FINITE ELEMENT APPROXIMATION OF THE $p$-LAPLACIAN

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Abstract. In this paper we consider the continuous piecewise linear finite element approximation of the following problem: Given $p \in (1, \infty)$, $f$, and $g$, find $u$ such that

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \quad u = g \quad \text{on} \quad \partial \Omega.$$ 

The finite element approximation is defined over $\Omega^h$, a union of regular triangles, yielding a polygonal approximation to $\Omega$. For sufficiently regular solutions $u$, achievable for a subclass of data $f$, $g$, and $\Omega$, we prove optimal error bounds for this approximation in the norm $W^{1,q}(\Omega^h)$, $q = p$ for $p < 2$ and $q \in [1, 2]$ for $p > 2$, under the additional assumption that $\Omega^h \subseteq \Omega$. Numerical results demonstrating these bounds are also presented.

1. INTRODUCTION

Let $\Omega$ be a bounded open set in $\mathbb{R}^2$ with a Lipschitz boundary $\partial \Omega$. Given $p \in (1, \infty)$, $f \in L^2(\Omega)$, and $g \in W^{1-1/p, p}(\partial \Omega)$, we consider the following problem:

$$(\mathcal{P}) \quad \text{Find } u \in W^{1,p}_g(\Omega) \equiv \{v \in W^{1,p}(\Omega) : v = g \text{ on } \partial \Omega\} \text{ such that }$$

$$\int_{\Omega} |\nabla u|^{p-2} (\nabla u, \nabla v)_{\mathbb{R}^2} d\Omega = \int_{\Omega} f v d\Omega \quad \forall v \in W^{1,p}_0(\Omega),$$

where $|v|^2 = (v, v)_{\mathbb{R}^2}$. Throughout we adopt the standard notation $W^{m,q}(D)$ for Sobolev spaces on $D$ with norm $\| \cdot \|_{W^{m,q}(D)}$ and seminorm $| \cdot |_{W^{m,q}(D)}$. We note that the seminorm $| \cdot |_{W^{1,q}(D)}$ and the norm $\| \cdot \|_{W^{1,q}(D)}$ are equivalent on $W^{1,q}_0(D)$.

Problem $(\mathcal{P})$ above is the weak formulation of the Dirichlet problem for the $p$-Laplacian

$$(1.1) \quad -\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega.$$ 

The well-posedness of $(\mathcal{P})$ is well established, and one can refer to, for example, Glowinski and Marrocco [5] or the account in Ciarlet [4]. Of course, one can study more general boundary conditions and the presence of lower-order terms in the differential operator. However, for ease of exposition, we just consider $(\mathcal{P})$, although most of our results can be adapted to more general
problems. From Glowinski and Marrocco [5], or Ciarlet [4], (3°) is equivalent to the following minimization problem:

(3°) Find \( u \in W^{1,p}_g(\Omega) \) such that

\[
J_\Omega(u) \leq J_\Omega(v) \quad \forall v \in W^{1,p}_g(\Omega),
\]

where

\[
J_\Omega(v) \equiv \frac{1}{p} \int_\Omega |\nabla v|^p \, d\Omega - \int_\Omega f v \, d\Omega.
\]

It is easily established that \( J_\Omega(\cdot) \) is strictly convex and continuous on \( W^{1,p}_g(\Omega) \). Further, \( J_\Omega(\cdot) \) is Gateaux differentiable with

\[
J_\Omega'(u)(v) = \frac{1}{p} \int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_\Omega f v \, d\Omega \quad \forall v \in W^{1,p}_0(\Omega).
\]

Hence, there exists a unique solution to (3°), and (3°) is equivalent to (â°), its Euler equation. In addition, we have that

\[
\|u\|_{W^{1,p}(\Omega)} < C\left[\|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\partial\Omega)}\right].
\]

The problem (3°) occurs in many mathematical models of physical processes: nonlinear diffusion and filtration, see Philip [8]; power-law materials, see Atkinson and Champion [1]; and quasi-Newtonian flows, see Atkinson and Jones [2], for example.

