CONTINUED FRACTIONS AND LINEAR RECURRENCES

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Dedicated to the memory of D. H. Lehmer

Abstract. We prove that the numerators and denominators of the convergents to a real irrational number $\theta$ satisfy a linear recurrence with constant coefficients if and only if $\theta$ is a quadratic irrational. The proof uses the Hadamard Quotient Theorem of A. van der Poorten.

Let $\theta$ be an irrational real number with simple continued fraction expansion $[a_0, a_1, a_2, \ldots]$. Define the numerators and denominators of the convergents to $\theta$ as follows:

(1) $p_{-2} = 0; \quad p_{-1} = 1; \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 0$

(2) $q_{-2} = 1; \quad q_{-1} = 0; \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 0$

By the classical theory of continued fractions (see, for example, [2, Chapter X]), we have

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

In this note, we consider the question of when the sequences $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ can satisfy a linear recurrence with constant coefficients. If, for example, $\theta = \sqrt{3}$, then $\theta = [1, 1, 2, 1, 2, 1, 2, \ldots]$, and it is easy to verify that $q_{n+4} = 4q_{n+2} - q_n$ for all $n \geq 0$. Our main result shows that this exemplifies the situation in general.

Theorem 1. Let $\theta$ be an irrational real number. Let its simple continued fraction expansion be $\theta = [a_0, a_1, \ldots]$, and let $(p_n)$ and $(q_n)$ be the sequence of numerators and denominators of the convergents to $\theta$, as defined above. Then the following four conditions are equivalent:

(a) $(p_n)_{n \geq 0}$ satisfies a linear recurrence with constant complex coefficients;

(b) $(q_n)_{n \geq 0}$ satisfies a linear recurrence with constant complex coefficients;

(c) $(a_n)_{n \geq 0}$ is an ultimately periodic sequence;

(d) $\theta$ is a quadratic irrational.
Our proof is simple, but uses a deep result of van der Poorten known as the Hadamard Quotient Theorem. We do not know how to give a short proof of the implication (b) \implies (c) from first principles.

**Proof.** The equivalence (c) \iff (d) is classical. We will prove the equivalence (b) \iff (c); the equivalence (a) \iff (c) will follow in a similar fashion.

(c) \implies (b): It is easy to see (cf. Frame [1]) that

\[
\begin{align*}
\begin{bmatrix}
  p_n \\
  q_n
\end{bmatrix}
= \begin{bmatrix}
  a_0 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  a_1 & 1 \\
  1 & 0
\end{bmatrix}
\cdots
\begin{bmatrix}
  a_n & 1 \\
  1 & 0
\end{bmatrix}.
\end{align*}
\]

Now if the sequence \((a_n)_{n \geq 0}\) is ultimately periodic, then there exists an integer \(r \geq 0\), and \(r\) integers \(b_0, b_1, \ldots, b_{r-1}\), and an integer \(s \geq 1\) and \(s\) positive integers \(c_0, c_1, \ldots, c_{s-1}\) such that

\[\theta = [b_0, b_1, \ldots, b_{r-1}, c_0, c_1, \ldots, c_{s-1}, c_0, c_1, \ldots, c_{s-1}, \ldots].\]

Now for each integer \(i\) modulo \(s\), define

\[M_i = \prod_{0 \leq j < s} \begin{bmatrix} c_{i+j} & 1 \\ 1 & 0 \end{bmatrix}.\]

Then for all \(n \geq r\), we have, by equation (3),

\[
\begin{align*}
\begin{bmatrix} p_{n+s} & p_{n+s-1} \\
q_{n+s} & q_{n+s-1}\end{bmatrix}
= \begin{bmatrix} p_n & p_{n-1} \\
q_n & q_{n-1}\end{bmatrix} M_{n-r}.
\end{align*}
\]

Since for all pairs \((i, j)\) it is possible to find matrices \(A, B\) such that \(M_i = AB\) and \(M_j = BA\), and since \(\text{Tr}(AB) = \text{Tr}(BA)\), it readily follows that \(t = \text{Tr}(M_i)\) does not depend on \(i\). Hence the characteristic polynomial of each \(M_i\) is \(X^2 - tX + (-1)^s\). Since every matrix satisfies its own characteristic polynomial, we see that \(M_{n-r}^2 - tM_{n-r} + (-1)^sI\) is the zero matrix. Combining this observation with equation (4), we get

\[
\begin{align*}
\begin{bmatrix} p_{n+2s} & p_{n+2s-1} \\
q_{n+2s} & q_{n+2s-1}\end{bmatrix} - t \begin{bmatrix} p_{n+s} & p_{n+s-1} \\
q_{n+s} & q_{n+s-1}\end{bmatrix} + (-1)^s \begin{bmatrix} p_n & p_{n-1} \\
q_n & q_{n-1}\end{bmatrix} = 0.
\end{align*}
\]

Therefore, \(q_{n+2s} - tq_{n+s} + (-1)^s q_n = 0\) for all \(n \geq r\), and hence the sequence \((q_n)_{n \geq 0}\) satisfies a linear recurrence with constant integral coefficients.

(b) \implies (c): The proof proceeds in two stages. First we show, by means of a theorem of van der Poorten, that if \((q_n)_{n \geq 0}\) satisfies a linear recurrence, then so does \((a_n)_{n \geq 0}\). Next we show that the \(a_n\) are bounded because otherwise the \(q_n\) would grow too rapidly. The periodicity of \((a_n)_{n \geq 0}\) then follows immediately.

