THE MINIMUM DISCRIMINANT
OF TOTALLY REAL ALGEBRAIC NUMBER FIELDS
OF DEGREE 9 WITH CUBIC SUBFIELDS

HIROYUKI FUJITA

Abstract. With the help of the computer language UBASIC86, the minimum discriminant \( d(K) \) of totally real algebraic number fields \( K \) of degree 9 with cubic subfields \( F \) is determined. It is given by \( d(K) = 16240385609 \). The defining equation for \( K \) is given by \( f(x) = x^9 - x^8 - 9x^7 + 4x^6 + 26x^5 - 2x^4 - 25x^3 - x^2 + 7x + 1 \), and \( K \) is uniquely determined by \( d(K) \) up to \( \mathbb{Q} \)-isomorphism. The field \( K \) has the cubic subfield \( F \) with \( d(F) = 49 \) defined by the polynomial \( f(x) = x^3 + x^2 - 2x - 1 \).

1. Introduction

Let \( K \) be a totally real algebraic number field of degree \( n \) with discriminant \( d(K) \), and \( r_1 \) be the number of real conjugate fields and \( 2r_2 \) the number of complex conjugate fields, so that we have \( n = r_1 + 2r_2 \).

It is an important problem to determine the minimum discriminant \( d(K) \) and the corresponding field \( K \) for each pair \( (r_1, r_2) \).

When \( n < 8 \), for all signatures \( (r_1, r_2) \) the field \( K \) with minimum discriminant \( d(K) \) is known. In the case \( n = 8 \), only the totally real case (i.e., \( r_2 = 0 \)) and the totally complex case (i.e., \( r_2 = 4 \)) have been determined (Pohst [1, 2], Pohst, Martinet, and Diaz y Diaz [4], Diaz y Diaz [11]).

In this paper we shall treat totally real fields of degree 9 (i.e., \( n = 9 \), \( r_2 = 0 \)). There are two cases. The first is when \( K \) has a cubic subfield, and the other is when \( K \) does not have any such subfields. Since it seems that the latter case is harder than the former, in this paper we shall deal only with the former case. We prove the following main theorem.

Theorem 1. Let \( K \) be a totally real algebraic number field of degree 9 with a cubic subfield \( F \) such that the discriminant \( d(K) \) satisfies

\[ d(K) \leq 16983563041. \]

Let \( f(x) = x^9 + a_1x^8 + a_2x^7 + a_3x^6 + a_4x^5 + a_5x^4 + a_6x^3 + a_7x^2 + a_8x + a_9 \) be a defining equation for the field \( K \). Then the complete list of such fields \( K \) is given as follows:

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Moreover, for each discriminant \( d(K) \) listed above, the field \( K \) is uniquely determined up to \( \mathbb{Q} \)-isomorphism.

In order to prove Theorem 1, we do the following:

1. First we determine an upper bound \( d_{\text{max}} \) and a lower bound \( d_{\text{min}} \) for the discriminant \( d(K) \).

2. Let \( F \) be a cubic subfield of \( K \). It is known that

\[
d(K) = d(F)^3 N(D(K/F)),
\]

where \( N(\cdot) \) is the norm of \( F/\mathbb{Q} \), and \( D(K/F) \) is the relative discriminant of \( K/F \). So we have

\[
d(F) \leq (d_{\text{max}})^{1/3}.
\]

For each field \( F \), we determine all fields \( K \) with \( d(K) \leq d_{\text{max}} \) such that \( K \) contains \( F \) as a subfield.

We must use a computer in order to calculate the discriminants, determinants, coefficients of polynomials and the other data. Since these values can be greater than \( 10^{14} \), we cannot calculate them with commonly used computer languages (for example FORTRAN, PASCAL, C). So we use the computer language UBASIC86 running on the personal computer NEC PC-9801 series. UBASIC86 is a high-precision BASIC, which is an excellent public-domain software written by Y. Kida [9]. We use the version 8.12 (October 1990). UBASIC86 can calculate with up to 2600 digits for integers and real numbers, and up to 2600 digits total for the real and imaginary parts of complex numbers. Since it does the job faster than the familiar languages, UBASIC86 is best for our purpose.