It is the purpose of this paper to analyze the finite element approximation of (3°). Let \( \Omega^h \) be a polygonal approximation to \( \Omega \) defined by \( \Omega^h \equiv \bigcup_{t \in T^h} \overline{\tau} \), where \( T^h \) is a partitioning of \( \Omega^h \) into a finite number of disjoint open regular triangles \( \tau \), each of maximum diameter bounded above by \( h \). In addition, for any two distinct triangles, their closures are either disjoint, or have a common vertex, or a common side. Let \( \{P_j\}_{j=1} \) be the vertices associated with the triangulation \( T^h \), where \( P_j \) has coordinates \((x_j, y_j)\). Throughout we assume that \( P_j \in \partial\Omega^h \) implies \( P_j \in \partial\Omega \), and that \( \Omega^h \subseteq \Omega \). We note that, owing to the elliptic degeneracy of the \( p \)-Laplacian and the limited regularity of the solution \( u \), see below, it is not a simple matter to extend the results in this paper to the case \( \Omega^h \not\subseteq \Omega \). Associated with \( T^h \) is the finite-dimensional space

\[
S^h = \{\chi \in C(\Omega^h) : \chi|_\tau \text{ is linear } \forall \tau \in T^h\} \subset W^{1,p}(\Omega^h).
\]

Let \( \pi_h : C(\Omega^h) \to S^h \) denote the interpolation operator such that for any \( v \in C(\Omega^h) \), the interpolant \( \pi_h v \in S^h \) satisfies \( \pi_h v(P_j) = v(P_j) \), \( j = 1, \ldots, J \). We recall the following standard approximation results. For \( m = 0 \) or 1, and for all \( \tau \in T^h \), we have (a) for \( q \in [1, \infty) \), \( s \in [1, \infty) \), provided \( W^{2,s}(\tau) \hookrightarrow W^{m,q}(\tau) \),

\[
|v - \pi_h v|_{W^{m,q}(\tau)} \leq C h^{2(1/q - 1/s)} h^{2-m} |v|_{W^{2,s}(\tau)} \quad \forall v \in W^{2,s}(\tau);
\]

and (b) for \( q > 2 \),

\[
|v - \pi_h v|_{W^{m,q}(\tau)} \leq C h^{1-m} |v|_{W^{1,q}(\tau)} \quad \forall v \in W^{1,q}(\tau).
\]

In (1.6a) we have noted the imbedding \( W^{2,1}(\tau) \hookrightarrow C(\overline{\tau}) \); see, for example, p. 300 in Kufner et al. [6].

The finite element approximation of (3°) that we wish to consider is:
(\mathcal{P}^h) \text{ Find } u^h \in S_g^h \text{ such that }

\begin{align*}
(1.7a) & \quad \int_{\Omega^h} |\nabla u^h|^p - 2 (\nabla u^h, \nabla v^h)_{\mathbb{R}^2} \, d\Omega^h = \int_{\Omega^h} f v^h \, d\Omega^h \quad \forall v^h \in S_0^h, \\
(1.7b) & \quad S_0^h \equiv \{ \chi \in S^h : \chi = 0 \text{ on } \partial\Omega^h \}
\end{align*}

where

\begin{align*}
(1.7c) & \quad S_g^h \equiv \{ \chi \in S^h : \chi = g^h \text{ on } \partial\Omega^h \},
\end{align*}

where \( g^h \in S^h \) is chosen to approximate the Dirichlet boundary data. If \( p > 2 \), then \( u \in W^{1,p}(\Omega) \) implies \( u \in C(\overline{\Omega}) \), and so we set \( g^h \equiv \pi_h u \). For the explicit error bounds derived in §3 for \( p < 2 \), we assume that \( u \in C(\overline{\Omega}) \), and so once again set \( g^h \equiv \pi_h u \). However, for the abstract analysis of this and the next section, \( g^h \) can be arbitrary. The corresponding minimization problem is:

