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# $L^1$-STABILITY OF STATIONARY DISCRETE SHOCKS

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ABSTRACT. The nonlinear stability in the $L^p$-norm, $p \geq 1$, of stationary weak discrete shocks for the Lax-Friedrichs scheme approximating general $m \times m$ systems of nonlinear hyperbolic conservation laws is proved, provided that the summations of the initial perturbations equal zero. The result is proved by using both a weighted estimate and characteristic energy method based on the internal structures of the discrete shocks and the essential monotonicity of the Lax-Friedrichs scheme.

## 1. INTRODUCTION

In this paper, we study the asymptotic stability of the Lax-Friedrichs scheme,

$$u_j^{n+1} - u_j^n + \tfrac{\lambda}{2}\big(f(u_{j+1}^n) - f(u_{j-1}^n)\big) = \tfrac{\nu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\,, \tag{1.1}$$

approximating general systems of nonlinear conservation laws,

$$u_t + f(u)_x = 0\,, \tag{1.2}$$

for a stationary shock solution,

$$u(x\,,t) = \begin{cases} u_-\,, & x < 0\,, \\ u_+\,, & x > 0\,, \end{cases} \tag{1.3a}$$

where $u = u(x\,,t) \in \mathbf{R}^m$, $f$ is a smooth nonlinear mapping from $\mathbf{R}^m$ to $\mathbf{R}^m$, $u_\pm$ are two constant vectors in $\mathbf{R}^m$ satisfying the Rankine-Hugoniot condition,

$$f(u_-) = f(u_+)\,, \tag{1.3b}$$

and the Lax entropy condition,

$$\lambda_k(u_+) < 0 < \lambda_k(u_-)\,, \tag{1.3c}$$

for a genuinely nonlinear field $k$; $u_j^n$ is an approximation of $u(x_j\,,t_n)$, $x_j = j\Delta x$ and $t_n = n\Delta t$, with $\Delta x$ and $\Delta t$ being the spatial and temporal grid sizes; $\nu$ is a constant satisfying $0 < \nu < 1$, and the temporal and spatial grid ratio $\lambda = \Delta t/\Delta x$ satisfies a Courant-Friedrichs-Levy condition,

$$\lambda \sup_\mu |\lambda_\mu(u)| \leq \nu\,. \tag{1.4}$$

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We shall assume that the system in (1.2) is strictly hyperbolic in the sense that at each state $u \in \mathbf{R}^m$ the Jacobian $\nabla f(u)$ has $m$ real and distinct eigenvalues, $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_m(u)$, with corresponding left and right eigenvectors $l_\mu(u)$ and $r_\mu(u)$, respectively, and that each characteristic field is either genuinely nonlinear or linear degenerate in the sense of Lax [5], i.e., for $\mu = 1, \dots, m$, the eigenvector $r_\mu$ satisfies $\nabla\lambda_\mu \cdot r_\mu \equiv 1$ or $\nabla\lambda_\mu \cdot r_\mu \equiv 0$. In the following, we normalize the eigenvectors so that $l_\mu(u) r_k(u) = \delta_{\mu k}$ and define the $m \times m$ matrices $L(u)$, $R(u)$ and $\Lambda(u)$ by

$$L(u) = \left(l_1(u)^\top, \dots, l_m(u)^\top\right)^\top, \qquad R(u) = \left(r_1(u), \dots, r_m(u)\right),$$
$$\Lambda(u) = \operatorname{diag}\left(\lambda_1(u), \dots, \lambda_m(u)\right).$$

The main goal of this paper is to show the following nonlinear stability result on the stationary shock profile solution $\phi_i$ of (1.1), i.e.,

$$\lambda\left(f(\phi_{j+1}) - f(\phi_{j-1})\right) = \nu(\phi_{j+1} - 2\phi_j + \phi_{j-1}), \tag{1.5a}$$

$$\phi_j \to u_\pm \quad \text{as } j \to \pm\infty, \tag{1.5b}$$

which is called a stationary discrete shock. Its existence has been proved by Majda and Ralston [8] and compressibility and asymptotic properties for it are established in [7].

**Theorem 1.1.** *Suppose that* (1.2) *is a strictly hyperbolic system and the $k$-characteristic field is genuinely nonlinear. Let $\phi_j$ be the stationary discrete shock profile* (1.5) *in $k$-field connecting $u_+$ to $u_-$. We assume*

$$\sum_{j=-\infty}^{\infty} (u_j^0 - \phi_j) = 0, \tag{1.6a}$$

$$\varepsilon = |u_+ - u_-| \le c_1, \tag{1.6b}$$

*and*

$$\sum_{j=-\infty}^{\infty} (1 + j^2)^2 |u_j^0 - \phi_j|^2 \le c_2, \tag{1.6c}$$

*for some (suitably small) positive constants $c_1$ and $c_2$. Then, there exists a unique global solution, $u_j^n$, to the Lax-Friedrichs scheme,* (1.1) *with initial data $u_j^0$, and it satisfies*

$$\lim_{n\to\infty} \sum_{j=-\infty}^{\infty} |u_j^n - \phi_j|^p = 0 \tag{1.7}$$

*for all $p > 1$ and*

$$\sup_{0\le n<\infty} \sum_{j=-\infty}^{\infty} |u_j^n - \phi_j| < \infty. \tag{1.8}$$