### 2. Cubic subfields

Let \( K \) be a totally real algebraic number field of degree 9 with discriminant \( d(K) \). We determine an upper bound and a lower bound for the discriminant \( d(K) \). A. Odlyzko [8] supplied a table of triples \( (A, B, E) \) and gave the following lower bounds for \( d(K) \):

\[
A r_1 B^{2r_2} e^{-E} < d(K),
\]

where \( e \) is Euler's constant, the base of the natural logarithms. One of them, which we will use, is given as follows:

\[
A = 29.534, \quad B = 14.616, \quad E = 8.2267.
\]

In our case \( (r_1 = 9, \ r_2 = 0) \) it follows by (3) and (4) that

\[
4392565450.66469 < d(K).
\]

So we take the lower bound \( d_{\text{min}} = 4392565451 \).

Let \( \mathbb{Q}(\zeta_m) \) be a cyclotomic field with \( m \)th root of unity \( \zeta_m \) and \( \mathbb{Q}(\zeta_m)^+ \) be the maximal real subfield. In the case \( m = 19 \), we have \( [\mathbb{Q}(\zeta_{19})^+ : \mathbb{Q}] = 9 \) and

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The minimum discriminant of algebraic number fields

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$d(Q(\zeta_{19})^+ ) = 19^8 = 16983563041$. So we have the upper bound

\[ d_{\text{max}} = 19^8 = 16983563041. \]

It is sufficient to determine all fields \( K \) such that

\[ (6) \quad d_{\text{min}} = 4392565451 \leq d(K) \leq d_{\text{max}} = 16983563041. \]

By (2), we need all cubic real fields \( F \) with the discriminants

\[ d(F) \leq [d_{\text{max}}^{1/3}] = 2570. \]

Let \( f(x) = x^3 + b_1 x^2 + b_2 x + b_3 \) be a defining equation for the field \( F \). Then the complete list of such fields \( F \) is given as in Table 1. The integral basis is given by \( 1, \alpha, (\alpha + \alpha^2)/2 \) in the cases \( d(F) = 961, 1304, 1772, 1849, 2089 \), and by \( 1, \alpha, (1 + \alpha^2)/2 \) in the case \( d(F) = 2292 \), and by \( 1, \alpha, \alpha^2 \) in the remaining cases.

3. Cubic extensions of cubic subfields

In this section we determine all totally real algebraic number fields \( K \) of degree 9 satisfying (6) which are cubic extensions of the cubic fields \( F \) given in Table 1.
Let $K$ be an algebraic number field and $F$ be a subfield of $K$. Let $\rho$ be a $\mathbb{Q}$-isomorphism of $K$, $\sigma$ be a $\mathbb{Q}$-isomorphism of $F$ and $\rho|_F$ be the restriction of $\rho$ to $F$. Then the $\sigma$-trace is defined by

$$\text{Tr}_{\sigma,K/F}(\theta) = \sum_{\rho \mid F} \rho \theta \quad (\theta \in K).$$

The following theorem is useful in finding a field $K$ which is an extension of its subfield $F$.

**Theorem 2** (Martinet [5]). Let $K$ be an algebraic number field of degree $n$, $F$ be an algebraic number field of degree $n'$ and $K$ be an extension of $F$ with degree $m$. Then there exists $\theta \in \mathcal{O}_K$, $\theta \not\in F$, such that

$$\sum_{i=1}^n |\theta(i)|^2 \leq \frac{1}{m} \sum_{\sigma} |\text{Tr}_{\sigma,K/F}(\theta)|^2 + \gamma_{n-n'} \left| \frac{d(K)}{mn'd(F)} \right|^{1/(n-n')} ,$$

where $\theta(i)$ are conjugates of $\theta$. Further, if $\theta$ satisfies this inequality, then $\theta + \delta$, for any $\delta \in F$, satisfies it also.

