A REFINEMENT OF H. C. WILLIAMS’ $q$th ROOT ALGORITHM

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Dedicated to the memory of D. H. Lehmer

Abstract. Let $p$ and $q$ be primes such that $p \equiv 1 \pmod{q}$. Let $a$ be an integer such that $a^{(p-1)/q} \equiv 1 \pmod{p}$. In 1972, H. C. Williams gave an algorithm which determines a solution of the congruence $x^q \equiv a \pmod{p}$ in $O(q^2 \log p)$ steps, once an integer $b$ has been found such that $(b^q - a)^{(p-1)/q} \not\equiv 0, 1 \pmod{p}$. A step is an arithmetic operation (mod $p$) or an arithmetic operation on $q$-bit integers. We present a refinement of this algorithm which determines a solution in $O(q^2 + O(q^2 \log p)$ steps, once $b$ has been determined. Thus the new algorithm is better when $q$ is small compared with $p$.

1. Introduction

Let $p$ and $q$ be primes and let $a$ be an integer not divisible by $p$. If $p \not\equiv 1 \pmod{q}$, the congruence

$$x^q \equiv a \pmod{p}$$

has one solution $x = a^u$, where $u$ and $v$ are integers such that $qu - (p-1)v = 1$. The integer $u$ is easily found by applying the Euclidean algorithm to $q$ and $p-1$. If $p \equiv 1 \pmod{q}$ and $a^{(p-1)/q} \not\equiv 1 \pmod{p}$, the congruence (1.1) has no solutions. If $p \equiv 1 \pmod{q}$ and $a^{(p-1)/q} \equiv 1 \pmod{p}$, (1.1) has $q$ solutions. H. C. Williams [14] has given an algorithm for finding a solution $x$ of (1.1) when $q$ is odd. Briefly, his algorithm may be described as follows: first determine by trial an integer $b$ such that $b^q - a$ is not a $q$th power residue of $p$; then use the formula

$$U_{j,m+n} \equiv \sum_{i=0}^{j} U_{i,n} U_{j-i,m} + (a - b^q) \sum_{i=1}^{q-1} U_{j+i,n} U_{q-i,m} \pmod{p}$$

$(j = 0, 1, \ldots, q-1; \ m = 1, 2, \ldots; \ n = 1, 2, \ldots)$

recursively, starting with the initial values

$$U_{0,1} = b, \quad U_{1,1} = 1, \quad U_{j,1} = 0 \quad (j = 2, \ldots, q-1),$$

to compute $x = U_{0,(p^n-1)/(p-1)q}$. Then $x$ is a solution of (1.1). Once $b$ has been determined, Williams’ algorithm requires $O(q^2 \log p)$ steps to solve (1.1).

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where by a step we mean an arithmetic operation in $GF(p)$ or an arithmetic operation on $q$-bit integers. All $q$ solutions of the congruence (1.1) are given by

$$x(b^q - a)^{(p-1)/q}, \quad j = 0, 1, \ldots, q - 1.$$ 

In this paper we present a refinement of Williams' algorithm which determines a solution of (1.1) in $O(q^4) + O(q^2 \log p)$ steps, once the integer $b$ has been found. Thus, our algorithm is better when $q$ is small compared with $p$, roughly when $q = O((\log p)^{1-\epsilon})$, where $0 < \epsilon < \frac{1}{2}$.

We remark that Williams' algorithm is a $q$th power version of an algorithm for computing square roots in $GF(p)$, which was published by Cipolla [5] in 1903 (see also [2], [9, pp. 132–134]). Shanks [12] has also given an algorithm for determining $q$th roots in $GF(p)$. His algorithm is an extension of an algorithm of Tonelli [13]. Adleman, Manders, and Miller [1] have shown, assuming the extended Riemann hypothesis, that there is a deterministic algorithm running in time $O(n \log^c(p + a))$ for some $c > 0$ such that on inputs $a, p, n$, where $p$ is prime, it outputs the least positive integer $x$ such that $x^n \equiv a \pmod{p}$ or "no" if no such $x$ exists. It is an open problem to find a polynomial-time algorithm—polynomial in $\log q$ and $\log p$—for $q$th roots in $GF(p)$.