*Remark* 1.1. One would expect that stability estimate (1.8) in Theorem 1.1 implies the following error estimate for the Lax-Friedrichs scheme (1.1) approximating systems of conservation laws (1.2) with stationary shock solution $u(x, t)$ of form (1.3):

$$\left\|u(\cdot, t) - u_h(\cdot, t)\right\|_{L^1} \le Ch, \tag{1.9}$$

where $u_h(x, t)$ is the approximate solution and $C$ is a positive constant independent of the grid size $h$. The error estimate in (1.9) shall be optimal. It has been achieved by Jennings in the scalar case [3]. It remains to combine some initial-layer estimates with (1.8) to obtain (1.9). This is left for the future. The $L^1$-norm is the natural norm in which to measure the stability of the shock waves; it is of both mathematical and physical significance. So far as we know, our $L^1$-stability result in Theorem 1.1 is the first one in $L^1$-stability of shock waves for systems of conservation laws.

*Remark* 1.2. We also study the nonlinear stability of nonstationary discrete shocks for the Lax-Friedrichs scheme approximating general $m \times m$ systems of nonlinear hyperbolic conservation laws. Both single discrete shock and multiple discrete shock are proved nonlinearly stable. This will appear in a forthcoming paper [7]. Because of the complicated structure of nonstationary discrete shocks, the analysis in [7] is technically much more involved. The main contribution of this article is to present a different (and simpler) method in the case of stationary discrete shock.

*Remark* 1.3. In the original Lax-Friedrichs scheme [5], $\nu = 1$. However, we do not expect asymptotic stability of the discrete shock profiles in this case. In fact, we can easily verify that stationary discrete shock profiles of the Lax-Friedrichs scheme for the scalar equations are not asymptotically stable. We note that the theorem of Jennings for the scalar equation also excludes the case $\nu = 1$ [3].

Our stability analysis is strongly motivated by the nonlinear stability of viscous shock waves for systems of viscous hyperbolic conservation laws of the form

$$u_t + f(u)_x = \nu u_{xx}, \qquad \nu > 0. \tag{1.10}$$

They have been extensively studied in the last three decades. Recently, important progress has been made by Goodman [1], Kawashima and Matsumura [4], Liu [6], and Szepessy and Xin [13] in the study of asymptotic stability of viscous shock profiles for a large class of viscous hyperbolic conservation laws. They showed that a weak viscous shock profile is nonlinearly stable in the $L^2$-norm in the sense that a small initial disturbance, under suitable restriction, will die out as time tends to infinity. In the scalar case, Osher and Ralston [10] proved $L^1$-stability for viscous shocks.

The study of existence and stability of discrete shocks is important in understanding the convergence behavior of numerical shock computations. Jennings [3] proved the stability of discrete shock profiles for the general first-order monotone schemes for the scalar conservation laws, see also [2]. The existence of discrete shock profiles of first-order accurate finite difference methods for systems of conservation laws was established by Majda and Ralston [8] by means of the center manifold theorem, see also [9]. Smyrlis [11] proved stability of a scalar stationary discrete shock wave for the Lax-Wendroff scheme. Szepessy [12] studied the existence and $L^2$-stability of stationary discrete shocks for a first-order implicit streamline diffusion finite element method for systems of conservation laws.

## 2. Stability analysis

In this section we proceed to prove Theorem 1.1—the nonlinear stability of stationary discrete shocks. We first use a weighted energy method to get

an a priori estimate in the $L^2$-norm. In contrast with the scalar case, owing to the coupling of waves from different characteristic families, even the linear stability analysis in the $L^1$-norm is very difficult. We overcome this difficulty by carefully choosing weights so that propagation of waves in the principal direction dominates waves in transversal wave directions. The former can be estimated in the $L^1$-norm by using the essential monotonicity of the scheme in the principal direction. This, together with the $L^2$-nonlinear stability analysis, yields the desired result.

We first reformulate the problem as follows. Let $u_j^n$ be a solution of the Lax-Friedrichs scheme (1.1) with initial data $u_j^0$ satisfying (1.6a), which is assumed to exist up to $n \le n_1 < +\infty$. Denote by $\phi_j$ the stationary discrete shock profile in the $k$-field whose existence has been proved in [8]. Setting

$$\bar{v}_j^n = \sum_{i=-\infty}^{j} (u_i^n - \phi_i)\,, \tag{2.1}$$

we obtain after subtracting (1.5a) from (1.1), summing up the resulting expression from $-\infty$ to $j$, and using some manipulations that

$$\begin{aligned} &\bar{v}_j^{n+1} - \bar{v}_j^n + \tfrac{\lambda}{2}\nabla f(\phi_{j+1})(\bar{v}_{j+1}^n - \bar{v}_j^n) + \tfrac{\lambda}{2}\nabla f(\phi_j)(\bar{v}_j^n - \bar{v}_{j-1}^n) \\ &\quad + \tfrac{\lambda}{2}Q(\phi_{j+1}, \bar{v}_{j+1}^n - \bar{v}_j^n) + \tfrac{\lambda}{2}Q(\phi_j, \bar{v}_j^n - \bar{v}_{j-1}^n) \\ &= \tfrac{\nu}{2}\left(\bar{v}_{j+1}^n - 2\bar{v}_j^n + \bar{v}_{j-1}^n\right), \end{aligned} \tag{2.2}$$