Let $F$ be a cubic field and $K$ a cubic extension of $F$. Let $\{\sigma_1 = 1, \sigma_2, \sigma_3\}$ be a $\mathbb{Q}$-isomorphism of $F$, and $\{1, \delta^{(1)}, e^{(1)}\}$ be an integral basis of $F$, and $\{\delta^{(j)}, e^{(j)}\} \ (j = 2, 3)$ be the conjugates. Then, using Theorem 2, we have the following inequality for some $\theta \in \mathcal{O}_K$, $\theta \not\in F$:

$$(7) \quad \sum_{i=1}^9 |\theta(i)|^2 \leq \frac{1}{3} \sum_{j=1}^3 (\text{Tr}_{\sigma_j,K/F}(\theta))^2 + \left( \frac{19^8}{3^3 d(F)} \right)^{1/6} \times \left( \frac{64}{3} \right)^{1/6} .$$

We put

$$T_2 = \left( \frac{19^8}{3^3 d(F)} \right)^{1/6} \times \left( \frac{64}{3} \right)^{1/6} .$$

Hence, $\theta$ is a root of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ which decomposes in $\mathcal{O}_F[x]$ into a product of three conjugate irreducible polynomials, say, $f^{(1)}(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$. We use the following notation:

$$f(x) = x^9 + a_1 x^8 + a_2 x^7 + a_3 x^6 + a_4 x^5 + a_5 x^4 + a_6 x^3 + a_7 x^2 + a_8 x + a_9 \quad (a_i \in \mathbb{Z}, \ 1 \leq i \leq 9),$$

$$f^{(1)}(x) = x^3 + a_1^{(1)} x^2 + a_2^{(1)} x + a_3^{(1)} \quad (a_i^{(1)} \in \mathcal{O}_F, \ 1 \leq i \leq 3),$$

$$f^{(2)}(x) = x^3 + a_1^{(2)} x^2 + a_2^{(2)} x + a_3^{(2)} \quad (a_i^{(2)} = \sigma_2(a_i^{(1)}), \ 1 \leq i \leq 3),$$

$$f^{(3)}(x) = x^3 + a_1^{(3)} x^2 + a_2^{(3)} x + a_3^{(3)} \quad (a_i^{(3)} = \sigma_3(a_i^{(1)}), \ 1 \leq i \leq 3).$$

Clearly, we have $\text{Tr}_{\sigma_j,K/F}(\theta) = -a_1^{(j)} \ (j = 1, 2, 3)$.

We order the roots $\theta^{(i)} \ (1 \leq i \leq 9)$ of $f(x)$ such that $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ are the roots of $f^{(1)}(x)$, $\theta^{(4)}$, $\theta^{(5)}$, $\theta^{(6)}$ are the roots of $f^{(2)}(x)$ and $\theta^{(7)}$, $\theta^{(8)}$, $\theta^{(9)}$ are the roots of $f^{(3)}(x)$. For each natural number $j$ we consider the power
sums

\[ S_j = \sum_{i=1}^{9} \theta^{(i)j}, \]
\[ s_j^{(1)} = \theta^{(1)j} + \theta^{(2)j} + \theta^{(3)j}, \]
\[ s_j^{(2)} = \theta^{(4)j} + \theta^{(5)j} + \theta^{(6)j}, \]
\[ s_j^{(3)} = \theta^{(7)j} + \theta^{(8)j} + \theta^{(9)j}. \]

Clearly, \( s_j^{(1)}, s_j^{(2)}, \) and \( s_j^{(3)} \) are conjugates in \( F \) and satisfy

\[ S_j = s_j^{(1)} + s_j^{(2)} + s_j^{(3)}, \quad |S_j| \leq |s_j^{(1)}| + |s_j^{(2)}| + |s_j^{(3)}| \leq \sum_{i=1}^{9} |\theta^{(i)j}|. \]

We determine the set of all coefficients \( (a_1^{(i)}, a_2^{(i)}, a_3^{(i)}) \) of \( f^{(i)}(x) \) \( (i = 1, 2, 3) \).