Algorithms for the more general problem of factoring polynomials over finite fields have been given by a number of authors, notably, Berlekamp [3], Moench [10], Rabin [11], and Cantor and Zassenhaus [4] (see also [8, Chapter 4]). Cantor and Zassenhaus give a heuristic argument to suggest that the expected running time of their algorithm to factor a polynomial of degree $n$ in $GF(p^m)$ is $O(n^3 + n^2 \log (pm))$.

2. IDEA OF ALGORITHM

Let $p$ and $q$ be primes with $q \mid p - 1$. Let $a$ be a nonzero element of $k = GF(p)$ which is the $q$th power of an element in $k$. We wish to determine a $q$th root of $a$. The algorithm constructs an extension field $K = k[\theta] \cong GF(p^q)$ together with an element $\alpha \in K$ which, when raised to the power $(p^q - 1)/(p - 1)$, gives $a$. It then follows that $\alpha^{(p^q - 1)/(q(p-1))} \in k$ is the desired $q$th root of $a$. This strategy, in rather disguised form, is used by Williams [14]. The contribution of this paper is a way to compute the high power of $\alpha$ somewhat more quickly than the usual repeated squaring algorithm does. The idea is to write the exponent in base $p$ and use automorphisms of $K/k$ to get the effect of raising elements to $p^e$th powers.

3. THE ALGORITHM

Let $p$ and $q$ be primes satisfying $p \equiv 1 \pmod{q}$. Let $a \in k \setminus \{0\}$ be such that $a^{(p-1)/q} = 1$. We first show that there exists $b \in k$ with $(b^q - a)^{(p-1)/q} \neq 0, 1$. Clearly, we can identify $k$ with the residues $\{1, 1 - a, 1 - 2a, \ldots, 1 - (p - 1)a\}$ modulo $p$. As $k$ contains $(p - 1)(q - 1)/q \geq q - 1 \geq 1$ elements which are not $q$th powers, we can let $l$ be the smallest nonnegative integer such that $1 - la$ is not a $q$th power of an element of $k$. Clearly, we have $l \geq 1$ and $1 - (l - 1)a = b^q$ for some $b \in k$. Then we have $b^q - a = 1 - la$, and so, as $1 - la$ is not a $q$th power, we have $(b^q - a)^{(p-1)/q} \neq 0, 1$. We set

$$c = (b^q - a)^{(p-1)/q}.$$
Clearly, $c$ is a primitive $q$th root of unity in $k$. Since $b^q - a$ is not a $q$th power in $k$, we can adjoin a $q$th root $\theta$ of this quantity to $k$ and obtain an extension field

$$K = k[\theta] = GF(p)[\theta] \simeq GF(p^q), \quad \text{where } \theta^q = b^q - a. \tag{3.2}$$

In $K$ we have $\theta^p = (\theta^q)^{(p-1)/q} = (b^q - a)^{(p-1)/q} = c\theta$, so that