where

$$Q(\phi, u - \phi) = f(u) - f(\phi) - \nabla f(\phi)(u - \phi) \tag{2.3a}$$

satisfies the estimate

$$|Q(\phi, u - \phi)| \le O(1)|u - \phi|^2 \tag{2.3b}$$

for $u$ on any bounded set. Using the notations

$$L_j = L(\phi_j)\,, \quad \Lambda_j = \Lambda(\phi_j)\,, \quad R_j = R(\phi_j)\,, \quad \theta_j^n = Q(\phi_j, \bar{v}_j^n - \bar{v}_{j-1}^n)\,,$$

we may rewrite system (2.2) in terms of characteristic variables

$$v_j^n = L_j \bar{v}_j^n$$

as

$$\begin{aligned} &v_j^{n+1} - v_j^n + \tfrac{\lambda}{2}\Lambda_{j+1}(v_{j+1}^n - v_j^n) + \tfrac{\lambda}{2}\Lambda_j(v_j^n - v_{j-1}^n) - \tfrac{\nu}{2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) \\ &\quad = \tfrac{\lambda}{2}\Lambda_j(L_{j+1} - L_{j-1})R_j v_j^n + e_j^n\,, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} e_j^n = {} & \tfrac{\lambda}{2}\,(\Lambda_{j+1} - \Lambda_j)\,(L_{j+1} - L_j)\,R_j v_j^n - \tfrac{\lambda}{2}\,L_j\,(R_{j+1} - R_j)\,\Lambda_{j+1}\,(v_{j+1}^n - v_j^n) \\ & + \tfrac{\lambda}{2}\,L_j\,(R_{j+1} - R_j)\,\Lambda_{j+1}\,(L_{j+1} - L_j)R_j v_j^n \\ & - \tfrac{\lambda}{2}\,\Lambda_j\,(L_j - L_{j-1})\,R_{j-1}\,(v_j^n - v_{j-1}^n) \\ & - \tfrac{\lambda}{2}\,\Lambda_j\,(L_j - L_{j-1})\,(R_j - R_{j-1})\,v_j^n + \tfrac{\nu}{2}\,L_j\,(R_{j+1} - R_j)\,(v_{j+1}^n - v_j^n) \\ & + \tfrac{\nu}{2}\,L_j\,(R_j - R_{j-1})\,(v_j^n - v_{j-1}^n) + \tfrac{\nu}{2}\,L_j\,(R_{j+1} - 2R_j + R_{j-1})\,v_j^n \\ & - \tfrac{\lambda}{2}\,L_j\,(\theta_{j+1}^n + \theta_j^n)\,. \end{aligned} \tag{2.5}$$

Before we derive our energy estimate, we first state the following theorem on compressibility and asymptotic behavior of stationary discrete shocks.

**Theorem 2.1** [7]. *Suppose that $u_-$ and $u_+$ satisfy* (1.3b–c) *and $|u_- u_+| = \varepsilon$ is small. Then there is a stationary discrete shock profile to* (1.5) *which satisfies, for all $j = 0, \pm 1, \dots,$*

$$\lambda_k(\phi_j) > \lambda_k(\phi_{j+1}) , \tag{2.6a}$$

$$c_1|\phi_j - \phi_{j+1}| \le \lambda_k(\phi_j) - \lambda_k(\phi_{j+1}) \le c_2|\phi_j - \phi_{j+1}| , \tag{2.6b}$$

$$|\phi_{j+1} - 2\phi_j + \phi_{j-1}| \le c_3\, \varepsilon\, |\phi_{j+1} - \phi_j| , \tag{2.6c}$$

*where $c_1$, $c_2$, and $c_3$ are positive constants independent of $\varepsilon$ and $j$.*

We choose weights

$$W_j = \operatorname{diag}\{w_{1,j}, w_{2,j}, \dots, w_{m,j}\} , \tag{2.7a}$$

as

$$w_{\mu,j} = \frac{c_{1,\mu}}{|\lambda_{\mu,j}|} \prod_{i=-\infty}^{j-1} \left(1 - c_{2,\mu} \frac{\lambda_{k,i} - \lambda_{k,i+1}}{\lambda_{\mu,i}}\right) , \qquad \mu \ne k \tag{2.7b}$$

and

$$w_{k,j} \equiv 1 , \tag{2.7c}$$

where $c_{1,\mu}$ and $c_{2,\mu}$ are suitable positive constants to be chosen. We denote

$$|v_j^n|_w = \left(v_j^n \cdot W_j v_j^n\right)^{1/2} .$$

The specific choice of weights in (2.7) is made to insure that waves propagating in the transversal directions can be dominated by waves propagating in the principal direction, which is controllable owing to the compressibility of the discrete shock profiles (2.6a). More precisely, we have the following lemma.

**Lemma 2.1.** *Let $W_j$ be the weights defined by* (2.7). *Then we can choose $c_{1,\mu}$ and $c_{2,\mu}$ appropriately so that*

$$\begin{aligned} &\lambda(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j) - \lambda W_j \Lambda_j (L_{j+1} - L_{j-1}) R_j \\ &\quad + \tfrac{\nu}{2}(W_{j+1} - 2W_j + W_{j-1}) \le -\tfrac{\lambda}{2}(\lambda_{k,j} - \lambda_{k,j+1}) W_j , \end{aligned} \tag{2.8}$$

