GENERALIZED FIBONACCI AND LUCAS SEQUENCES
AND ROOTFINDING METHODS

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Dedicated to the memory of D. H. Lehmer

Abstract. Consider the sequences \( \{u_n\}\) and \( \{v_n\}\) generated by
\[
\begin{align*}
  u_{n+1} &= pu_n - qu_{n-1} \\
  v_{n+1} &= pv_n - qv_{n-1},
\end{align*}
\]
where \(u_0 = 0, u_1 = 1, v_0 = 2, v_1 = p,\) with \(p\) and \(q\) real and nonzero. The Fibonacci sequence and
the Lucas sequence are special cases of \(\{u_n\}\) and \(\{v_n\}\), respectively. Define
\[
\begin{align*}
  r_n &= u_{n+d}/u_n, \\
  R_n &= v_{n+d}/v_n,
\end{align*}
\]
where \(d\) is a positive integer. McCabe and Phillips showed that for \(d = 1\), applying one step of Aitken acceleration
to any appropriate triple of elements of \(\{r_n\}\) yields another element of \(\{r_n\}\). They also proved for \(d = 1\)
that if a step of the Newton-Raphson method or the secant method is applied to elements of \(\{r_n\}\) in solving the characteristic
equation \(x^2 - px + q = 0,\) then the result is an element of \(\{r_n\}\).

The above results are obtained for \(d > 1\). It is shown that if any of
the above methods is applied to elements of \(\{R_n\}\), then the result is an element
of \(\{R_n\}\). The application of certain higher-order iterative procedures, such as
Halley's method, to elements of \(\{r_n\}\) and \(\{R_n\}\) is also investigated.

Fibonacci and Lucas numbers appear repeatedly in the works of the father of
computational number theory, D. H. Lehmer, who contributed also to numerical
analysis, notably [5]. To his memory is dedicated this extension of results of
solving nonlinear equations to ratios of generalized Fibonacci numbers.

1. Introduction

Let \(p\) and \(q\) be real and nonzero. Define the generalized Fibonacci sequence
\[
(1.1) \quad u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = pu_n - qu_{n-1}, \quad n \geq 1,
\]
and the generalized Lucas sequence
\[
(1.2) \quad v_0 = 2, \quad v_1 = p, \quad v_{n+1} = pv_n - qv_{n-1}, \quad n \geq 1.
\]
Let \(d\) be a natural number. If \(u_n \neq 0\), define the ratio
\[
(1.3) \quad r_n = u_{n+d}/u_n.
\]
If \(v_n \neq 0\), define the ratio
\[
(1.4) \quad R_n = v_{n+d}/v_n.
\]

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Related to the recurrence relation appearing in (1.1) and (1.2) is the characteristic equation

\[(1.5) \quad x^2 - px + q = 0.\]

If the equation has two real and unequal roots, then when \(d = 1\), the sequences of ratios \(\{r_n\}\) and \(\{R_n\}\) converge to the root of larger modulus. If there is a double root, then the sequences \(\{r_n\}\) and \(\{R_n\}\) converge to this root. McCabe and Phillips determined the condition for a generalized Fibonacci sequence to have no zero members; a necessary condition is that equation (1.5) have complex roots ([6, p. 554]). Their analysis can be adapted readily to generalized Lucas numbers, by Lemma 3 below.

If \(\alpha\) and \(\beta\) are the roots of (1.5), then they satisfy ([3, equation (1.4)])

\[(1.6) \quad \alpha + \beta = p, \quad \alpha\beta = q, \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 - 4q.\]

If \(\alpha = \beta\), then

\[(1.7) \quad 2\alpha = p, \quad \alpha^2 = q = (p/2)^2, \quad p^2 - 4q = 4\alpha^2 - 4\alpha^2 = 0.\]

**Lemma 1** ([3, equations (2.6), (2.7)]). If \(\alpha\) and \(\beta\) are the distinct roots of (1.5) and \(n \geq 0\), then

\[u_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad \text{and} \quad v_n = \alpha^n + \beta^n.\]

**Lemma 2.** If \(\alpha\) is the double root of (1.5) and \(n \geq 0\), then \(u_n = n(p/2)^{n-1}\) and \(v_n = 2(p/2)^n\).

