ZAREMBA'S CONJECTURE AND SUMS OF THE DIVISOR FUNCTION

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Dedicated to the memory of D. H. Lehmer

Abstract. Zaremba conjectured that given any integer \( m > 1 \), there exists an integer \( a < m \) with \( a \) relatively prime to \( m \) such that the simple continued fraction \([0, c_1, \ldots, c_r]\) for \( a/m \) has \( c_i \leq B \) for \( i = 1, 2, \ldots, r \), where \( B \) is a small absolute constant (say \( B = 5 \)). Zaremba was only able to prove an estimate of the form \( c_i \leq C \log m \) for an absolute constant \( C \). His first proof only applied to the case where \( m \) is a prime; later he gave a very much more complicated proof for the case of composite \( m \). Building upon some earlier work which implies Zaremba's estimate in the case of prime \( m \), the present paper gives a much simpler proof of the corresponding estimate for composite \( m \).

1. Introduction

Apparently, Zaremba [5, pp. 69 and 76] was the first to state the following:

Conjecture. Given any integer \( m > 1 \), there is a constant \( B \) such that for some integer \( a < m \) with \( a \) relatively prime to \( m \) the simple continued fraction \([0, c_1, \ldots, c_r]\) for \( a/m \) has \( c_i \leq B \) for \( i = 1, 2, \ldots, r \).

This conjecture is still unproved, though numerical evidence suggests that \( B = 5 \) would suffice. The best result known replaces the inequality in the conjecture by \( c_i \leq C \log m \) for some constant \( C \); this was first proved by Zaremba [5, Theorem 4.6 with \( s = 2 \), p. 74] for prime values of \( m \). Later, Zaremba [6] gave a very much more complicated proof for composite values of \( m \).

As a byproduct of a more general investigation, I proved in an earlier paper [1, p. 154] that the inequality in the conjecture can be replaced by \( c_i \leq 4(m/\varphi(m))^2 \log m \), where \( \varphi(m) \) is Euler's function. Of course, this implies \( c_i \leq C \log m \) if \( m \) is prime, but only gives \( c_i \leq C \log m (\log \log m)^2 \) in general. In the present paper, I show how the argument of [1] can be refined to eliminate the log log factors. The result is

Theorem 1. Given any integer \( m > 1 \), there is an integer \( a < m \) with a relatively prime to \( m \) such that the simple continued fraction \([0, c_1, \ldots, c_r]\) for \( a/m \) has \( c_i \leq 3 \log m \) for \( i = 1, 2, \ldots, r \).

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The proof is much simpler than the proof of the corresponding result in Zaremba [6]. I am grateful to Harald Niederreiter for suggesting that it would be worthwhile to publish this simpler proof.

2. Proof of Theorem 1

Let \( \|x\| \) denote the distance from \( x \) to the nearest integer. We shall actually prove the following sharpening of the case \( n = 2 \) of the theorem in [1].

**Theorem 2.** Given any integer \( m \geq 8 \), there exist integers \( a_1, a_2 \) relatively prime to \( m \) such that

\[
\prod_{i=1}^{2} \|ka_i/m\| > (3m \log m)^{-1} \quad \text{for each } k, \ 1 \leq k < m.
\]

As in [1], it is easy to deduce Theorem 1 from Theorem 2: We may assume \( a_1 = 1 \) and \( a_2 = a \) in Theorem 2, since we may replace \( a_i \) by \( ba_i \) \((i = 1, 2)\), where \( ba_1 \equiv 1 \mod m \). Thus, Theorem 2 implies that for any \( m \geq 8 \) there exists an integer \( a < m \) with \( a \) relatively prime to \( m \) such that

\[
(1) \quad k\|ka/m\| > (3 \log m)^{-1} \quad \text{for each } k, \ 1 \leq k < m.
\]

If \([0, c_1, \ldots, c_r]\) is the simple continued fraction for \( a/m \) with convergents \( \rho_i/q_i \) \((0 \leq i \leq r)\), then we have \( q_i\|q_ia/m\| < 1/c_{i+1} \) for \( i = 0, 1, \ldots, r - 1 \). Therefore, (1) implies Theorem 1. (For \( m < 8 \) it is easy to verify Theorem 1 by calculation.)

We begin the proof of Theorem 2 with some definitions taken from [1, p. 155]. Given any integer \( m > 1 \) and positive integers \( a_1, a_2, \) we let \( L \) denote a positive real number which we shall specify later. We say that the pair \( a_1, a_2 \) is exceptional (with respect to \( m \) and \( L \)) if

\[
(2) \quad \prod_{i=1}^{2} \|ka_i/m\| > L^{-1} \quad \text{for each } k, \ 1 \leq k < m.
\]

Obviously, the pair \( a_1, a_2 \) can be exceptional only if each \( a_i \) is relatively prime to \( m \). If for some \( k, 1 \leq k < m \), the inequality in (2) is false, then we say that \( k \) excludes the pair \( a_1, a_2 \). We shall estimate the integer \( J = J(k) = J(k, m, L) = \) number of pairs \( a_1, a_2 \) with each \( a_i \) relatively prime to \( m \) which are excluded by \( k \) and which satisfy \( 1 \leq a_1 < a_2 \leq m/2 \). The requirement that \( a_1 \) and \( a_2 \) be different is convenient later on.