$$\theta^{p^n} = c^n\theta, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

Now define $x \in K$ by

$$x = (b - \theta)^{(p^q-1)/(p-1)}. \tag{3.4}$$

As $(p^q - 1)/(p - 1) = 1 + p + p^2 + \cdots + p^{q-1}$, we have

$$x^q = \prod_{j=0}^{q-1} (b - \theta)^{p^j}. \tag{3.5}$$

Next we observe that $(b - \theta)^p = b^p - \theta^p = b - c\theta$, so that

$$(b - \theta)^{p^j} = b - c^j\theta, \quad j = 0, 1, 2, \ldots. \tag{3.6}$$

As $c$ is a primitive $q$th root of unity in $k$, we have

$$\prod_{j=0}^{q-1} (b - c^j\theta) = b^q - \theta^q = a, \tag{3.7}$$

so that by (3.5), (3.6), and (3.7), we see that $x^q = a$. Since the equation $\gamma^q = c$ has at most $q$ solutions in the field $K$, and since it has exactly $q$ solutions in the subfield $k$, every solution must belong to $k$. Thus, in particular, we have $x \in k$. We have thus shown that $x = (b - \theta)^{(p^q-1)/(p-1)}$ is a $q$th root of $a$ in $k$. We remark that H. C. Williams' algorithm is equivalent to computing $\frac{1}{q} \text{tr}_{K/k}((b - \theta)^{(p^q-1)/(p-1)})$, which is also a $q$th root of $a$. Note also that $N_{K/k}(b - \theta) = a$.

In order to compute $x$, we write it in the form

$$x = E_1^{(p-1)/q} E_2, \tag{3.8}$$

where

$$E_1 = (b - \theta)^{(p-1)\sigma^{-2}}, \quad E_2 = (b - \theta)^{(p^q-1)/(q(p-1)-(p-1)^q)/q}. \tag{3.9}$$

First we consider $E_1$. Applying the binomial theorem to $(p - 1)\sigma^{-2}$, and appealing to (3.6), we obtain

$$E_1 = \prod_{i=0}^{q-2} (b - c^i\theta)^{(i-1)\sigma^{-2}}, \tag{3.10}$$

say

$$E_1 = \sum_{i=0}^{q-1} a_i\theta^i. \tag{3.11}$$
where \( a_i \in k \), \( i = 0, 1, \ldots, q - 1 \). Now define \( a_i(j) \in k \) for \( i = 0, 1, \ldots, q - 1 \) and \( j = 1, 2, 3, \ldots \) by

\[
\sum_{i=0}^{q-1} a_i(j) \theta^i = \left( \sum_{i=0}^{q-1} a_i \theta^i \right)^j,
\]

so that

\[
a_i(1) = a_i, \quad i = 0, 1, \ldots, q - 1,
\]

and

\[
E_1^{(p-1)/q} = \sum_{i=0}^{q-1} a_i((p - 1)/q) \theta^i.
\]

Next we consider \( E_2 \). Again, by the binomial theorem and (3.6), we obtain

\[
E_2 = \prod_{i=1}^{q-1} \left( b - c^{q-i-1} \right)^{1 - (-1)^{(i+1)/q}/q}.
\]

It is easily proved by induction on \( i \) that the exponent \((1 - (-1)^{(q-1)})/q\) is an integer. Thus we have

\[
E_2 = \sum_{i=0}^{q-1} b_i \theta^i,
\]

where \( b_i \in k \), \( i = 0, 1, \ldots, q - 1 \). From (3.8), (3.14), and (3.16), we deduce

\[
x = \left( \sum_{i=0}^{q-1} a_i((p - 1)/q) \theta^i \right) \left( \sum_{j=0}^{q-1} b_j \theta^j \right),
\]

that is

\[
x = a_0((p - 1)/q) b_0 + (b^q - a) \sum_{i=1}^{q-1} a_i((p - 1)/q) b_{q-i}.
\]

Formula (3.17) is the expression we use to calculate \( x \). We can now give the algorithm.

**Algorithm to determine all solutions \( x \) of the congruence \( x^q \equiv a \pmod{p} \).**

**Input.** \( p, q \) primes satisfying \( p \equiv 1 \pmod{q} \). \( a \) an integer not divisible by \( p \).

**Step 1.** Compute \( a^{(p-1)/q} \) in \( k = GF(p) \). If \( a^{(p-1)/q} \neq 1 \), then \( x^q = a \) has no solutions in \( k \) and the algorithm terminates. Otherwise, \( x^q = a \) has \( q \) solutions in \( k \) and the algorithm continues with Step 2.