We put \( a_j^{(i)} = a_{j0} + a_{j1} \alpha_i + a_{j2} \beta_i \) \( (a_{j0}, a_{j1}, a_{j2} \in \mathbb{Z}, i, j = 1, 2, 3) \). Since \( \text{Tr}_{\sigma_i K/F}(\beta) = 3\beta \), \( \beta \in F \), and changing \( \theta \) to \(-\theta \) or \( \theta + \beta \), \( \beta \in F \), we may assume without loss of generality that

\[ a_{10}, a_{11}, a_{12} = 0 \text{ or } 1 \text{ or } 2. \]

By (7),

\[ S_2 = \frac{1}{3} \sum_{i=1}^{3} (\text{Tr}_{\sigma_i K/F}(\theta))^2 + T_2 = T_3. \]

By Siegel [6], we have \( 14 \leq S_2 \). Since \( S_2 = s_2^{(1)} + s_2^{(2)} + s_2^{(3)} \), we have

\[ 0 \leq s_2^{(1)} + s_2^{(2)} + s_2^{(3)} \leq T_3. \]

So we have

\[ -\frac{1}{2}(T_3 - a_1^{(i)2}) \leq a_2^{(i)} \leq -\frac{1}{2}(-a_1^{(i)2}) \quad (i = 1, 2, 3). \]

By the inequality of the geometric and arithmetic means, we have

\[ |a_3^{(i)}| \leq \left( \frac{a_1^{(i)2} - 2a_2^{(i)}}{3} \right)^{3/2} \quad (i = 1, 2, 3). \]

We put

\[ y_2i = -\frac{1}{2}(T_3 - a_1^{(i)2}), \quad x_2i = \frac{a_1^{(i)2}}{2}, \quad x_3i = \left( \frac{a_1^{(i)2} - 2a_2^{(i)}}{3} \right)^{3/2} \quad (i = 1, 2, 3). \]

We first treat the case that we can take \( \{1, \alpha, \alpha^2\} \) as \( \mathbb{Z} \)-basis of \( \mathcal{O}_F \), i.e.,

\[ \delta^{(i)} = \alpha^{(i)}, \quad e^{(i)} = \alpha^{(i)2} \quad (i = 1, 2, 3, \alpha^{(1)} < \alpha^{(2)} < \alpha^{(3)}). \]

From the inequalities (8)–(10), we have the following inequalities for the integers \( a_{10}, a_{11}, a_{12} \):

\[ 0 \leq a_{10}, a_{11}, a_{12} \leq 2. \]
For \( a_{20}, a_{21}, a_{22} \), we have the following inequalities:

\[
\frac{1}{\alpha(2) - \alpha(3)} \left( \frac{y_{21} - x_{22}}{\alpha(1) - \alpha(2)} - \frac{x_{21} - y_{23}}{\alpha(1) - \alpha(3)} \right) \leq a_{22} \leq \frac{1}{\alpha(2) - \alpha(3)} \left( \frac{x_{21} - y_{22}}{\alpha(1) - \alpha(2)} - \frac{y_{21} - x_{23}}{\alpha(1) - \alpha(3)} \right),
\]

(12)

\[
\begin{align*}
\frac{x_{21} - y_{22}}{\alpha(1) - \alpha(3)} - a_{22}(\alpha(1) + \alpha(2)) \\
\leq a_{21} \leq \frac{y_{21} - x_{22}}{\alpha(1) - \alpha(2)} - a_{22}(\alpha(1) + \alpha(2)), \\
\frac{x_{21} - y_{23}}{\alpha(1) - \alpha(3)} - a_{22}(\alpha(1) + \alpha(3)) \\
\leq a_{21} \leq \frac{y_{21} - x_{23}}{\alpha(1) - \alpha(3)} - a_{22}(\alpha(1) + \alpha(3)),
\end{align*}
\]

(13)

\[
\begin{align*}
y_{21} - a_{21} \alpha(1) - a_{22} \alpha(1)^2 & \leq a_{20} \leq x_{21} - a_{21} \alpha(1) - a_{22} \alpha(1)^2, \\
y_{22} - a_{21} \alpha(2) - a_{22} \alpha(2)^2 & \leq a_{20} \leq x_{22} - a_{21} \alpha(2) - a_{22} \alpha(2)^2, \\
y_{23} - a_{21} \alpha(3) - a_{22} \alpha(3)^2 & \leq a_{20} \leq x_{23} - a_{21} \alpha(3) - a_{22} \alpha(3)^2.
\end{align*}
\]