If \(d > 1\), and the roots of (1.5) are real, then the sequences of ratios \(\{r_n = u_{n+d}/u_n\}\) and \(\{R_n = v_{n+d}/v_n\}\) will converge to the \(d\)th power of a root of (1.5). In other words, the sequences of ratios \(\{r_n\}\) and \(\{R_n\}\) converge to a root of

\[(1.8) \quad x^2 - (\alpha^d + \beta^d)x + (\alpha\beta)^d = x^2 - v_dx + q^d = 0,\]

by Lemmas 1 and 2 and (1.6) and (1.7).

Define the Aitken transformation by

\[(1.9) \quad A(x, x', x'') = (xx'' - x'^2)/(x - 2x' + x'').\]

Define the secant transformation \(S(x, x')\) for equation (1.8) by

\[(1.10) \quad S(x, x') = \frac{x(x'^2 - v_dx' + q^d) - x'(x^2 - v_dx + q^d)}{(x'^2 - v_dx' + q^d) - (x^2 - v_dx + q^d)} = \frac{xx' - q^d}{x + x' - v_d},\]

and the Newton-Raphson transformation \(N(x)\) for equation (1.8) by

\[(1.11) \quad N(x) = x - (x^2 - v_dx + q^d)/(2x - v_d) = (x^2 - q^d)/(2x - v_d).\]

McCabe and Phillips proved that, if \(d = 1\), then

(i) \(A(r_{n-t}, r_n, r_{n+t}) = r_2n\) if \(r_{2n} \neq 0\),
(ii) \(S(r_n, r_m) = r_{n+m}\) if \(r_{n+m} \neq 0\),
(iii) \(N(r_n) = r_{2n}\) if \(r_{2n} \neq 0\).

It is now possible to state the extensions. As long as division by zero is avoided, then

(i) \(A(r_{n-t}, r_n, r_{n+t}) = r_{2n},\quad A(R_{n-t}, R_n, R_{n+t}) = r_{2n}\),
(ii) \(S(r_n, r_m) = r_{n+m},\quad S(R_n, R_m) = r_{n+m}\),
(iii) \(N(r_n) = r_{2n},\quad N(R_n) = r_{2n}\),
for any natural number $d$. The idea of considering $d > 1$ is due to Jamieson [4], who applied it only to the ordinary Fibonacci sequence.

The other extension is to apply the Halley transformation $H(x)$, which is a third-order refinement of the Newton-Raphson transformation:

$$H(r_n) = r_{3n}, \quad H(R_n) = R_{3n}.$$  

Note that in the latter case the image is a ratio of generalized Lucas numbers. The Newton-Raphson and Halley transformations are two members of a certain infinite family of transformations; proofs applicable to the infinite family will be given.

Applying any of these transformations to elements of the sequence $\{R_n\}$, where (1.5) has a double root $\alpha$, gives rise to division by zero. In this situation $R_n = (p/2)^d = \alpha^d$ for every $n \geq 1$; i.e., $R_n$ is the root of (1.8), by Lemma 2 and (1.7). In this case the ratios are constant, so the sequence is trivial. In the sequel the transformations will be applied to $R_n$ under the assumption that (1.5) has distinct roots.

Section 2 contains a list of elementary relationships about generalized Fibonacci and Lucas numbers. In §3 the Aitken transformation is studied. Section 4 is devoted to the secant transformation. Section 5 begins with the presentation of the Halley transformation. Then an infinite family of transformations, which includes those of Newton-Raphson and Halley, is investigated.