**Step 2.** Try \( b = 1, 2, 3 \ldots \) until the first integer \( b \) is found such that \((b^q - a)^{(p-1)/q} \neq 0, 1 \), and set \( c = (b^q - a)^{(p-1)/q} \).

**Step 3.** In \( K = k[\theta] = \{ c_0 + c_1 \theta + \ldots + c_{q-1} \theta^{q-1} \} \), where \( \theta^q = b^q - a \), compute the quantities \( X_i = (b - c^i \theta)^{(i+1)/q} \) for \( i = 0, 1, \ldots, q - 2 \) and \( Y_i = (b - c^{q-i-1} \theta)^{(1-(q-1)/q)} \) for \( i = 1, \ldots, q - 1 \). Then compute the products \( E_1 = \prod_{i=0}^{q-2} X_i = \sum_{i=0}^{q-1} a_i \theta^i \) and \( E_2 = \prod_{i=1}^{q-1} Y_i = \sum_{i=0}^{q-1} b_i \theta^i \) to obtain \( a_0, a_1, \ldots, a_{q-1}, b_0, b_1, \ldots, b_{q-1} \in k \).
Step 4. Use the recurrence relation in $k$,

$$a_i(m + n) = \sum_{j=0}^{i} a_j(m)a_{i-j}(n) + (b^q - a) \sum_{j=i+1}^{q-1} a_j(m)a_{q+i-j}(n)$$

subject to the initial conditions

$$a_i(1) = a_i \quad (i = 0, 1, \ldots, q - 1),$$

to calculate $a_i((p - 1)/q)$ $(i = 0, 1, \ldots, q - 1)$. 

Output. A solution $x$ of the congruence $x^q \equiv a \pmod{p}$ is given by

$$x = a_0((p - 1)/q)b_0 + (b^q - a) \sum_{i=1}^{q-1} a_i((p - 1)/q)b_{q-i}. $$

All solutions are given by $x_j = c^jx$, $j = 0, 1, \ldots, q - 1$.

We conclude this section by determining the running time of the algorithm. Recall that a step is an arithmetic operation in $k = GF(p)$ or an arithmetic operation on $q$-bit integers. Note that arithmetic operations in $K = GF(p^q)$ take $O(q^2)$ steps.

Step 1. The calculation of $a^{(p-1)/q}$ can be carried out in $O(\log p)$ steps in $k$ by the repeated squaring technique.

Step 2. Let $N$ denote the number of $(x, y) \in k \times k$ with $x^q - y^q = a$ and $B$ the number of values of $b \in k$ for which $(b^q - a)^{(p-1)/q} = 0$ or 1. Then we have

$$N = \sum_{x \in k} \sum_{y \in k} 1 = \sum_{x \in k} 1 + \sum_{x \in k} \sum_{y \neq 0 \pmod{p}} 1 = q + q \sum_{x \in k} \sum_{y \neq 0 \pmod{p}} 1 = q + q(B - q).$$

From the work of Davenport and Hasse [6, p. 174] we have

$$|N - p| \leq q - 1 + ((q - 1)^2 - (q - 1))\sqrt{p},$$

so that

$$|qB - q(q - 1) - p| \leq q - 1 + (q - 1)(q - 2)\sqrt{p}. $$

Hence, we have

$$B \leq \frac{p}{q} + \frac{(q^2 - 1)}{q} + \frac{(q - 1)(q - 2)}{q} \sqrt{p} \leq \frac{p}{q} + 2q \sqrt{p}$$

and

$$B \geq \frac{p}{q} + \frac{(q - 1)^2}{q} - \frac{(q - 1)(q - 2)}{q} \sqrt{p} \geq \frac{p}{q} - q \sqrt{p}, $$

so that

$$\left|\frac{B}{p} - \frac{1}{q}\right| \leq \frac{2q}{\sqrt{p}}.$$
Thus, for $q$ small compared with $p$, say for example $q \leq p^{1/4}$, a random value of $b$ does not satisfy $(b^q - a)^{(p-1)/q} \neq 0, 1$ with probability

$$\frac{B}{p} = \frac{1}{q} + O\left(\frac{q}{\sqrt{p}}\right) = \frac{1}{q} + O(p^{-1/4}).$$

Thus finding an appropriate value of $b$ is usually quite fast in practice.