(14)

For \( a_{30}, a_{31}, a_{32} \) we have the following inequalities:

\[
|a_{32}| \leq \frac{1}{\alpha(2) - \alpha(3)} \left( \frac{x_{31} + x_{32}}{\alpha(1) - \alpha(2)} + \frac{x_{31} + x_{33}}{\alpha(1) - \alpha(3)} \right),
\]

(15)

\[
\begin{align*}
\frac{x_{31} + x_{32}}{\alpha(1) - \alpha(2)} - a_{32}(\alpha(1) + \alpha(2)) \\
\leq a_{31} \leq \frac{-x_{31} + x_{32}}{\alpha(1) - \alpha(2)} - a_{32}(\alpha(1) + \alpha(2)), \\
\frac{x_{31} + x_{33}}{\alpha(1) - \alpha(3)} - a_{32}(\alpha(1) + \alpha(3)) \\
\leq a_{31} \leq \frac{-x_{31} + x_{33}}{\alpha(1) - \alpha(3)} - a_{32}(\alpha(1) + \alpha(3)),
\end{align*}
\]

(16)

\[
\begin{align*}
-x_{31} - a_{31} \alpha(1) - a_{32} \alpha(1)^2 & \leq a_{30} \leq x_{31} - a_{31} \alpha(1) - a_{32} \alpha(1)^2, \\
x_{32} - a_{31} \alpha(2) - a_{32} \alpha(2)^2 & \leq a_{30} \leq x_{32} - a_{31} \alpha(2) - a_{32} \alpha(2)^2, \\
x_{33} - a_{31} \alpha(3) - a_{32} \alpha(3)^2 & \leq a_{30} \leq x_{33} - a_{31} \alpha(3) - a_{32} \alpha(3)^2.
\end{align*}
\]

(17)

In the remaining two cases we can take \( \{1, \alpha, (1 + \alpha^2)/2\} \) or \( \{1, \alpha, (\alpha + \alpha^2)/2\} \) as a \( \mathbb{Z} \)-basis of \( \mathcal{O}_F \). Also in these cases, from the inequalities (8)–(10) we get the inequalities for \( a_{10}, a_{11}, a_{12}, \ldots, a_{32} \).

Since \( K \) is totally real, we have

\[
\begin{vmatrix}
 s_0^{(i)} & s_1^{(i)} & s_2^{(i)} \\
 s_1^{(i)} & s_2^{(i)} & s_3^{(i)} \\
 s_2^{(i)} & s_3^{(i)} & s_4^{(i)}
\end{vmatrix} > 0 \quad (i = 1, 2, 3).
\]

(18)

By the inequality of Newton,

\[
3a_1^{(i)}a_3^{(i)} \leq a_2^{(i)^2} \quad (i = 1, 2, 3).
\]

(19)

The following theorem is useful to determine the discriminant \( d(f) \) of a polynomial \( f(x) \) and to check that \( f(x) \) has only real zeros.
Theorem 3 (Pohst [2]). Let \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in \mathbb{R}[x] \) be an irreducible polynomial. Then \( f(x) \) has only real zeros if and only if the symmetric matrix \( A = (a_{ij}) \) with entries
\[
a_{ij} = j(n - i)a_ia_j - n \sum_{k=1}^{[n/2]} (2k + i - j)a_{i+k}a_{j-k}
\]
\((i, j = 1, \ldots, n-1; \ i \geq j; \ a_0 = 1; \ a_m = 0 \ if \ m > n \ or \ m < 0)\)
is positive definite. The discriminant \( d(f) \) of \( f(x) \) is given by
\[
d(f) = n^{2-n} \det(A).
\]

From Theorem 3 we have the matrix \( A \) given by
\[
A = \begin{pmatrix}
2a_1^{(1)} - 6a_2^{(1)} & a_1^{(1)}a_2^{(1)} - 9a_3^{(1)} \\
a_1^{(2)}a_2^{(1)} - 9a_3^{(1)} & 2a_2^{(2)} - 6a_1^{(1)}a_3^{(1)}
\end{pmatrix} \quad (i = 1, 2, 3).
\]