2. Properties of generalized Fibonacci and Lucas numbers

For $n > 0$ define $v_n = \alpha^{-n} + \beta^{-n}$. Then by (1.6) and Lemma 1,

$$q^n v_n = (\alpha \beta)^n v_n = \beta^n + \alpha^n = v_n.$$  

Similarly, if equation (1.5) has distinct roots, define $u_n = (\alpha^{-n} - \beta^{-n})/\alpha - \beta)$. Then by (1.6) and Lemma 1 ([3, equation (2.17)])

$$q^n u_n = (\alpha \beta)^n u_n = (\beta^n - \alpha^n)/(\alpha - \beta) = -u_n.$$  

Formula (2.2) is applicable also if equation (1.5) has a double root, for if $u_{-n}$ is defined by $-n(p/2)^{-n-1}$, then $q^n u_n = -n(p/2)^{-n-1}(p/2)2^n = -n(p/2)^{n-1} = -u_n$.

It is easy to verify that the recurrence relations in (1.1) and (1.2) are valid also for negative subscripts.

Lemma 3 ([3, equation (4.10)]). If $n$ is an integer, then $u_{2n} = u_n v_n$.

Lemma 4. If $n$, $m$, and $e$ are integers, then

(a) $u_{n+e} u_{n-e} - u_n^2 = -q^{n-e} u_e^2$,

(b) $u_{n+e} u_{m} - u_n u_{n+m+e} = -q^m u_e u_{n-m}$,

(c) $u_{n+e} u_{m+e} - q^e u_n u_m = u_e u_{n+m+e}$,

(d) $u_{n+e} - q^e u_{n-e} = v_n u_e$,

(e) $u_{n+e} - v_n u_n = -q^e u_{n-e}$.

On the right side of statements (a)–(d) of the following lemma, there appears the factor $p^2 - 4q$. If (1.5) has a double root, then $p^2 - 4q = 0$, by (1.7). It suffices to show in the case of a double root, accordingly, that the left side of each of these statements vanishes.
Lemma 5. If \( n, m, \) and \( e \) are integers, then

(a) \( v_{n+e}v_{n-e} - v_n^2 = q^{n-e}(p^2 - 4q)u_e^2, \)
(b) \( v_{n+e}v_m - v_nv_{m+e} = q^m(p^2 - 4q)u_eu_{n-m}, \)
(c) \( v_{n+e}v_{m+e} - q^e v_nv_m = (p^2 - 4q)u_eu_{n+m+e}, \)
(d) \( v_{n+e} - q^e v_{n-e} = (p^2 - 4q)u_nu_e, \)
(e) \( v_{n+e} - v_nv_n = -q^e v_{n-e}. \)

Lemma 6. If \( n, m, \) and \( e \) are integers, then

\[ u_{n+e}v_m - u_nv_{m+e} = q^m u_e v_{n-m}. \]

Lemma 7 ([3, equation (4.13)]). If \( n \) is an integer, then

\[ u_n(v_n^2 - q^n) = u_{3n}. \]

3. The Aitken Transformation

Theorem 1. Let \( n > t \geq 0 \) be integers, and assume that division by zero does not occur. Then (A) \( A(r_{n-t}, r_n, r_{n+t}) = r_{2n} \); (B) if equation (1.5) has distinct roots, then \( A(R_{n-t}, R_n, R_{n+t}) = r_{2n} \).

Proof. We prove only part (A). The proof of part (B) is similar. By (1.3) and (1.9),

\[
A(r_{n-t}, r_n, r_{n+t}) = \frac{r_{n-t}r_{n+t} - r_n^2}{r_{n-t} - 2r_n + r_{n+t}}
= \frac{(u_{n-t+d}/u_{n-t})(u_{n+t+d}/u_{n+t}) - (u_{n+d}/u_n)^2}{u_{n-t+d}/u_{n-t} - 2u_{n+d}/u_n + u_{n+t+d}/u_{n+t}}
= \frac{u_{n-t+d}u_{n+t+d}u_n^2 - u_{n-t}u_{n+t}u_{n+t+d}}{u_n[u_{n+t+d}u_{n+t}u_{n+t+d} - 2u_{n+d}u_{n-t}u_{n+t} + u_{n+t+d}u_{n-t}u_{n+t+d}]} - q^{n-t+d}u_n^2u_{n+t+d} + q^{n-t}u_{n+t}^2u_n^2
\]

by Lemmas 4(a) and 4(b),

\[
= \frac{u_t(u_{n+d}^2 - q^d u_{n-t}^2)}{u_nu_d(u_{n+t} - q^t u_{n-t})} = \frac{u_tu_du_{n+d}}{u_nu_d v_n u_t},
\]

by Lemmas 4(c) and 4(d),

\[ = u_{2n+d}/u_{2n} = r_{2n}, \]

by Lemma 3 and then (1.3). \( \square \)