Let us recall a familiar definition: if the sequence of complex numbers \((u_n)_{n \geq 0}\) satisfies a linear recurrence with constant complex coefficients

\[u_n = \sum_{1 \leq i \leq d} e_i u_{n-i}\]

for all \(n\) sufficiently large, and \(d\) is chosen to be as small as possible, then \(X^d - \sum_{1 \leq i \leq d} e_i X^{d-i}\) is said to be the *minimal polynomial* for the linear recur-
rence. Also recall that a sequence of complex numbers \((u_n)_{n \geq 0}\) satisfies a linear recurrence with constant coefficients if and only if the formal series \(\sum_{n \geq 0} u_n X^n\) represents a rational function of \(X\).

Define the two formal series \(F = \sum_{n \geq 0} (q_{n+2} - q_n) X^n\) and \(G = \sum_{n \geq 0} q_{n+1} X^n\). Clearly \(F\) and \(G\) represent rational functions. We now use the following theorem of van der Poorten \([4, 5, 6]\):

**Theorem 2 (Hadamard Quotient Theorem).** Let \(F = \sum_{i \geq 0} f_i X^i\) and \(G = \sum_{i \geq 0} g_i X^i\) be formal series representing rational functions in \(C(X)\). Suppose that the \(f_i\) and \(g_i\) are complex numbers such that \(g_i \neq 0\) and \(f_i/g_i\) is an integer for all \(i \geq 0\). Then \(\sum_{i \geq 0} (f_i/g_i) X^i\) also represents a rational function.

Since \(q_{n+2} = a_{n+2}q_{n+1} + q_n\) for all \(n \geq 0\), it follows from this theorem that \(\sum_{n \geq 0} a_{n+2} X^n\) represents a rational function, and hence the sequence of partial quotients \((a_n)_{n \geq 0}\) also satisfies a linear recurrence with constant coefficients.

We now require the following lemma:

**Lemma 3.** Suppose that \((y_n)_{n \geq 0}\) and \((z_n)_{n \geq 0}\) are sequences of complex numbers, each satisfying a linear recurrence, with the property that the minimal polynomial of \((z_n)_{n \geq 0}\) divides the minimal polynomial of \((y_n)_{n \geq 0}\). Let \(d\) denote the degree of the minimal polynomial of \((y_n)_{n \geq 0}\). Then there exist constants \(c > 0\) and \(n_0\) such that for all \(n \geq n_0\) we have

\[
\max(|y_{n-d+1}|, |y_{n-d+2}|, \ldots, |y_n|) > c|z_n|.
\]

**Proof.** Put \(Y = \sum_{n \geq 0} y_n X^n = f/g\) with \(\gcd(f, g) = 1\) and \(\deg g = d\), and \(Z = \sum_{n \geq 0} z_n X^n = h/g\); here \(f, g, h \in C[X]\). Since \(\gcd(f, g) = 1\), we can find a polynomial \(k = \sum_{0 \leq i < d} k_i X^i\) of degree \(< d\) such that \(kf \equiv h \pmod{g}\). Then \(Z = kY + m\), for a polynomial \(m\), and \(z_n = \sum_{0 \leq i < d} k_i y_{n-i}\) for \(n > n_0 = \deg m\). It follows that

\[
|z_n| \leq \left( \sum_{0 \leq i < d} |k_i| \right) \max(|y_{n-d+1}|, |y_{n-d+2}|, \ldots, |y_n|),
\]

and the lemma follows, with \(c = (1 + \sum_{0 \leq i < d} |k_i|)^{-1}\).

Since \((a_n)_{n \geq 0}\) satisfies a linear recurrence, we may express \(a_n\) as a generalized power sum

\[
a_n = \sum_{1 \leq i \leq d} A_i(n) \alpha_i^n,
\]

for all \(n\) sufficiently large. Here the \(\alpha_i\) are distinct nonzero complex numbers (the “characteristic roots”) and the \(A_i(n)\) are polynomials in \(n\).

Now take \(y_n = a_n\) and \(z_n = n^l |\alpha|^n\), where \(\alpha = \alpha_i\) and \(l = \deg A_i\) for some \(i\). Then the hypothesis of Lemma 3 holds, and we conclude that at least one of \(a_{n-d+1}, a_{n-d+2}, \ldots, a_n\) is greater than \(c n^l |\alpha|^n\), for all \(n\) sufficiently large. Then, using equation (2), we have

\[
q_{dm} \geq \prod_{1 \leq j \leq dm} a_j > c' \cdot c^m \cdot d^{lm} \cdot (m!)^l \cdot (|\alpha|^d)^{m(m+1)/2}
\]
for some positive constant $c'$ and all $m \geq 1$. But $(q_n)_{n \geq 0}$ satisfies a linear recurrence, and therefore $\log q_{dm} = O(dm)$. It follows that $|\alpha_i| \leq 1$ for all $i$, and further that $\deg A_i = 0$ for those $i$ with $|\alpha_i| = 1$. Hence the sequence $(a_n)_{n \geq 0}$ is bounded. But a simple argument using the pigeonhole principle (see, for example, [3, Part VIII, Problem 158]) shows that any bounded integer sequence satisfying a linear recurrence is ultimately periodic. This completes the proof. □

Bibliography


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