So we have the following inequalities:
\[
(20) \quad 2a_1^{(i)} - 6a_2^{(i)} > 0, \quad \det(A) > 0 \quad (i = 1, 2, 3).
\]

By the inequalities (11)–(20), we determine the set of \((a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{32})\) and the set of all coefficients \((a_1^{(i)}, a_2^{(i)}, a_3^{(i)})\) of \( f^{(i)}(x) \) \((i = 1, 2, 3)\). Since \( f(x) = f^{(1)}(x)f^{(2)}(x)f^{(3)}(x) \), we determine the set of all coefficients \((a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)\) of the polynomial \( f(x) \).

We examine the irreducibility of the polynomial \( f(x) \). We find the zeros \( \theta^{(i)} \) \((1 \leq i \leq 9)\) of \( f(x) \) by Newton’s method. If \( f(x) \) is reducible, then \( f(x) \) is divisible by a first-degree polynomial or a cubic polynomial. Hence, if \( f(x) \) does not satisfy the following two conditions (21) and (22), then \( f(x) \) is irreducible.

\[
(21) \quad \text{There is a } \theta^{(i)} \text{ such that } \theta^{(i)} \in \mathbb{Z}.
\]

\[
(22) \quad \text{There are } \theta^{(i)} \text{ and } \theta^{(j)} \text{ and } \theta^{(k)} \text{ such that } \theta^{(i)} + \theta^{(j)} + \theta^{(k)} = 0, \quad \theta^{(i)} \theta^{(j)} + \theta^{(j)} \theta^{(k)} + \theta^{(k)} \theta^{(i)} \in \mathbb{Z}.
\]

Then we determine the set of all coefficients \((a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)\) such that \( f(x) \) is irreducible. We denote by \( N_1 \) the number of such polynomials \( f(x) \) for each cubic field \( F \). Then we have Table 2.

<table>
<thead>
<tr>
<th>( d(F) )</th>
<th>( N_1 )</th>
<th>( d(F) )</th>
<th>( N_1 )</th>
<th>( d(F) )</th>
<th>( N_1 )</th>
<th>( d(F) )</th>
<th>( N_1 )</th>
<th>( d(F) )</th>
<th>( N_1 )</th>
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<tr>
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<td>993</td>
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<td>1593</td>
<td>6</td>
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</table>

For all the remaining cases we have \( N_1 = 0 \).
4. Minimum discriminant of the totally real fields of degree 9

Finally, we determine the minimum discriminant $d(K)$ of the fields $K$ obtained in §3. Using Theorem 3, we obtain the discriminant $d(f)$ of each polynomial $f(x)$ found in §3. In general, $d(f)$ is not equal to $d(K)$; it is known that $d(f) = m^2d(K)$ $(m > 0 \in \mathbb{Z})$. So we shall determine $m$. We decompose $d(f) = m_2^2m_2d(F)^3$, where $m_2$ is squarefree. If $f(x)$ satisfies the inequality

\[(23) \quad m_2d(F)^3 > d_{\text{max}},\]

then we can exclude such $f(x)$. We denote by $N2$ the number of $f(x)$ such that $f(x)$ does not satisfy (23). Then we have Table 3.

<table>
<thead>
<tr>
<th>$d(F)$</th>
<th>$N2$</th>
<th>$d(F)$</th>
<th>$N2$</th>
<th>$d(F)$</th>
<th>$N2$</th>
<th>$d(F)$</th>
<th>$N2$</th>
</tr>
</thead>
<tbody>
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<td>1229</td>
<td>2</td>
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<td>2</td>
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<tr>
<td>148</td>
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<td>257</td>
<td>8</td>
<td>473</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For all the remaining cases we have $N2 = 0$.

In these cases we check whether $p | (d(f)/d(K))$ or $p \nmid (d(f)/d(K))$ for all primes $p$ such that $p | m_1$. In order to do this, the following theorem is useful.