4. The secant transformation

Theorem 2. Let \( n \) and \( m \) be positive integers, and assume that division by zero does not occur. Then (A) \( S(r_n, r_m) = r_{n+m} \); (B) if equation (1.5) has distinct roots, then \( S(R_n, R_m) = r_{n+m} \).

Proof. We prove only part (B). The proof of part (A) is similar. By (1.4) and (1.10),

\[
S_d(R_n, R_m) = \frac{R_nR_m - q^d}{R_n + R_m - v_d} = \frac{(v_{n+d}/v_n)(v_{m+d}/v_m) - q^d}{v_{n+d}/v_n + v_{m+d}/v_m - v_d}
\]
\[ \frac{v_{n+d}v_{m+d} - q^d v_n v_m}{v_{n+d}v_m + v_n(v_{m+d} - v_d v_m)} = \frac{(p^2 - 4q)u_d u_{n+m+d}}{v_{n+d}v_m - q^d v_n v_{m-d}} , \]

by Lemmas 5(c) and 5(e),

\[ \frac{(p^2 - 4q)u_d u_{n+m+d}}{(p^2 - 4q)u_d u_{n+m}} = \frac{u_{n+m+d}}{u_{n+m}} = r_{n+m} , \]

by Lemma 5(c) and then (1.3). □

5. THE NEWTON-RAPHSON AND HALLEY TRANSFORMATIONS

The Halley transformation for the equation \( f(x) = 0 \) is given by ([1, p. 131])

\[ H(x) = x - f(x)/[f'(x) - f(x)f''(x)/2f'(x)] . \]

Applying the Halley transformation to equation (1.8) yields

\[ H(x) = x - \frac{x^2 - v_d x + q^d}{(2x - v_d) - (x^2 - v_d x + q^d)/(2x - v_d)} \]

\[ = \frac{x^3 - 3q^d x + v_d q^d}{3x^2 - 3v_d x + v_d^2 - q^d} . \]

An infinite family of transformations, which includes those of Newton-Raphson and Halley, will now be investigated. To this end, define the homogeneous polynomials in \( y \) and \( z \) by

\[ u_d^h q^{-f} T_{h,f,d}(y, z) = -\sum_{k=0}^{h} \binom{h}{k} (-y)^k z^{h-k} u_{dk-f} . \]

Lemma 8. For \( i = 0, 1, 2, ..., h \) define

\[ E(i) = u_d^i q^{-it} \sum_{k=0}^{h-i} \binom{h-i}{k} (-u_i)^k u_{t+d}^{h-i-k} u_{dk-f-it} . \]

Then \( E(i) \) is independent of \( i \).

Proof. It suffices to show that if \( 0 \leq i \leq h - 1 \), then \( E(i) = E(i + 1) \). By definition, \( \binom{j}{k} = 0 \) if \( k < 0 \) or \( k > j \). Thus

\[ E(i) = u_d^i q^{-it} \sum_{k=0}^{h-i} \left[ \binom{h-i-1}{k} + \binom{h-i-1}{k-1} \right] (-u_i)^k u_{t+d}^{h-i-k} u_{dk-f-it} \]

\[ = u_d^i q^{-it} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-u_i)^k u_{t+d}^{h-i-k} u_{dk-f-it} \]

\[ + u_d^i q^{-it} \sum_{j=0}^{h-i-1} \binom{h-i-1}{j} (-u_i)^{j+1} u_{t+d}^{h-i-j-1} u_{d_j+d-f-it} \]

\[ = u_d^i q^{-it} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-u_i)^k u_{t+d}^{h-i-k-1} (u_{t+d} u_{dk-f-it} - u_i u_{dk+d-f-it}) \]