*provided that $\varepsilon$ is suitably small.*

We delay the proof of Lemma 2.1 until the end of this section. Assuming Lemma 2.1, we can estimate the solution to (2.4) as follows. Taking the scalar product of system (2.4) with $2v_j^n W_j$, and using summation by parts, we obtain

$$\begin{aligned} &\sum_j |v_j^{n+1}|_w^2 - \sum_j |v_j^n|_w^2 + \tfrac{\nu}{2} \sum_j (v_{j+1}^n - v_j^n)\cdot(W_j + W_{j+1})(v_{j+1}^n - v_j^n) \\ &\quad = \sum_j v_j^n \cdot (\lambda(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j) + \tfrac{\nu}{2}(W_{j+1} - 2W_j + W_{j-1}) \\ &\qquad\qquad + \lambda W_j \Lambda_j (L_{j+1} - L_{j-1}) R_j) v_j^n \\ &\quad + \sum_j |v_j^{n+1} - v_j^n|_w^2 - \lambda \sum_j v_j^n \cdot (W_j - W_{j+1}) \Lambda_{j+1} (v_{j+1}^n - v_j^n) \\ &\quad + 2 \sum_j v_j^n \cdot W_j e_j^n , \end{aligned} \tag{2.9}$$

where we have used the identities

$$2\sum_j v_j^n\cdot W_j(v_{j+1}^n-2v_j^n+v_j^n)=-\sum_j(v_{j+1}^n-v_j^n)\cdot(W_j+W_{j+1})(v_{j+1}^n-v_j^n)$$
$$+\sum_j v_j^n\cdot(W_{j+1}-2W_j+W_{j-1})v_j^n$$

and

$$|v_j^{n+1}|_w^2-|v_j^n|_w^2=2v_j^n\cdot W_j(v_j^{n+1}-v_j^n)+|v_j^{n+1}-v_j^n|_w^2\,.$$

Taking into account Lemma 2.1, we obtain from (2.9) that

$$\begin{aligned}&\sum_j|v_j^{n+1}|_w^2-\sum_j|v_j^n|_w^2\\&\quad+\tfrac{\lambda}{2}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})\,|v_j^n|_w^2+\tfrac{\nu}{2}\sum_j|v_{j+1}^n-v_j^n|_w^2\\&\le\sum_j|v_j^{n+1}-v_j^n|_w^2-\lambda\sum_j v_j^n\cdot(W_j-W_{j+1})\Lambda_{j+1}(v_{j+1}^n-v_j^n)\\&\quad+2\sum_j v_j^n\cdot W_je_j^n\,.\end{aligned}\tag{2.10}$$

Set

$$M(n_1)=\sup_{n\le n_1}\left(\sum_j|v_j^n|^2\right)^{1/2}\tag{2.11}$$

and assume that $M(n_1)$ is small. Clearly, we have

$$\sup_{n,j}|v_j^n|\le M(n_1)\,.\tag{2.12}$$

It follows from equation (2.4) that

$$\begin{aligned}|v_j^{n+1}-v_j^n|_w\le\tfrac12\big(\lambda|\Lambda|_{L^\infty}+\nu+O(\varepsilon)+M(n_1)\big)\big(|v_{j+1}^n-v_j^n|_w+|v_j^n-v_{j-1}^n|_w\big)\\+O(1)\big|(\lambda_{k,j}-\lambda_{k,j+1})v_j^n\big|_w\,,\end{aligned}$$

where we have used the bound (see (2.3))

$$|\theta_j^n|\le O(1)\big(|v_j^n-v_{j+1}^n|^2+(\lambda_{k,j}-\lambda_{k,j+1})^2|v_j^n|^2\big)\,.\tag{2.13}$$

Consequently, we have

$$\begin{aligned}\sum_j|v_j^{n+1}-v_j^n|_w^2\le\tfrac{\lambda}{8}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})\,|v_j^n|_w^2\\+\big((\lambda|\Lambda|_{L^\infty}+\nu)^2+O(\varepsilon)+O(1)M(n_1)\big)\\\times\sum_j|v_{j+1}^n-v_j^n|_w^2\,,\end{aligned}\tag{2.14}$$

where we have used Theorem 2.1.

Next, using (2.5), (2.13), and Theorem 2.1, one can get after some careful manipulations that

$$\begin{aligned}\sum_j|v_j^n\cdot W_je_j^n|\le\tfrac{\lambda}{8}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})\,|v_j^n|_w^2\\+\big(O(\varepsilon)+O(1)\,M(n_1)\big)\sum_j|v_{j+1}^n-v_j^n|_w^2\,.\end{aligned}\tag{2.15}$$

In fact, two typical terms involved in establishing (2.15) can be estimated as follows:

$$\begin{aligned}
&\left|v_j^n \cdot W_j L_j (R_{j+1} - R_j) \Lambda_{j+1} (v_{j+1}^n - v_j^n)\right| \\
&\quad \le O(1)(\lambda_{k,j} - \lambda_{k,j+1})\left|v_j^n \cdot W_j (v_{j+1}^n - v_j^n)\right| \\
&\quad \le \tfrac{\lambda}{16}(\lambda_{k,j} - \lambda_{k,j+1})|v_j^n|_w^2 + O(\varepsilon)|v_{j+1}^n - v_j^n|_w^2 ,
\end{aligned}$$

and

$$\begin{aligned}
\left|v_j^n \cdot W_j L_j \theta_j^n\right| &\le O(1)|W_j v_j^n|\,|v_j^n - v_{j+1}^n|^2 + O(\varepsilon)(\lambda_{k,j} - \lambda_{k,j+1})|v_j^n|_w^2 \\
&\le O(1)\, M(n_1)|v_j^n - v_{j+1}^n|_w^2 + O(\varepsilon)(\lambda_{k,j} - \lambda_{k,j+1})|v_j^n|_w^2 .
\end{aligned}$$