(\mathcal{E}^h) \text{ Find } u^h \in S_g^h \text{ such that }

\begin{align*}
(1.8) & \quad J_{\Omega^h}(u^h) \leq J_{\Omega^h}(v^h) \quad \forall v^h \in S_g^h.
\end{align*}

The well-posedness of \((\mathcal{P}^h) \equiv (\mathcal{E}^h)\) follows in an analogous way to that of \((\mathcal{P})\) and \((\mathcal{E})\), see Glowinski and Marrocco [5] or Ciarlet [4], and

\begin{align*}
(1.9) & \quad \|u^h\|_{L^p(\Omega^h)} \leq C[\|f\|_{L^p(\Omega^h)} + \|g^h\|_{L^p(\Omega^h)}].
\end{align*}

We note that for \( p = 2 \), problem \((\mathcal{P})\) reduces to the weak formulation of the linear Laplacian, and hence the regularity of \( u \) and the finite element error analysis are well established in this case. For \( p \neq 2 \), the regularity of \( u \) is less well established, as \((1.1)\) is then a degenerate quasi-linear elliptic problem. It is well known, see Example 3.1 in §3, that \( u \) has limited regularity for infinitely smooth data \( f, g, \text{ and } \Omega \). Therefore, there is no benefit in considering higher-order finite element approximations, and hence our restriction to continuous piecewise linear elements from the outset. Lieberman [7] has proved that if \( \partial\Omega \in C^{1,\beta} \), then \( g \) is the trace of a function \( \in C^{1,\gamma}(\Omega) \) for \( \beta, \gamma \in (0, 1) \), and if \( f \in L^\infty(\Omega) \), then \( u \in C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \). However, for explicit finite element error bounds one requires global regularity results on the second, or maybe higher, derivatives of \( u \). Unfortunately, such results are not available at present in the literature, but it is an active area of research worldwide.

The following error bounds were proved in Glowinski and Marrocco [5] for the case \( \Omega^h \equiv \Omega \) and \( g \equiv 0 \) :

If \( u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \), then

\begin{align*}
(1.10) & \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq \left\{ \begin{array}{ll}
Ch^{1/(3-p)} & \text{if } p \leq 2, \\
Ch^{1/(p-1)} & \text{if } p \geq 2,
\end{array} \right.
\end{align*}

where throughout this paper \( C \) denotes a generic positive constant independent of \( h \). Chow [3], employing an approach of Tyukhtin [9], improved these error bounds. He proved that

\begin{align*}
(1.11a) & \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq C\|u - v^h\|_{W^{1,p}(\Omega)}^{p/2} \quad \forall v^h \in S_0^h \quad \text{if } p \leq 2, \\
\end{align*}

and

\begin{align*}
(1.11b) & \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq C(\|\pi_h u\|_{W^{1,p}(\Omega)})^{2/p} \quad \|u - \pi_h u\|_{W^{1,p}(\Omega)}^{2/p} \quad \text{if } p > 2,
\end{align*}
and hence, if \( u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega) \), it follows form (1.6a) that

\[
||u - u^h||_{W^{1,p}(\Omega)} \leq \begin{cases} 
Ch^{p/2} & \text{if } p \leq 2, \\
Ch^{2/p} & \text{if } p > 2.
\end{cases}
\]

It is the purpose of this paper to prove optimal error bounds. The layout is as follows. In the next section we prove an abstract error bound for the approximation \((\mathcal{P}^h)\) of \((\mathcal{P})\). In §3 we study the case \( p \in (1, 2) \) and prove an optimal \( W^{1,p} \) error bound, that is, \( O(h) \), provided that \( u \in W^{3,1}(\Omega) \cap C^{2,\frac{2-p}{p}}(\overline{\Omega}) \). Thus, this optimal error bound requires a stronger regularity assumption on \( u \) than that for the bound (1.12) in the case \( p < 2 \). In §4 we study the case \( p > 2 \) and first show that the bound (1.12) for \( p > 2 \) can be achieved under the weaker regularity requirement \( u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \). Second, under the additional assumption \(|f| \lesssim p > 0 \) a.e. in \( \Omega \), we prove an optimal \( W^{1,4/3} \) error bound. We note that the above regularity requirements on \( u \) for these optimal error bounds are achievable for a subclass of data \( f, g \), and \( \Omega \). In §5 we show that the error bounds derived in the previous sections hold for the fully practical scheme of employing numerical integration on the right-hand side of (1.7) if \( f \) is sufficiently smooth. Finally, we report on some numerical examples, which confirm these optimal error bounds.