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\[ u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-u_t)^k u_{t+d}^{h-i-k} u_{d_{k-f-(i+1)t}}, \]

by Lemma 4(b),

\[ = E(i+1). \]

**Theorem 3.** If \( u_d \neq 0 \), then \( T_{h,f,d}(u_t, u_{t+d}) = u_{h+f} \).

**Proof.** By Lemma 8,

\[ u_d^h q^f T_{h,f,d}(u_t, u_{t+d}) = -E(0) = -E(h) = -u_d^h q^h u_{-h-f}. \]

By (2.2),

\[ T_{h,f,d}(u_t, u_{t+d}) = u_{h+f}. \]

**Lemma 9.** For \( 0 < i < h \), \( i \) even, define

\[ F(i) = u_d^i q^{it} \sum_{k=0}^{h-i} \binom{h-i}{k} (-v_t)^k u_{t+d}^{h-i-k} u_{d_k-f-it}. \]

For \( 0 < s < h \), \( s \) odd, define

\[ G(s) = -u_d^s q^{st} \sum_{k=0}^{h-s} \binom{h-s}{k} (-v_t)^k u_{t+d}^{h-s-k} u_{d_k-f-st}. \]

Then \( F(i) = G(i+1) \) if \( i < h \), and \( G(i+1) = (p^2 - 4q) F(i+2) \) if \( i < h - 1 \).

**Proof.** We have

\[ F(i) = u_d^i q^{it} \sum_{k=0}^{h-i} \left[ \binom{h-i-1}{k} + \binom{h-i-1}{k-1} \right] (-v_t)^k u_{t+d}^{h-i-k} u_{d_k-f-it} \]

\[ = u_d^i q^{it} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-v_t)^k u_{t+d}^{h-i-k} \left( u_{t+d} u_{d_k-f-it} - v_t u_{d_k+d-f-it} \right) \]

\[ = -u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-1} \binom{h-i-1}{k} (-v_t)^k u_{t+d}^{h-i-k} u_{d_k-f-(i+1)t}, \]

by Lemma 6,

\[ = G(i+1). \]

Continuing,

\[ G(i+1) = -u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-2} \left[ \binom{h-i-2}{k} + \binom{h-i-2}{k-1} \right] (-v_t)^k u_{t+d}^{h-i-k} u_{d_k-f-(i+1)t} \]

\[ = -u_d^{i+1} q^{(i+1)t} \sum_{k=0}^{h-i-2} \binom{h-i-2}{k} (-v_t)^k u_{t+d}^{h-i-k} \left( u_{t+d} u_{d_k-f-(i+1)t} - v_t u_{d_k+d-f-(i+1)t} \right) \]

\[ = u_d^{i+2} q^{(i+2)t} (p^2 - 4q) \sum_{k=0}^{h-i-2} \binom{h-i-2}{k} (-v_t)^k u_{t+d}^{h-i-k} u_{d_k-f-(i+2)t}. \]
by Lemma 5(b),

\[ \frac{1}{(p^2 - 4q)F(i + 2)}. \]

**Theorem 4.** Assume \( u_d \neq 0 \). If \( h \) is even, then

\[ T_{h,f,d}(v_t, v_{t+d}) = (p^2 - 4q)^{h/2}u_{ht+f}. \]

If \( h \) is odd, then

\[ T_{h,f,d}(v_t, v_{t+d}) = (p^2 - 4q)^{(h-1)/2}v_{ht+f}. \]

**Proof.** Apply Lemma 9 \([h/2]\) times:

If \( h \) is even, then

\[ u_d^h q^{-f} T_{h,f,d}(v_t, v_{t+d}) = -F(0) = -(p^2 - 4q)F(2) = -(p^2 - 4q)^2 F(4) \]

\[ = \ldots = -(p^2 - 4q)^{h/2} F(h) = -u_d^h q^{ht} (p^2 - 4q)^{h/2} u_{-ht-f}. \]