Finally, we collect the estimates (2.10), (2.14), and (2.15) to obtain

$$\begin{aligned}
&\sum_j |v_j^{n+1}|_w^2 - \sum_j |v_j^n|_w^2 + \tfrac{\lambda}{4}\sum_j (\lambda_{k,j} - \lambda_{k,j+1})\,|v_j^n|_w^2 \\
&\qquad + \left(\nu - (\lambda|\Lambda|_{L^\infty} + \nu)^2 - O(\varepsilon) - O(1)\, M(n)\right)\sum_j |v_{j+1}^n - v_j^n|_w^2 \\
&\qquad \le O(1)\, M(n_1)\sum_j (\lambda_{k,j} - \lambda_{k,j+1})\,|v_j^n|_w^2 .
\end{aligned} \tag{2.16}$$

Since $\nu < 1$ and our weights are bounded both above and below by some positive constants, and by taking $\varepsilon$ and $\lambda$ suitably small, we have proved the following basic a priori estimate.

**Proposition 2.1** (A priori estimate). *Let $v_j^n$ be a solution of* (2.4) *for $n \le n_1$. Then there exists a positive constant $C$ independent of $n_1$ and $\varepsilon$ such that for all $n \le n_1$*

$$\begin{aligned}
&\sum_j |v_j^n|^2 + \sum_{n_2 \le n}\sum_j |v_j^{n_2} - v_{j+1}^{n_2}|^2 \\
&\qquad + \sum_{n_2 \le n}\sum_j |\lambda_{k,j}^{n_2} - \lambda_{k,j+1}^{n_2}|\,|v_j^{n_2}|^2 \le C\sum_j |v_j^0|^2 ,
\end{aligned} \tag{2.17}$$

*provided that $\varepsilon$, $\lambda$, and $M(n_1)$ are suitably small.*

Since (2.4) is a uniform discrete parabolic system, it follows from Proposition 2.1 and a standard continuity argument that the following proposition holds.

**Proposition 2.2.** *Assume that $\varepsilon$ and $M(0)$ are suitably small. Then the problem* (2.4) *has a unique global solution $v_j^n$ satisfying, for any $n \ge 0$,*

$$\sup_n \sum_j |v_j^n|^2 + \sum_{j,n} |v_j^n - v_{j+1}^n|^2 + \sum_{j,n} |\lambda_{p,j} - \lambda_{k,j+1}|\,|v_j^n|^2 \le CM^2(0) , \tag{2.18}$$

*where $C$ is a positive constant independent of $n$ and $j$.*

We now turn our attention to the $L^1$-stability analysis. We first rewrite (2.4) as

$$\begin{aligned}
&v_j^{n+1} - v_j^n + \tfrac{\lambda}{2}\Lambda_{j+1}(v_{j+1}^n - v_j^n) + \tfrac{\lambda}{2}\Lambda_j(v_j^n - v_{j-1}^n) - \tfrac{\nu}{2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) \\
&\qquad = \tfrac{\lambda}{2}\Lambda_j(L_{j+1} - L_{j-1})R_j v_j^n + B_j^n(v_{j+1}^n - v_j^n) + C_j^n(v_j^n - v_{j-1}^n) + \tilde{e}_j^n ,
\end{aligned} \tag{2.19}$$

where $B_j^n$ and $C_j^n$ are matrices given by

$$B_j = \tfrac{1}{2}L_j(R_{j+1}-R_j)(\nu-\lambda\Lambda_{j+1}) , \qquad C_j = \tfrac{1}{2}(\nu+\lambda\Lambda_j)L_j(R_j-R_{j-1}) , \tag{2.20a}$$

and $\tilde{e}_j^n$ is a vector given by

$$\begin{aligned} \tilde{e}_j^n = {} & \tfrac{\lambda}{2}(\Lambda_{j+1}-\Lambda_j)(L_{j+1}-L_j)R_jv_j^n \\ & + \tfrac{\lambda}{2}L_j(R_{j+1}-R_j)\Lambda_{j+1}(L_{j+1}-L_j)R_jv_j^n \\ & - \tfrac{\lambda}{2}\Lambda_j(L_j-L_{j-1})(R_j-R_{j-1})v_j^n \\ & + \tfrac{\nu}{2}L_j(R_{j+1}-2R_j+R_{j-1})v_j^n - \tfrac{\lambda}{2}L_j(\theta_{j+1}^n+\theta_j^n) . \end{aligned} \tag{2.20b}$$

In the rest of this section, abusing notations a little bit, we will denote by $|A|$ the matrix (vector) whose components are the absolute values of the corresponding components of a given matrix (vector) $A$, and by $\operatorname{diag}(A)$ the diagonal matrix consisting of the diagonal elements of a given matrix $A$, i.e.,

$$|A| = \left(|a_{i,j}|\right) , \quad \text{and} \quad \operatorname{diag}(A) = \operatorname{diag}(a_{11}, \dots, a_{mm}) \quad \text{for } A = (a_{i,j}) .$$

