ON A PROBLEM OF ERDŐS CONCERNING PRIMITIVE SEQUENCES

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Dedicated to Paul Erdős on the occasion of his 80th birthday

Abstract. A sequence \( A = \{a_i\} \) of positive integers \( a_1 < a_2 < \cdots \) is said to be primitive if no term of \( A \) divides any other. Let \( \Omega(a) \) denote the number of prime factors of \( a \) counted with multiplicity. Let \( p(a) \) denote the least prime factor of \( a \) and \( A(p) \) denote the set of \( a \in A \) with \( p(a) = p \). The set \( A(p) \) is called homogeneous if there is some integer \( s_p \) such that either \( A(p) = \emptyset \) or \( \Omega(a) = s_p \) for all \( a \in A(p) \). Clearly, if \( A(p) \) is homogeneous, then \( A(p) \) is primitive. The main result of this paper is that if \( A \) is a positive integer sequence such that \( 1 \notin A \) and each \( A(p) \) is homogeneous, then

\[
\sum_{a \leq n, a \in A} \frac{1}{a \log a} \leq \sum_{p \leq n, p \text{ prime}} \frac{1}{p \log p} \quad \text{for } n > 1.
\]

This would then partially settle a question of Erdős who asked if this inequality holds for any primitive sequence \( A \).

1. Introduction

A sequence \( A = \{a_i\} \) of positive integers \( a_1 < a_2 < \cdots \) is said to be primitive if no term of \( A \) divides any other (cf. [3] or [5]). We denote by \( p_m \) the \( m \)th prime, by \( p \) a variable prime and by \( p(a) \) the least prime factor of \( a \). We define the degree of an integer \( a \), denoted by \( \Omega(a) \), to be the number of prime factors of \( a \) counted with multiplicity. The degree of an integer sequence \( A \), denoted by \( d^\circ(A) \), is defined as the maximum degree of its terms. We take \( d^\circ(A) = 0 \) if \( A = \{1\} \) or \( \emptyset \).

For a primitive sequence \( A \) with \( d^\circ(A) > 0 \) we define

\[
f(A) = \sum_{a \in A} \frac{1}{a \log a}.
\]

We take \( f(A) = 0 \) if \( d^\circ(A) = 0 \). Erdős [1] proved that there exists an absolute constant \( C \) such that \( f(A) \leq C \) for any primitive sequence \( A \). Recently he [2] has asked if the inequality

\[
\sum_{a \leq n, a \in A} \frac{1}{a \log a} \leq \sum_{p \leq n, p \text{ prime}} \frac{1}{p \log p} \quad \text{for } n > 1
\]

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is always true for any primitive sequence $A$. Zhang [8] proved that if $A$ is primitive with $d^*(A) \leq 4$, then the inequality is true. Erdős and Zhang [4] proved that $f(A) < 1.84$ for any primitive sequence $A$, and gave a necessary and sufficient condition for the inequality (1), namely $\sum_{b \in B} \frac{1}{(b \log b)} \leq \sum \frac{1}{(p \log p)}$ for any primitive sequence $B$. Clearly, if (1) is true then $C = \sum \frac{1}{(p \log p)} < 1.64$.

In this paper we partially settle this question of Erdős in another direction. To give our result, we need some more notation and concepts. Let $A(p)$ denote the set of $a \in A$ with $p(a) = p$. A sequence $B$ is called homogeneous if either $B = \emptyset$ or $\Omega(b) = d^*(B)$ for all $b \in B$. Clearly, if $B$ is homogeneous, then $B$ is primitive. Now we state our main result as the following

**Theorem.** If $A$ is a positive integer sequence such that $1 \notin A$ and each $A(p)$ is homogeneous, then the inequality (1) is true.

The basic idea for proving the theorem is the same as that used in [8]; i.e., we consider the least prime factors of the terms of $A$. The key point of this paper is to prove that, for a given prime $p$, if $B = B(p)$ is homogeneous and nonempty, then

$$\sum_{b \in B} \frac{1}{b \log b} \leq \frac{1}{p \log p}.$$  

It is clear that (2) immediately implies the theorem. In fact we have the stronger result where “$a \leq n$” is replaced in (1) with “$(a, n!) > 1$”.