By (2.2), \( T_{h,f,d}(v_t, v_{t+d}) = (p^2 - 4q)^{h/2}u_{ht+f}. \)

If \( h \) is odd, then

\[ u_d^h q^{-f} T_{h,f,d}(v_t, v_{t+d}) = -F(0) = -(p^2 - 4q)F(2) \]

\[ = \ldots = -(p^2 - 4q)^{(h-1)/2} F(h-1) \]

\[ = -(p^2 - 4q)^{(h-1)/2} G(h) = (p^2 - 4q)^{(h-1)/2} u_d^h q^{ht} u_{-ht-f}. \]

By (2.1), \( T_{h,f,d}(v_t, v_{t+d}) = (p^2 - 4q)^{(h-1)/2}v_{ht+f}. \)

Define

\[ g_h(z/y) = \frac{-q^d \sum_{k=0}^{h} \binom{h}{k} \left( \frac{z}{-y} \right)^{h-k} u_d(k-1)}{-\sum_{k=0}^{h} \binom{h}{k} \left( \frac{z}{-y} \right)^{h-k} u_d(k)}. \]

Multiply the numerator and the denominator of the fraction by \( u_d^h (-y)^h \):

\[ g_h(z/y) = \frac{-u_d^h q^d \sum_{k=0}^{h} \binom{h}{k} (-y)^k z^{h-k} u_d(k-1)}{-u_d^h \sum_{k=0}^{h} \binom{h}{k} (-y)^k z^{h-k} u_d(k)}. \]

The immediate consequences of Theorems 3 and 4 are:

**Theorem 5.** (a) Assume that \( u_d \neq 0 \) and \( u_{ht} \neq 0 \). Then \( g_h(u_{t+d}/u_t) = u_{ht+d}/u_{ht} \).

(b) Assume that \( u_d \neq 0 \), \( v_t \neq 0 \), and \( v_{ht} \neq 0 \). Then

\[ g_h(v_{t+d}/v_t) = \begin{cases} u_{ht+d}/u_{ht}, & \text{h even}, \\ v_{ht+d}/v_{ht}, & \text{h odd}. \end{cases} \]

**Theorem 6.** If \( n \) is a positive integer, and division by zero does not occur, then \( N(r_n) = N(R_n) = r_{2n} \).

**Proof.** In view of Theorem 5, it suffices to show that \( g_2(z/y) = N(z/y) \), where \( N(x) \) is given by equation (1.11). By (5.3),

\[ g_2(z/y) = \frac{-q^d (z^2 u_d + y^2 u_d)}{-(2yz u_d + y^2 u_d)} = \frac{z^2 u_d - q^d y^2 u_d}{2yz u_d - y^2 u_d v_d}. \]
by (2.2) and Lemma 3,
\[ \frac{(z/y)^2 - q^d}{2z/y - v_d} = N(z/y). \]

**Theorem 7.** If \( n \) is a positive integer, and division by zero does not occur, then \( H(r_n) = r_3^n \) and \( H(R_n) = R_3^n \).

**Proof.** In view of Theorem 5, it suffices to show that \( g_3(z/y) = H(z/y) \), where \( H(x) \) is given by equation (5.1). By (5.3),
\[
g_3(z/y) = -q^d(z^3u_d + 3y^2zu_d - y^3u_2d) \\
= (-3yz^2u_d + 3y^2zu_2d - y^3u_3d) \\
= \frac{z^3u_d - 3y^2zq^du_d + y^3q^dudv_d}{3yz^2u_d - 3y^2zu_dv_d + y^3u_d(v_d^2 - q^d)},
\]
by (2.2), Lemma 3, and Lemma 7,
\[
= \frac{(z/y)^3 - 3q^d(z/y) + q^dvd}{3(z/y)^2 - 3(z/y)vd + v_d^2 - q^d} = H(z/y). \]

**Remark.** Theorem 3, with \( f = 0 \) and \( d = 1 \), resembles a formula given by H. Siebeck, cited in [2, p. 394].

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**Bibliography**