We first estimate \( J(k, m, L) \) in the case where the greatest common divisor \((k, m)\) is 1. Such a \( k \) excludes the pair \( a_1, a_2 \) if and only if \( k \) excludes the pair \( ka_1, ka_2 \); therefore.

\[
(3) \quad J(k) = J(1) \quad \text{whenever } (k, m) = 1.
\]

We shall prove

\[
(4) \quad J(1) < \frac{\varphi(m)^2}{2L} (\log(m^2/L) + \log \log m).
\]

In order to do this, we need to define the following sums \( D(x, r, m) \) of the divisor function \( d(n) \) (= the number of positive integer divisors of the positive
integer $n$) over arithmetic progressions with difference $m$:

$$D(x, r, m) = \sum_{n \leq x \atop n \equiv r \mod m} d(n).$$

A pair $a_1, a_2$ with $a_i \leq m/2$ ($i = 1, 2$) is excluded by $k = 1$ if

$$a_1 a_2 \leq m^2 / L. \tag{5}$$

The number of ways of writing any positive integer $n \leq m^2 / L$ as $a_1 a_2$ is just $d(n)$, and the factors are both relatively prime to $m$ if and only if $n$ is relatively prime to $m$. Hence, the number of pairs $a_1, a_2$ satisfying (5) and the additional conditions $(a_i, m) = 1$ ($i = 1, 2$) and $1 \leq a_1 < a_2 \leq m/2$ does not exceed

$$\frac{1}{2} \sum_{(n, m) = 1 \atop n \leq m^2 / L} d(n) = \frac{1}{2} \sum_{(r, m) = 1 \atop r = 1}^m D(m^2 / L, r, m)$$

(the factor of $\frac{1}{2}$ comes from the fact that $d(n)$ counts each factorization $n = a_1 a_2$ with distinct $a_1$ and $a_2$ twice; this is where our assumption that $a_1$ and $a_2$ are distinct is convenient). Thus, we have proved

$$J(1, m, L) \leq \frac{1}{2} \sum_{(r, m) = 1 \atop r = 1}^m D(m^2 / L, r, m). \tag{6}$$

In order to estimate the sum in (6), we need some results of D. H. Lehmer [4] concerning the sums $H(x, r, m)$ defined by

$$H(x, r, m) = \sum_{n \leq x \atop n \equiv r \mod m} 1/n.$$ 

Lehmer [4, p. 126] proved the existence of the generalized Euler constants $\gamma(r, m)$ defined for any integers $r$ and $m > 0$ by

$$\gamma(r, m) = \lim_{x \to \infty} (H(x, r, m) - m^{-1} \log x). \tag{7}$$

Clearly, Euler's constant $\gamma$ is $\gamma(0, 1)$, and $\gamma(r, m)$ is a periodic function of $r$ with period $m$.

**Lemma 1.** For any integers $r, m$ with $m > 0$ and $0 \leq r < m$, we have

$$0 < H(x, r, m) - m^{-1} \log x - \gamma(r, m) < 1/x$$

for all $x \geq m$.

**Proof.** This follows easily from the proof of the existence of the limit in (7), as given by Lehmer [4, p. 126]. □

In order to state our next two lemmas, it is convenient to define the arithmetical functions $v(n)$ and $w(n)$ by

$$v(n) = -\sum_{d | n} \mu(d) d^{-1} \log d$$

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(here, $\mu(d)$ is the Möbius function and the sum is taken over all positive integer divisors $d$ of $n$) and

$$w(n) = n\nu(n)/\varphi(n) = \sum_{p|n} \frac{\log p}{(p-1)}$$

(here, the sum is taken over all prime divisors $p$ of $n$).

**Lemma 2.** For every positive integer $m$,

$$\sum_{\substack{r=1 \\ (r, m)=1}}^{m} \gamma(r, m) = \varphi(m)m^{-1}(\gamma + w(m)).$$

**Proof.** This is equation (16) of Lehmer [4, p. 132]. \qed

**Lemma 3.** For every integer $m \geq 8$,

$$\gamma + w(m) < (m/\varphi(m))\log \log m.$$

**Proof.** Theorem 5 of Davenport [2, p. 294] states

$$\limsup_{m \to \infty} \frac{\varphi(m)}{\log \log m} = \frac{1}{4},$$

which implies the lemma for all large $m$. Some simple calculations (using $\gamma = 0.577\ldots$) gives the inequality as stated. \qed

Our final lemma gives an upper bound on the sum $D(x, r, m)$ when $r$ is relatively prime to $m$.