**2. Proof of the theorem**

We first define two functions:

$$w(s, m) = \sum_{\Omega(a) = s-1, p(a) \geq p_{m+1}} \frac{1}{a \log(p_{m+1}a)}$$

for integers $s \geq 2$, $m \geq 0$, and

$$h(m) = \sum_{i > m} \frac{1}{p_i \log(i-1)}$$

for integers $m \geq 2$.

We need nine lemmas.

**Lemma 1.** We have $p_n > n \log n$ for $n \geq 1$ and $p_n < n(\log n + \log \log n)$ for $n \geq 6$.

These results may be found in [6] and [7].

**Lemma 2.** We have $h(m) < 1/\log m$ for $m \geq 2$.

**Proof.** Note that for each $i \geq 3$, we have

$$\frac{1}{i \log i \log(i-1)} < \frac{\log(i/(i-1))}{\log i \log(i-1)} = \frac{1}{\log(i-1)} - \frac{1}{\log i}.$$ 

Thus, from Lemma 1,

$$h(m) < \sum \frac{1}{i \log i \log(i-1)} < \sum \left( \frac{1}{\log(i-1)} - \frac{1}{\log i} \right) = \frac{1}{\log m}. \quad \Box$$

In the following we define $i(a) = i$ if the largest prime factor of $a$ is $p_i$. 

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Lemma 3. For $m \geq 2$, $s \geq 1$, we have

$$\sum_{p(a) > p_m, \Omega(a) = s} \frac{1}{a \log(i(a) - 1)} \leq h(m) < \frac{1}{\log m}.$$ 

Proof. We proceed by induction on $s$. If $s = 1$, then this is just Lemma 2. Assume the lemma for $s$. For the $s + 1$ case, we have, by Lemma 2,

$$\sum_{p(b) > p_m, \Omega(b) = s} \frac{1}{b \log(i(b) - 1)} < \sum_{b \log(i(b) - 1)} \leq h(m) < \frac{1}{\log m}. \quad \Box$$

Lemma 4. For $i \geq 2$, $B \geq 2$, we have

$$\sum_{j > i} \frac{1}{p_j \log(Bp_j)} < \frac{\log(1 + \log B/\log i)}{\log B} \leq \min \left\{ \frac{1}{\log i}, \frac{1}{e \log i} + \frac{1}{e \log B} \right\},$$

where $e = 2.718 \cdots$ is the base of the natural logarithms.

Proof. We have, by Lemma 1,

$$\sum_{j > i} \frac{1}{p_j \log(Bp_j)} < \int_i^\infty \frac{dx}{x \log x \log(Bx)} = \frac{\log(1 + \log B/\log i)}{\log B} \leq \min \left\{ \frac{1}{\log i}, \frac{1}{e \log i} + \frac{1}{e \log B} \right\},$$

observing that the last inequality follows from

$$\log(1 + x) < x \quad \text{and} \quad \log x = 1 + \log(1 + (x - e)/e) \leq x/e$$

for all $x > 0$. \quad \Box

Lemma 5. For $m \geq 2$, $B \geq 2$, $s \geq 2$, we have

$$\sum_{p(u) > p_m, \Omega(u) = s} \frac{1}{u \log(Bu)} < (e^{-1} + \cdots + e^{1-s})h(m) + e^{1-s} \sum_{j > m} \frac{1}{p_j \log(Bp_j)}.$$ 

Proof. We proceed by induction on $s$. If $s = 2$, then we have, by Lemma 4,

$$\sum_{p(u) > p_m, \Omega(u) = 2} \frac{1}{u \log(Bu)} = \sum_{j > m} \frac{1}{p_j} \sum_{k \geq j} \frac{1}{p_k \log(Bp_j p_k)}$$

$$< e^{-1} h(m) + e^{-1} \sum_{j > m} \frac{1}{p_j \log(Bp_j)}.$$
For the $s + 1$ case, we have, by Lemmas 3 and 4 and the $s$ case,
\[
\sum_{p(b) > p_m, \Omega(b) = s + 1} \frac{1}{u \log(Bu)} = \sum_{p(b) > p_m, \Omega(b) = s} \frac{1}{b} \sum_{j > i(b)} \frac{1}{p_j \log(Bb p_j)} < \sum_{p(b) > p_m, \Omega(b) = s} \frac{e^{-1}}{b} \left( \frac{1}{\log(i(b) - 1)} + \frac{1}{\log(Bb)} \right) \]
\[
< (e^{-1} + \cdots + e^{-s})h(m) + e^{-s} \sum_{j > m} \frac{1}{p_j \log(Bp_j)}. \quad \square
\]