**Theorem 4** (Zassenhaus [3]). Let $F$ be an algebraic number field, $\mathcal{O}_F$ be the ring of integers in $F$. Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ $(a_i \in \mathcal{O}_F, 1 \leq i \leq n)$ be an irreducible polynomial over $F$ with the discriminant $d(f)$ and $\alpha$ be a zero of $f(x)$ and $K = F(\alpha)$ be an extension of $F$ with degree $n$. Let the decomposition of the ideal $(d(f))$ in $\mathcal{O}_F$ be as follows:

\[
d(f) = \prod_{i=1}^{s_0} \mathfrak{p}_i^{e_i} \prod_{i=s_0+1}^s \mathfrak{p}_i \quad (e_i > 1, \quad 1 \leq i \leq s_0; \quad \mathfrak{p}_i \text{ distinct prime ideals}).
\]

Let

\[
f(x) \equiv \prod_{1 \leq i \leq s_0} f_{i j}(x)^{e_{ij}} \pmod{\mathfrak{p}_i} \quad (1 \leq i \leq s_0)
\]

be an irreducible factorization of $f(x) \pmod{\mathfrak{p}_i}$. We define $d_i(x)$, $g_i(x)$, and $h_i(x)$ by

\[
d_i(x) = \prod_{1 \leq i \leq s} f_{i j}(x), \quad g_i(x) = \prod_{1 \leq i \leq s} f_{i j}(x)^{e_{ij}-1},
\]

\[
\pi_i h_i(x) = (d_i(x)g_i(x) - f(x))\eta_i \quad \text{in } \mathcal{O}_F,
\]

where $\pi_i \in \mathfrak{p}_i$, $\pi \notin \mathfrak{p}_i^2$, and $\eta_i \notin \mathfrak{p}_i$. Then $\mathfrak{p}_i^2 \nmid (d(f)/d(K)) \quad (1 \leq i \leq s_0)$ if and only if $\text{G.C.D.}(d_i(x), g_i(x), h_i(x)) = 1 \pmod{\mathfrak{p}_i}$, and for $\mathfrak{p}_i \quad (s_0 + 1 \leq i \leq s)$ we have $\mathfrak{p}_i \nmid (d(f)/d(K))$.

Using this theorem, we find all primes $p$ such that $p | m_1$ and $p \nmid (d(f)/d(K))$. Then, if $p_1^{2v_{p_1}(m_1)} \cdots p_j^{2v_{p_j}(m_j)} m_2d(F)^3 > d_{\text{max}}$, where $p_i | m_1$, $p_i \nmid (d(f)/d(K))$
(i = 1, . . . , j) and \( p_i^{2v_p(m_i)} \) is the largest power of \( p \) dividing \( m_i \), then we can exclude \( f(x) \).

We apply Theorem 4 in the case \( f(x) = x^9 + 2x^8 - 8x^7 - 14x^6 + 22x^5 + 30x^4 - 24x^3 - 20x^2 + 10x + 2 \), \( d(f) = 522901049600 = 148^3 \times 2^2 \times 5^2 \times 1613 \), \( d(F) = 148 \), and \( m_1 = 2 \times 5 \). If \( p = 2 \), then

\[
f(x) \equiv x^9 \pmod{2}, \quad d(x) = x, \quad g(x) = x^8, \quad h(x) = -x^8 + 4x^7 + 7x^6 - 11x^5 - 15x^4 + 12x^3 + 10x^2 - 5x - 1.
\]

Since \( \text{G.C.D.}(d(x), g(x), h(x)) = 1 \) \( \pmod{2} \), we have \( 2 \nmid (d(f)/d(K)) \). If \( p = 5 \), then

\[
f(x) \equiv (x^2 + x + 2)^2(x^2 + 2x - 1)(x^3 - 2x^2 + 2x + 2) \pmod{5}, \quad d(x) = (x^2 + x + 2)(x^2 + 2x - 1)(x^3 - 2x^2 + 2x + 2), \quad g(x) = (x^2 + x + 2), \quad h(x) = 2x^7 + 4x^6 - 3x^5 + 10x^3 + 10x^2 - 2x - 2.
\]

Since \( \text{G.C.D.}(d(x), g(x), h(x)) = 1 \), we have \( 5 \nmid (d(f)/d(K)) \). Consequently, \( m = 1 \), \( d(K) = d(f) = 529010496600 > d_{\text{max}} \) and we exclude \( f(x) \). By this method we can exclude 602 cases.