We now rewrite (2.19) as

$$\begin{aligned} & v_j^{n+1} - \tfrac{1}{2}\left(\nu+\lambda\Lambda_j - 2\operatorname{diag}C_j\right)v_{j-1}^n - \tfrac{1}{2}\left(\nu-\lambda\Lambda_{j+1}+2\operatorname{diag}B_j\right)v_{j+1}^n \\ & \quad - \left(1-\nu+\tfrac{\lambda}{2}(\Lambda_{j+1}-\Lambda_j) - \operatorname{diag}(B_j-C_j)\right)v_j^n \\ & = \tfrac{\lambda}{2}\Lambda_j(L_{j+1}-L_{j-1})R_jv_j^n + (B_j-\operatorname{diag}B_j)\,(v_{j+1}^n-v_j^n) \\ & \quad + (C_j-\operatorname{diag}C_j)\,(v_j^n-v_{j-1}^n) + \tilde{e}_j^n . \end{aligned} \tag{2.21}$$

By the definition of the matrices $B_j$ and $C_j$, each component in these matrices has a bound of order $O(\varepsilon)$, by virtue of Theorem 2.1. Consequently, the matrices on the left-hand side of (2.21) are all diagonal and positive for small $\varepsilon$ and $\lambda$. This implies immediately that

$$\begin{aligned} & |v_j^{n+1}| - \tfrac{1}{2}\left(\nu+\lambda\Lambda_j - 2\operatorname{diag}C_j\right)|v_{j-1}^n| - \tfrac{1}{2}\left(\nu-\lambda\Lambda_{j+1}+2\operatorname{diag}B_j\right)|v_{j+1}^n| \\ & \quad - \left(1-\nu+\tfrac{\lambda}{2}(\Lambda_{j+1}-\Lambda_j) - \operatorname{diag}(B_j-C_j)\right)|v_j^n| \\ & \le \tfrac{\lambda}{2}\left|\Lambda_j(L_{j+1}-L_{j-1})R_j\right|\,|v_j^n| + \left|B_j-\operatorname{diag}B_j\right|\,|v_{j+1}^n-v_j^n| \\ & \quad + \left|C_j-\operatorname{diag}C_j\right|\,|v_j^n-v_{j-1}^n| + |\tilde{e}_j^n| , \end{aligned} \tag{2.22}$$

which can be rewritten as

$$\begin{aligned} & |v_j^{n+1}| - |v_j^n| + \tfrac{\lambda}{2}\Lambda_{j+1}\left(|v_{j+1}^n|-|v_j^n|\right) + \tfrac{\lambda}{2}\Lambda_j\left(|v_j^n|-|v_{j-1}^n|\right) \\ & \quad - \tfrac{\nu}{2}\left(|v_{j+1}^n| - 2|v_j^n| + |v_{j-1}^n|\right) \\ & \quad - \operatorname{diag}B_j\left(|v_{j+1}^n|-|v_j^n|\right) + \operatorname{diag}C_j\left(|v_j^n|-|v_{j-1}^n|\right) \\ & \le \tfrac{\lambda}{2}\left|\Lambda_j(L_{j+1}-L_{j-1})R_j\right|\,|v_j^n| + \left|B_j-\operatorname{diag}B_j\right|\,\left(|v_{j+1}^n|+|v_j^n|\right) \\ & \quad + \left|C_j-\operatorname{diag}C_j\right|\,\left(|v_j^n|+|v_{j-1}^n|\right) + |\tilde{e}_j^n| , \end{aligned} \tag{2.23}$$

where the vector inequality is understood componentwise. Multiplying both sides of (2.23) by $W_j$ defined in (2.7), summing up with respect to $j$, and

using summation by parts, we obtain

$$
\begin{aligned}
&\sum_j \sum_\mu (w_{\mu,j}|v_{\mu,j}^{n+1}| - w_{\mu,j}|v_{\mu,j}^n|) \\
&\quad \le \sum_j \sum_\mu (\lambda(w_{\mu,j+1}\lambda_{\mu,j+1} - w_{\mu,j}\lambda_{\mu,j}) \\
&\qquad + \tfrac{1}{2}(\nu - \lambda\lambda_{\mu,j})(w_{\mu,j+1} - 2w_{\mu,j} + w_{\mu,j-1}) \\
&\qquad\qquad - \tfrac{\lambda}{2}(w_{\mu,j+1} - w_{\mu,j})(\lambda_{\mu,j+1} - \lambda_{\mu,j}))|v_{\mu,j}^n| \\
&\quad + \big||D_j|\,|v_j^n|\big| + \big|W_j|\tilde{e}_j^n|\big|,
\end{aligned}
\tag{2.24}
$$

where

$$
\begin{aligned}
D_j = {} & \tfrac{\lambda}{2} W_j \big|\Lambda_j(L_{j+1} - L_{j-1})R_j\big| + W_{j-1}\operatorname{diag}(B_{j-1} - B_j) \\
& + W_{j+1}\operatorname{diag}(C_{j+1} - C_j) + W_{j-1}\big|B_{j-1} - \operatorname{diag} B_{j-1}\big| \\
& + W_j\big|B_j - \operatorname{diag} B_j\big| + W_j\big|C_j - \operatorname{diag} C_j\big| + W_{j+1}\big|C_{j+1} - \operatorname{diag} C_{j+1}\big|.
\end{aligned}
\tag{2.25}
$$

Now our main task is to bound the terms on the right-hand side of (2.24). This is achieved by choosing appropriate weights $W_j$; more precisely, we have the following lemma.