**Lemma 6.** For $m \geq 5$, $s \geq 2$, we have $w(s, m) < 1/\log p_{m+1}$.

*Proof.* We have, by Lemmas 2, 4, and 5,
\[
w(s, m) < W(s, m),
\]
where
\[
W(s, m) = \frac{e^{-1} + \cdots + e^{1-s}}{\log m} + \frac{e^{1-s}}{\log p_{m+1}}.
\]
By Lemma 1 we have
\[
\frac{\log p_{m+1}}{\log m} < \frac{\log(m + 1) + \log(\log(m + 1) + \log \log(m + 1))}{\log m} \leq \frac{\log 6 + \log(\log 6 + \log \log 6)}{\log 5} = 1.65 \cdots < e - 1.
\]
Thus,
\[
W(s, m) - W(s + 1, m) = e^{-s} \left( \frac{e - 1}{\log p_{m+1}} - \frac{1}{\log m} \right) > 0
\]
for $m \geq 5$, $s \geq 2$. Therefore,
\[
w(s, m) < W(s, m) \leq W(2, m) = \frac{1}{e \log m} + \frac{1}{e \log p_{m+1}} < \frac{e - 1}{e \log p_{m+1}} + \frac{1}{e \log p_{m+1}} = \frac{1}{\log p_{m+1}}. \quad \square
\]

**Lemma 7.** For $0 \leq m \leq 4$, we have $w(2, m) < 1/\log p_{m+1}$.

*Proof.* We have, by Lemma 4,
\[
w(2, m) < w(m) \quad \text{for } 0 \leq m \leq 4,
\]
where
\[
w(m) = \frac{1}{p_{m+1} \log(p_{m+1}^2)} + \frac{1}{p_{m+2} \log(p_{m+1} p_{m+2})} + \frac{1}{\log p_{m+1}} \log \left( 1 + \frac{\log p_{m+1}}{\log(m + 2)} \right) \quad \text{for } 1 \leq m \leq 4
\]
and
\[
w(0) = \frac{1}{2 \log 4} + \frac{1}{3 \log 6} + \frac{1}{5 \log 10} + \frac{1}{\log 2} \log \left( 1 + \frac{\log 2}{\log 3} \right).
\]
By calculation we have Table 1.
Thus, \( w(2, m) < w(m) < 1 / \log p_{m+1} \) for \( 0 \leq m \leq 4 \). □

**Lemma 8.1.** For \( s \geq 3, 2 \leq m \leq 4 \), we have \( w(s, m) < 1 / \log p_{m+1} \).

**Proof.** For a fixed \( m \), put

\[
\gamma_s = (e^{-1} + \cdots + e^{2-s})h(m) + e^{2-s}w(m),
\]

where \( w(m) \) is the upper bound of \( w(2, m) \), defined in the proof of Lemma 7. Then by Lemma 5 we have for \( s \geq 3 \) that

\[
w(s, m) < (e^{-1} + \cdots + e^{2-s})h(m) + e^{2-s}w(2, m) < \gamma_s.
\]

If \( h(m)/w(m) < e - 1 \) and \( m \leq 4 \), then we have, from Table 1,

\[
\gamma_s < ((e^{-1} + \cdots + e^{2-s})(e - 1) + e^{2-s})w(m) = w(m) < 1 / \log p_{m+1}.
\]

For \( m = 4 \), we have, by Lemma 2,

\[
h(4) = \sum_{i=5}^{10} \frac{1}{p_i \log(i-1)} + h(10) < 0.6442,
\]

using \( h(10) < 1 / \log 10 \). Thus, \( h(4)/w(4) < 1.7 < e - 1 \), so that the case \( m = 4 \) is done.