Using the Dirichlet Discriminant Theorem, we have the following proposition.

**Proposition 1.** Let \( K \) be an algebraic number field of degree 9 and \( F \) be its subfield of degree 3. Let \( p \) be a prime number such that \( p \) ramifies in \( K \) and \( p \) is a prime ideal in \( F \). We denote by \( v_p(m) \) the \( p \)-index of \( m \) (i.e., \( p^{\nu_p(m)} \mid m \) and \( p^{\nu_p(m) + 1} \nmid m \)). Then, if \( p \neq 2, 3 \) then \( v_p(d(K)) = 3 \) or \( 6 \), if \( p = 2 \) then \( v_p(d(K)) \geq 6 \), if \( p = 3 \) then \( v_p(d(K)) = 3 \) or \( \geq 9 \).

Since the proof is easy, we shall omit it.

We apply Proposition 1 in the case \( f(x) = x^9 - x^8 - 1lx^7 + 12x^6 + 36x^5 - 41x^4 - 31x^3 + 33x^2 + 2x - 1 \), \( d(f) = 119414482370560 = 2^{15} \times 5 \times 151 \times 1693 \), \( d(F) = 169 \), \( m_1 = 128 = 2^7 \), and \( d(f)/m_1^2 = 7288481590 \).

The ideal \( (2) \) is a prime ideal in \( F \). Since \( 2 \mid d(K) \) and \( 2 \) ramifies in \( K \). We have \( d(K) \geq 2^5 \times 7288481590 \geq d_{\text{max}} \). So \( f(x) \) is excluded. By this method we can exclude 6 cases.

**Proposition 2** (Takeuchi [7]). Let \( K \) be an algebraic number field and \( F \) be a subfield. Let \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in \mathcal{O}_F[x] \) be the defining polynomial for \( K \) over \( F \). Let \( p \) be a prime ideal of \( \mathcal{O}_F \). If \( p \mid a_i \quad (m < i \leq n) \) and \( p^2 \nmid a_n \), then the ramification index of \( p \) for \( K/F \) is at least \( n - m \).

We apply Proposition 2 in the case \( f(x) = x^9 + 5x^8 - 2x^7 - 37x^6 - 20x^5 + 78x^4 + 52x^3 - 40x^2 - 16x + 8 \), \( d(f) = 20181511264415744 = 2^{18} \times 65449 \times 493 \), \( d(F) = 49 \), \( m_1 = 512 = 2^9 \), and \( d(f)/m_1^2 = 770009401 \).

We know that \( F = \mathbb{Q}(\alpha) \), where \( \alpha \) is a zero of \( x^3 + x^2 - 2x - 1 \). Then \( x^3 + (1 + \alpha^2)x^2 + (-6 + 2\alpha^2)x + 2 - 2\alpha - 2\alpha^2 \) is an irreducible polynomial of \( K \) over \( F \). Since the ideal \( (2) \) is a prime ideal in \( F \), we see that \( 2 \mid (-6 + 2\alpha^2), 2 \mid (2 - 2\alpha - 2\alpha^2) \) and \( 2 \nmid (2 - 2\alpha - 2\alpha^2) \). So 2 ramifies in \( K \). By Proposition 1, we have \( d(K) \geq 2^6 \times 770009401 > d_{\text{max}} \). So \( f(x) \) is excluded. By this method we can exclude 106 cases.
We denote by $N_k$ the number of $f(x)$ such that the discriminant $d(K)$ of the field $K$ given by $f(x)$ satisfies (6). Then we have Table 4.

To determine if two fields $K$ with the same $d(K)$, but given by different polynomials $f(x)$, are isomorphic, we use the method of Takeuchi [7]. Consequently, we see that the fields $K$ are uniquely determined up to $\mathbb{Q}$-isomorphism. So the proof of Theorem 1 is completed.

**Bibliography**


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