - x), & \alpha \in (1, \infty), (-\infty, -1), \\
(\alpha - x)(x - \beta), & \alpha \in (1, \infty), \beta \in (-\infty, -1), \text{ and } -\beta \leq \alpha, \\
(x - \alpha)^2 + \beta^2, & \alpha \in \mathbb{R}^{+}, \beta \in \mathbb{R}, \text{ and } \alpha^2 + \beta^2 \geq 1,
\end{array} \right. \]

where the expressions in parentheses refer to the case of nonincreasing ratio.

Setting in the preceding theorem

\[ g_1(z) = z + \tilde{a}, \quad \tilde{a} \in (-1, 1), \quad \text{i.e., } |g_1(e^{i\varphi})|^2 = 1 + \tilde{a}^2 + 2\tilde{a}x, \]

resp. for \( b > 0 \)

\[ g_2(z) = z^2 + (1 + 2b)^{-1}, \quad \text{i.e., } |g_2(e^{i\varphi})|^2 = 4(b^2 + (1 + 2b)x^2)/(2b + 1)^2, \]

where \( x = \cos \varphi \), we obtain the results of Kumar [5, 6] concerning the functions of the second kind, and the results of Akrivis [1] on the norm of the error functional \( R_n(\cdot, w_\alpha, \beta, |g_1(e^{i\varphi})|^2) \), \( j = 1, 2 \). If we put

\[ g_2(z) = z^2 + \frac{2\delta}{\beta} z + \left(1 - \frac{2\alpha}{\beta}\right) \]

with \( 0 < \alpha < \beta \), \( \beta \neq 2\alpha \), and \( |\delta| < \beta - \alpha \), which gives

\[ \frac{\beta^2}{4} |g_2(e^{i\varphi})|^2 = \beta(\beta - 2\alpha)x^2 + 2\delta(\beta - \alpha)x + \alpha^2 + \delta^2, \]

we obtain the results of Notaris [8] on the norm of the error functional, using his conditions (2.31)–(2.42) on the parameters \( \alpha, \beta, \gamma, \delta \) under which the function \( w(x, |g_2(e^{i\varphi})|^2)/w(-x, |g_2(e^{i\varphi})|^2) \) is strictly increasing, resp. strictly decreasing.

**Acknowledgment**

I would like to thank the referee for a careful reading of the manuscript and for his valuable comments.
BIBLIOGRAPHY


Institut für Mathematik, Johannes Kepler Universität Linz, A-4040 Linz, Austria