For \( m = 3 \) we have

\[
h(3) = 1/(7 \log 3) + h(4) < 0.7743,
\]

and

\[
h(3)/w(3) < 1.7 < e - 1.
\]

Thus the \( m = 3 \) case is done.

For \( m = 2 \), since

\[
h(2) = 1/(5 \log 2) + h(3) < 1.063,
\]

we use the upper bound \( H = 1.063 \) for \( h(2) \) and we see that

\[
H/w(2) > e - 1.
\]

However, we then have

\[
\gamma_s < (e^{-1} + \cdots + e^{2-s})H + e^{2-s} \frac{H}{e - 1} = \frac{H}{e - 1} < 0.62 < 1 / \log 5,
\]

so that the \( m = 2 \) case is done. □

**Lemma 8.2.** We have \( w(s, 1) < 1 / \log p_2 \) for \( s \geq 3 \).

**Proof.** We have \( w(s, 1) = u(s) + v(s) \), where

\[
u(s) = \frac{1}{3} \sum_{\Omega(b)=s-2 \atop p(b) \geq p_2} \frac{1}{b \log(9b)} \quad \text{and} \quad v(s) = \sum_{\Omega(b)=s-1 \atop p(b) \geq p_3} \frac{1}{b \log(3b)}.
\]
Taking
\[ h(2) < \sum_{i=3}^{25} \frac{1}{p_i \log(i-1)} + \frac{1}{\log 25} < 1.0396 \]
and
\[ \sum_{i=2}^{25} \frac{1}{p_i \log(3p_i)} < \sum_{i=3}^{25} \frac{1}{p_i \log(3p_i)} + \frac{1}{\log 25} < 0.5779 < \frac{1.0396}{e-1}, \]
we have, by Lemma 5,
\[ v(s) < 1.0396(e^{-1} + \cdots + e^{2-s}) + 0.5779e^{2-s} < \frac{1.0396}{e-1} < 0.6051 < \frac{2/3}{\log 3}. \]
Since \( w(2, 1) < 1/\log 3 \) by Lemma 7 and \( u(s) < w(s-1, 1)/3 \), we have, for \( s \geq 3 \),
\[ w(s, 1) < w(s-1, 1)/3 + v(s) < (1/3)/\log 3 + (2/3)/\log 3 = 1/\log 3. \]

Lemma 8.3. We have \( w(s, 0) < 1/\log 2 \) for \( s \geq 3 \).

Proof. Put
\[ u_i(s) = \frac{1}{p_i} \sum_{\Omega(b)=s-2, p(b) \geq p_i} \frac{1}{b \log(2pib)} \text{ for } 1 \leq i \leq 9 \]
and
\[ v_i(s) = \sum_{\Omega(b)=s-1, p(b) \geq p_i} \frac{1}{b \log(2b)} \text{ for } 1 \leq i \leq 10. \]
Then for \( 1 \leq i \leq 9 \), we have
\[ v_i(s) = u_i(s) + v_{i+1}(s) \]
and
\[ u_i(s) < \frac{v_i(s-1)}{p_i}. \]

Let \( N = 800 \). Put
\[ h = \sum_{i=10}^{N} \frac{1}{p_i \log(i-1)} + \frac{1}{\log N} < 0.403693 \]
and
\[ g = \sum_{i=10}^{N} \frac{1}{p_i \log(2p_i)} + \frac{1}{\log N} < 0.306441. \]
Then
\[ h(9) < h \quad \text{and} \quad \sum_{i=9}^{N} \frac{1}{p_i \log(2p_i)} < g. \]
We have, by Lemma 5, \( v_{10}(s) < V_{10}(s) \), where
\[ V_{10}(s) = (e^{-1} + \cdots + e^{2-s})h + e^{2-s}g. \]
By calculation we get the upper bounds of \( V_{10}(s) \), for \( 3 \leq s \leq 9 \), listed in Table 2, which serve as upper bounds of \( v_{10}(s) \) for \( 3 \leq s \leq 9 \).
By Lemma 4 we have

\[ u_i(3) < \frac{1}{p_i} \left( \sum_{j=1}^{N} \frac{1}{p_j \log(2p_j p_j)} + \frac{1}{\log N} \right). \]

By calculation we get the upper bounds of \( u_i(3) \) for \( 1 \leq i \leq 9 \), listed in Table 2.