Step 3. First we observe that all of the values of $b^i$ ($i = 0, 1, \ldots, q-1$) and $c^i$ ($i = 0, 1, \ldots, (q-1)^2$) can be computed in $O(q^2)$ arithmetic operations in $k$.

Next we remark that as

$$\binom{q-2}{i} \leq 2^{q-2} < 2^q < 10^q$$

for $i = 0, 1, \ldots, q-2$, each entry in the first $q-2$ rows of Pascal’s triangle can be represented as a $q$-bit integer, and so $O(q^2)$ additions of $q$-bit integers are required to compute all the binomial coefficients $\binom{q-2}{i}$ ($i = 0, 1, \ldots, q-2$) from Pascal’s triangle.

Knowing the values of $c^i$ ($i = 0, 1, \ldots, q-2$) and $\binom{q-2}{i}$ ($i = 0, 1, \ldots, q-2$), we can, when $q - i$ is even, compute each quantity $(b - c^i\theta)^{(-1)^{q-i}(q-2)} = (b - c^i\theta)^{\binom{q-2}{i}}$ by repeated squarings in $K$ in $O(q^2 \log(q-2))$ steps. Knowing the values of $b^i$ ($i = 0, 1, \ldots, q-1$), $(c^i)^j$ ($i = 0, 1, \ldots, q-1$; $j = 0, 1, \ldots, q-1$) and $\binom{q-2}{i}$ ($i = 0, 1, \ldots, q-2$), as

\begin{equation}
(b - c^i\theta)^{-1} = a^{p-2}(b^{q-1} + b^{q-2}c^i\theta + \ldots + (c^i)^{q-1}\theta^{q-1}),
\end{equation}

we can, when $q - i$ is odd, compute each quantity

$$(b - c^i\theta)^{(-1)^{q-i}(q-2)} = (b - c^i\theta)^{-\binom{q-2}{i}}$$

$$= (a^{p-2}(b^{q-1} + b^{q-2}c^i\theta + \ldots + (c^i)^{q-1}\theta^{q-1}))^{\binom{q-2}{i}}$$

by repeated squarings in $K$ in

$$O(\log p) + O(q) + O\left(q^2 \log\left(\frac{q-2}{i}\right)\right)$$

steps. Hence, all of

$$X_i = (b - c^i\theta)^{(-1)^{q-i}(q-2)} \quad (i = 0, 1, \ldots, q-2)$$

can be computed in

$$O(q^2) + \sum_{i=0}^{q-2} \left(O(\log p) + O(q) + O\left(q^2 \log\left(\frac{q-2}{i}\right)\right)\right) = O(q \log p) + O(q^4)$$

steps, as

$$\sum_{i=0}^{n} \log\left(\frac{n}{i}\right) \sim \frac{1}{2} n^2, \quad \text{as } n \to \infty.$$

see [7]. Multiplying the $X_i$ together in $K$ to obtain $E_1 = \prod_{i=0}^{q-2} X_i = \sum_{i=0}^{q-1} a_i\theta^i$ takes a further $O(q)$ multiplications in $K$, that is, $O(q^3)$ steps. Hence, $a_0, a_1, \ldots, a_{q-1}$ can be computed in $O(q \log p) + O(q^4)$ steps. A similar calculation shows that $b_0, b_1, \ldots, b_{q-1}$ can also be computed in $O(q \log p) + O(q^4)$ steps.
Step 4. The quantities $a_i((p-1)/q)$ ($i = 0, 1, \ldots, q-1$) can be computed from the values of the $a_i$ ($i = 0, 1, \ldots, q-1$) using (3.18) in $O(q^2 \log p)$ steps, since each use of the recurrence relation (3.18) requires $O(q)$ operations and each of the $q$ recurrence relations must be applied $O(\log(p-1)/q)$ times in the repeated doubling technique.