**Lemma 4.** For any integers $r, m$ with $r$ relatively prime to $m$ and $m \geq 8$, we have

$$D(x, r, m) < \varphi(m)m^{-2}x\log x + 2xm^{-1}\log \log m.$$  

**Proof.** We adapt the standard proof of Dirichlet’s theorem on summing $d(n)$ for $n \leq x$. The sum $D(x, r, m)$ is the number of lattice points $(u, v)$ with $uv \equiv r \mod m$ lying below the curve $uv = x$ in the first quadrant of the $u, v$ plane. By using the symmetry in the line $u = v$, if we define $T = [x^{1/2}]$, then we have

$$D(x, r, m) < 2 \sum_{i=1}^{T} F_i(x),$$

where $F_i(x)$ denotes the number of integers $v$ such that $iv \equiv r \mod m$ and $iv \leq x$; we have strict inequality here since we are double counting the lattice points in the square of side $T$ formed by portions of the $u$- and $v$-axes. (For a more elaborate version of this argument, which leads to a $O$-estimate analogous to the one for the usual Dirichlet divisor problem, see Satz 2 of Kopetzky [3]. The simple inequality of Lemma 4 suffices for our purposes, since the more detailed argument does not affect the main term.) If $r$ is relatively prime to $m$, then $iv \equiv r \mod m$ is solvable if and only if $i$ is also relatively prime to $m$, and in that case there is exactly one solution $v \mod m$. It follows that $F_i(x) = 0$ unless $i$ is relatively prime to $m$ and that

$$F_i(x) \leq x(im)^{-1} \quad \text{for } (i, m) = 1.$$
Now (9) implies
\[ \sum_{i=1}^{T} F_i(x) \leq \frac{x}{m} \sum_{r=1}^{m} H(T, r, m). \]

Finally, Lemmas 1, 2, and 3 give the inequality in Lemma 4. □

It follows from (3), (6) and Lemma 4 that
\[ J(k, m, L) < \frac{\varphi(m)^2}{2L} \log(m^2L^{-1}) + m\varphi(m)L^{-1} \log \log m \]
holds for all \( k \) with \( k \) relatively prime to \( m \). By the argument in [1, pp. 156–157], the inequality in (10) is still true if \( k \) is not relatively prime to \( m \) (indeed, in that case we can even insert a factor of \( 8/9 \) on the right-hand side of (10)).

We can now complete the proof of Theorem 2 (and so of Theorem 1) as in [1, p. 157]: Clearly, (2) holds if and only if the inequality in (2) is true for each \( k \leq m/2 \). The total number of pairs \( a_1, a_2 \) with each \( a_i \) relatively prime to \( m \) and \( 1 \leq a_1 < a_2 \leq m/2 \) is
\[ \left( \frac{\varphi(m)/2}{2} \right) > \frac{\varphi(m)^2}{8}. \]

By (10) and the definition of \( J(k, m, L) \), an exceptional pair \( a_1, a_2 \) certainly exists if
\[ \frac{\varphi(m)^2}{8} > \frac{1}{2} m\left( \frac{1}{2} \varphi(m)^2L^{-1} \log(m^2L^{-1}) + m\varphi(m)L^{-1} \log \log m \right). \]

Computation (using the well-known fact that \( \limsup m(\varphi(m)\log \log m)^{-1} = e^\gamma = 1.781\ldots \)) shows that (11) is true for \( m \geq 8 \) if \( L \geq 3m \log m \). This completes the proof of Theorem 2.

### 3. Generalizations

It was pointed out in [1, pp. 154–155] that something like Theorem 2 can be proved in the case of \( n \) integers. The main result of [1] was

**Theorem 3.** Given any integers \( d > 4n \) and \( n > 1 \), there exist integers \( a_1, \ldots, a_n \) relatively prime to \( m \) such that
\[ \prod_{i=1}^{n} \|ka_i/m\| > 4^{-n}(\varphi(m)/m)^{n(m^2L^{-1})^{-1}} \quad \text{for each} \quad k, \quad 1 \leq k < m. \]

In view of the connection of Theorems 1 and 2 above, this can be regarded as an \( n \)-dimensional generalization of a weakened form of Zaremba’s conjecture. In [1, p. 155], I proposed the following general conjecture; Zaremba’s conjecture is the case \( n = 2 \).

**Conjecture.** For each \( n \geq 2 \), the lower bound in (12) can be replaced by \( c(n)(m \log^{n-2} m)^{-1} \).

The proof of Theorem 2 above removed the factors \( \varphi(m)/m \) in the case \( n = 2 \) of (12). One might hope to achieve the same result for arbitrary \( n \) by generalizing the proof of Theorem 2; this would require working with the
generalized divisor functions \( d_n(t) \) = the number of ways of writing the positive integer \( t \) as a product of \( n \) positive integer factors.

To conclude, I repeat another speculation from [1, p. 155]: It is possible that the lower bound in (12) could be replaced by \( c(n)m^{-1} \) for \( n = 3 \), or even for all \( n \geq 2 \). A small amount of computer testing of this for \( n = 3 \) was reported in [1, p. 155]. Further computer experiments might be worthwhile.

**Bibliography**


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