Since we now have upper bounds for \( v_{10}(3) \) and \( u_i(3) \), we can, by equation (3), get upper bounds of \( v_i(3) \) for \( i = 9, 8, \ldots, 2, 1 \). Then, by equation (3), inequality (4) and the upper bounds of \( v_{10}(s) \), we can get upper bounds of \( v_i(s) \) for \( i = 9, 8, \ldots, 2, 1 \); \( s = 4, 5, \ldots, 9 \).

In this way we get upper bounds (listed in Table 2) of

\[ w(s, 0) = v_1(s) < 1/\log 2 \quad \text{for} \quad 3 \leq s \leq 9. \]

In the above calculations we also get the upper bounds of \( v_i(9) \), for \( 1 \leq i \leq 10 \), listed in Table 2.

Let \( k_1 = 1/\log 2 \) and

\[ k_i = \frac{\prod_{j=1}^{i-1} (1 - 1/p_j)}{\log 2} \quad \text{for} \quad 2 \leq i \leq 10. \]

We list the values of \( k_i \), for \( 1 \leq i \leq 10 \), in Table 2.

**Table 2. Upper bounds of \( V_{10}(s) \), \( u_i(3) \), \( w(s, 0) = v_1(s) \) and \( v_i(9) \); and values of \( k_i \)**

<table>
<thead>
<tr>
<th>( s ) or ( i )</th>
<th>( V_{10}(s) )</th>
<th>( u_i(3) )</th>
<th>( w(s, 0) = v_1(s) )</th>
<th>( v_i(9) )</th>
<th>( k_i )</th>
</tr>
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<tr>
<td>1</td>
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<td></td>
<td>1.4412</td>
<td>1.4426</td>
<td></td>
</tr>
<tr>
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<td>0.1885</td>
<td></td>
<td>0.7204</td>
<td>0.7213</td>
<td></td>
</tr>
<tr>
<td>3</td>
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<td>0.0843</td>
<td>1.1049</td>
<td>0.4795</td>
<td>0.4808</td>
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<tr>
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<td>0.3835</td>
<td>0.3847</td>
</tr>
<tr>
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<td>0.2385</td>
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</tr>
<tr>
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<td>0.0228</td>
<td>1.4224</td>
<td>0.2987</td>
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<tr>
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<td>0.0164</td>
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<td>0.2767</td>
</tr>
<tr>
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<td>0.2595</td>
<td>0.2604</td>
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<td>1.4412</td>
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<td></td>
<td></td>
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<td>0.2360</td>
<td></td>
</tr>
</tbody>
</table>

We see that

\[ v_i(9) < k_i \quad \text{for} \quad 1 \leq i \leq 10. \]

Since \( V_{10}(9) < k_{10} \) and \( V_{10}(s + 1) - V_{10}(s) = e^{1-s}(h - (e - 1)g) < 0 \), we have

\[ v_{10}(s) < V_{10}(s) < k_{10} \quad \text{for} \quad s \geq 9. \]

For \( i = 9 \) down to 1, for \( s = 9, 10, \ldots \), we have, by (3), (4), (5), and (6),

\[ v_i(s + 1) < \frac{v_i(s)}{p_i} + v_{i+1}(s + 1) < \frac{k_i}{p_i} + k_{i+1} = k_i. \]

Thus, \( w(s, 0) = v_1(s) < k_1 = 1/\log 2 \quad \text{for} \quad s \geq 9. \)

Combining Lemmas 8.1, 8.2, and 8.3, we have the following
Lemma 8. We have $w(s, m) < 1/\log p_{m+1}$ for $s \geq 3$, $0 \leq m \leq 4$.

Lemma 9. For a given prime $p$, if $B = B(p)$ is homogeneous and nonempty, then

$$\sum_{b \in B} \frac{1}{b \log b} < \frac{1}{p \log p}.$$ 

Proof. This follows from Lemmas 6, 7, and 8. □

As we have seen above, Lemma 9 immediately implies the theorem.

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