The calculation of the solution $x$ of (1.1) from the values of the $a_i((p-1)/q)$ ($i = 0, 1, \ldots, q-1$) and $b_i$ ($i = 0, 1, \ldots, q-1$) using (3.20) takes $O(q)$ steps, and the calculation of the other solutions $x_{c_0}^j$ ($j = 1, 2, \ldots, q-1$) can be done in $O(q)$ steps. Hence the algorithm determines all the solutions of (1.1) in $O(q^2) + O(q^2 \log p)$ steps, once a suitable $b$ has been determined in Step 2.

We remark that this algorithm (suitably modified) can be used to compute $q$th roots in $GF(p^n)$, when $q$ divides $p^n - 1$.

4. Example

Following the suggestion of the referee, we present a small example to illustrate our algorithm, which the interested reader can easily check by hand. The algorithm is easily programmed to solve (1.1) for large values of $p$ and values of $q$ small compared with $p$.

We determine all the solutions $x$ of the congruence

$$x^3 \equiv 2 \pmod{31},$$

using our refinement to the algorithm of H. C. Williams. Here, $p = 31$, $q = 3$, $a = 2$, $(p-1)/q = 10$, $(p-1)^{q-2} = 30$ and $(p^q-1)/(p-1)-(p-1)^{q-1}/q = 31$. As

$$a^{(p-1)/q} = 2^{10} \equiv 32^2 \equiv 1^2 \equiv 1 \pmod{p},$$

the congruence (4.1) is solvable. We can take $b = 2$, $c = 25$, as

$$(b^q-a)^{(p-1)/q} = (2^3-2)^{10} = 6^{10} = 36^5 \equiv 55 \equiv 3125 \equiv 25 \pmod{p}. $$

Also, $\theta$ is a root of $\theta^q = b^q - a$, that is, $\theta^3 = 6$. We perform calculations in $k = GF(31)$ and $K = GF(31)[\theta] \approx GF(31^3)$.

Appealing to (3.9), (3.10), and (3.21), we have

$$E_1 = (2-\theta)^{30} = (2-\theta)^{-1}(2-25\theta) = (2+\theta+16\theta^2)(2+6\theta) = 22+14\theta+7\theta^2,$$

so that $a_0 = 22$, $a_1 = 14$, $a_2 = 7$. Making use of the recurrence relations

$$\left\{\begin{array}{l}
a_i(m+n) = \sum_{j=0}^{i} a_j(m)a_{i-j}(n) + 6 \sum_{j=i+1}^{2} a_j(m)a_{3+i-j}(n), \\
a_i(1) = a_i,
\end{array}\right. \quad m, n = 1, 2, \ldots,$$

we obtain the values in Table 1 (next page).

Next, from (3.9) and (3.15), we have $E_2 = 2-25\theta$, so that $b_0 = 2$, $b_1 = 6$, $b_2 = 0$. Finally, appealing to (3.20), we obtain

$$x = a_0(10)b_0 + 6(a_1(10)b_2 + a_2(10)b_1) = 9 \times 2 + 6 \times 19 \times 6 = 20.$$
Table 1

<table>
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<th>a_2(j)</th>
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<td>7</td>
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</tr>
<tr>
<td>10</td>
<td>9</td>
<td>4</td>
<td>19</td>
</tr>
</tbody>
</table>

We note that \( x = 20 \) is indeed a solution of (4.1), as \( 20^3 \equiv (-11)^3 \equiv (-121)11 \equiv 3 \times 11 = 33 \equiv 2 \) (mod 31). All solutions of (4.1) are given by \( x \equiv 20 \cdot 25^j \) (mod 31), \( j = 0, 1, 2 \), that is, \( x \equiv 20, 4, 7 \) (mod 31).

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