**Lemma 2.2.** *Let $W_j$ be the weights defined by* (2.7). *Then we can choose $c_{1,\mu}$ and $c_{2,\mu}$ appropriately so that*

$$
\begin{aligned}
&\sum_\mu (\lambda(w_{\mu,j+1}\lambda_{\mu,j+1} - w_{\mu,j}\lambda_{\mu,j}) \\
&\qquad + \tfrac{1}{2}(\nu - \lambda\lambda_{\mu,j})(w_{\mu,j+1} - 2w_{\mu,j} + w_{\mu,j-1}) \\
&\qquad\qquad - \tfrac{\lambda}{2}(w_{\mu,j+1} - w_{\mu,j})(\lambda_{\mu,j+1} - \lambda_{\mu,j}))|v_{\mu,j}^n| + \big||D_j|\,|v_j^n|\big| \\
&\quad \le -\tfrac{\lambda}{2}(\lambda_{k,j} - \lambda_{k,j+1}) \sum_\mu w_{\mu,j}\,|v_{\mu,j}^n|,
\end{aligned}
\tag{2.26}
$$

*provided that $\varepsilon$ is suitably small.*

Assuming the lemma for a moment, we have from (2.24) and (2.26) that

$$
\begin{aligned}
&\sum_j \sum_\mu (w_{\mu,j}\,|v_{\mu,j}^{n+1}| - w_{\mu,j}\,|v_{\mu,j}^n|) \\
&\quad + \tfrac{\lambda}{2} \sum_j (\lambda_{k,j} - \lambda_{k,j+1}) \sum_\mu w_{\mu,j}\,|v_{\mu,j}^n| \le O(1) \sum_j |\tilde{e}_j^n|.
\end{aligned}
\tag{2.27}
$$

Direct computation using Theorem 2.1 shows that

$$
|\tilde{e}_j^n| \le O(\varepsilon)\,(\lambda_{k,j} - \lambda_{k,j+1})\,|v_j^n| + O(1)\,|v_j^n - v_{j+1}^n|^2,
$$

which, together with (2.27), implies that

$$
\begin{aligned}
&\sup_n \sum_j \sum_\mu w_{\mu,j}|v_{\mu,j}^n| + \sum_{j,n} (\lambda_{k,j} - \lambda_{k,j+1}) \sum_\mu w_{\mu,j}|v_{\mu,j}^n| \\
&\quad \le \sum_j |v_j^0| + O(1) \sum_{j,n} |v_j^n - v_{j+1}^n|^2.
\end{aligned}
\tag{2.28}
$$

But Proposition 2.2 shows that the last term on the right-hand side of (2.28) is bounded above by $O(1)\,M(0)^2$. Thus, we have shown

**Proposition 2.3.** *Assume that $\varepsilon$ and $M(0)$ are suitably small. Then the problem* (2.4) *has a unique global solution $v_j^n$ satisfying, for any $n \geq 0$,*

$$\sup_n \sum_j |v_j^n| + \sum_{j,n} |\lambda_{p,j} - \lambda_{k,j+1}|\,|v_j^n| \leq O(1) \sum_j \left(|v_j^0| + |v_j^0|^2\right). \tag{2.29}$$

With Propositions 2.2 and 2.3 at hand, we can obtain the nonlinear stability result quite easily.

*Proof of Theorem* 1.1. First, it is not difficult to verify that condition (1.6) implies $M(0)$ being small. Thus, the hypotheses in Propositions 2.2 and 2.3 are fulfilled under the conditions (1.6). It follows from Proposition 2.2 that there exists a unique global solution, $u_j^n$, to the Lax-Friedrichs scheme (1.1), owing to the relation

$$u_j^n = \phi_j + \bar{v}_j^n - \bar{v}_{j-1}^n,$$

which follows from (2.1), and

$$\sum_{n=1}^{\infty} \left( \sum_j |\bar{v}_j^n - \bar{v}_{j+1}^n|^2 \right) < +\infty,$$

which implies

$$\lim_{n\to\infty} \sum_j |\bar{v}_j^n - \bar{v}_{j+1}^n|^2 = 0.$$

Using (2.1) again, we have

$$\lim_{n\to\infty} \sum_j |u_j^n - \phi_j|^2 = \lim_{n\to\infty} \sum_j |\bar{v}_j^n - \bar{v}_{j+1}^n|^2 = 0. \tag{2.30}$$

From Proposition 2.3 and (2.1), we have

$$\sum_j |u_j^n - \phi_j| \leq 2 \sum_j |\bar{v}_j^n| < \infty,$$

which, together with (2.30), yields the desired estimates (1.7) and (1.8). This completes the proof of Theorem 1.1. □

Finally, we turn to the proofs of Lemma 2.1 and Lemma 2.2.

*Proof of Lemma* 2.1. It can be easily verified, by using (2.7), that

$$\begin{aligned} W_{j+1} - W_j &= O(1)\,(\lambda_{k,j} - \lambda_{k,j+1}) W_j, \\ W_{j+1} - 2W_j + W_{j-1} &= O(\varepsilon)\,(\lambda_{k,j} - \lambda_{k,j+1}) W_j. \end{aligned} \tag{2.31}$$

Let $c_{2,k} = 1$ and $C_2$ denote the diagonal matrix

$$C_2 = \operatorname{diag}\{c_{2,1}, c_{2,2}, \cdots, c_{2,m}\}.$$

Then our weights in (2.7) are a solution to the following difference equation:

$$W_j(\Lambda_j - s) - W_{j+1}(\Lambda_{j+1} - s) = (\lambda_{k,j} - \lambda_{k,j+1})\, C_2 W_j. \tag{2.32}$$

As a consequence of (2.31) and (2.32), the left-hand side of (2.8) becomes

$$-\lambda(\lambda_{k,j} - \lambda_{k,j+1})\, C_2 W_j - \tfrac{\lambda}{2} W_j \Lambda_j (L_{j+1} - L_{j-1}) R_j + O(\varepsilon)\,(\lambda_{k,j} - \lambda_{k,j+1}) W_j. \tag{2.33}$$

We now choose $c_{2,\mu}$ suitably large so that the matrix inequality

$$\begin{aligned}(2.34)\qquad &-\lambda(\lambda_{k,j}-\lambda_{k,j+1})\,C_2\,W_j-\tfrac{\lambda}{2}W_j\Lambda_j(L_{j+1}-L_{j-1})R_j\\ &\quad+O(\varepsilon)\,(\lambda_{k,j}-\lambda_{k,j+1})W_j\le-\tfrac{3\lambda}{4}(\lambda_{k,j}-\lambda_{k,j+1})W_j\end{aligned}$$

holds for suitably small $\varepsilon$, where we have used Theorem 2.1 and the fact that $\lambda_{k,j}=O(1)\,\varepsilon$. Combining (2.33) and (2.34) proves Lemma 2.1. □

*Proof of Lemma* 2.2. The definition of our weight, (2.7), implies that

$$\begin{aligned}&\sum_\mu(w_{\mu,j}\lambda_{\mu,j}-w_{\mu,j+1}\lambda_{\mu,j+1})|v^n_{\mu,j}|\\ &\qquad=(\lambda_{k,j}-\lambda_{k,j+1})\Big(|v^n_{k,j}|+\sum_{\mu\neq k}c_{2,\mu}w_{\mu,j}|v^n_{\mu,j}|\Big).\end{aligned}$$

Thus, the right-hand side of (2.26) becomes

$$\begin{aligned}(2.35)\qquad &-\lambda(\lambda_{k,j}-\lambda_{k,j+1})\Big(|v^n_{k,j}|+\sum_{\mu\neq k}c_{2,\mu}w_{\mu,j}|v^n_{\mu,j}|\Big)\\ &\quad+O(\varepsilon)(\lambda_{k,j}-\lambda_{k,j+1})\sum_\mu|v^n_{\mu,j}|+\big||D_j||v^n_j|\big|.\end{aligned}$$

Now, we estimate $D_j$ of (2.25). Since the $(k,k)$-element of the matrix

$$\Lambda_j(L_{j+1}-L_{j-1})R_j$$

is of order $O(\varepsilon)(\lambda_{k,j}-\lambda_{k,j+1})$ and the remaining elements are of order $O(1)(\lambda_{k,j}-\lambda_{k,j+1})$, we have

$$\begin{aligned}(2.36)\qquad &W_j\big|\Lambda_j(L_{j+1}-L_{j-1})R_j\big|\,|v^n_j|\\ &\qquad\le O(1)\Big(\varepsilon+\sum_{\mu\neq k}c_{1,\mu,j}\Big)(\lambda_{k,j}-\lambda_{k,j+1})\sum_\mu|v^n_{\mu,j}|,\end{aligned}$$

where we have used

$$\sup_{j,n}w_{\mu,j}=O(1)\,c_{1,\mu}.$$

In view of (2.20a), we obtain

$$(2.37)\qquad \operatorname{diag}(B_{j-1}-B_j)+\operatorname{diag}(C_{j+1}-C_j)=O(\varepsilon)\,(\lambda_{k,j}-\lambda_{k,j+1}).$$

As a consequence of (2.20a) and the fact that the diagonal elements of the matrices

$$W_j\big|B_j-\operatorname{diag}B_j\big|\quad\text{and}\quad W_j\big|C_j-\operatorname{diag}C_j\big|$$

are zero and the remaining elements of the matrices are of order $O(1)(\lambda_{k,j}-\lambda_{k,j+1})$, we have

$$(2.38)\qquad \big||D_j|\,|v^n_j|\big|\le O(1)(\lambda_{k,j}-\lambda_{k,j+1})\Big(\Big(\varepsilon+\sum_{\mu\neq k}c_{1,\mu}\Big)|v^n_{k,j}|+\sum_{\mu\neq k}|v^n_{\mu,j}|\Big).$$

Collecting all the estimates (2.35) and (2.38), we may conclude that the left-hand side of (2.26) is bounded by

$$\begin{aligned}
&-\lambda(\lambda_{k,j}-\lambda_{k,j+1})\Big(|v^n_{k,j}|+\sum_{\mu\neq k} c_{2,\mu} w_{\mu,j}|v^n_{\mu,j}|\Big)\\
&\qquad + O(1)(\lambda_{k,j}-\lambda_{k,j+1})\Big(\Big(\varepsilon+\sum_{\mu\neq k} c_{1,\mu}\Big)|v^n_{k,j}|+\sum_{\mu\neq k}|v^n_{\mu,j}|\Big)\\
&\qquad \le -\tfrac{\lambda}{2}(\lambda_{k,j}-\lambda_{k,j+1})\sum_{\mu} w_{\mu,j}|v^n_{\mu,j}|
\end{aligned}$$

for suitably large $c_{2,\mu}$, $\mu \neq k$, and suitably small $c_{1,\mu}$, $\mu \neq k$. The proof of Lemma 2.2 is complete. □

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