2. AN ABSTRACT ERROR BOUND

We first prove a lemma, which is a generalization of Lemmas 5.1, 5.2, 5.3 and 5.4 in Glowinski and Marrocco [5].

**Lemma 2.1.** For all \( p > 1 \) and \( \delta \geq 0 \) there exist positive constants \( C_1 \) and \( C_2 \) such that for all \( \xi, \eta \in \mathbb{R}^2, \xi \neq \eta \),

\[
(2.1a) \quad ||\xi|^{p-2}\xi - |\eta|^{p-2}\eta|| \leq C_1|\xi - \eta|^{1-\delta}(|\xi| + |\eta|)^{p-2+\delta}
\]

and

\[
(2.1b) \quad (||\xi|^{p-2}\xi - |\eta|^{p-2}\eta|, \xi - \eta)|_{\mathbb{R}^2} \geq C_2|\xi - \eta|^{2+\delta}(|\xi| + |\eta|)^{p-2-\delta}.
\]

**Proof.** The approach is similar to that in Glowinski and Marrocco [5]. For all \( \xi, \eta \in \mathbb{R}^2, \xi \neq \eta \), let

\[
(2.2) \quad G_1(\xi, \eta) = ||\xi|^{p-2}\xi - |\eta|^{p-2}\eta||/[|\xi - \eta|^{1-\delta}(|\xi| + |\eta|)^{p-2+\delta}].
\]

We wish to prove that \( G_1 \) is bounded above. For any \( \varepsilon > 0 \), \( G_1 \) is continuous on

\[
D_\varepsilon \equiv \{(\xi, \eta): |\xi - \eta| \geq \varepsilon \text{ and } (|\xi| + |\eta|) \leq 1/\varepsilon\}.
\]

In addition, we note that for all \( \xi, \eta \in \mathbb{R}^2, \xi \neq \eta \),

\[
(2.3) \quad G_1(\xi, \eta) \equiv G_1(\eta, \xi), \quad G_1(\lambda\xi, \lambda\eta) \equiv G_1(\xi, \eta) \quad \text{for all } \lambda \in \mathbb{R}^+, \quad G_1(0, \eta) = 1 \quad \text{and} \quad G_1(A\xi, A\eta) \equiv G_1(\xi, \eta) \quad \text{if } A^TA = I,
\]

i.e., \( A \) is a rotation matrix. Therefore, without loss of generality we can take \( \xi = e_1 \equiv (1, 0) \). Since

\[
(2.4) \quad G_1(e_1, \eta) \to 1 \quad \text{as } |\eta| \to \infty,
\]

it remains to show that \( \limsup G_1(e_1, \eta) < \infty \) as \( |e_1 - \eta| \to 0 \).
Let \( \eta = (1 + \rho \cos \theta, \rho \sin \theta) \). Then a simple calculation yields that
\[
\lim_{\rho \to 0} G_1(e_1, \eta) = \begin{cases} 
0 & \text{if } \delta > 0, \\
2^{2-p}(1 + (p-2)\cos^2 \theta)^{1/2} & \text{if } \delta = 0.
\end{cases}
\]
Hence the desired result (2.1a).

Similarly, we prove (2.1b). Let
\[
G_2(\xi, \eta) \equiv |\xi - \eta|^{2+\delta}(|\xi| + |\eta|)^{p-2-\delta}/(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2}.
\]
From Glowinski and Marrocco [5] we have that
\[
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} > 0 \quad \text{if } \xi \neq \eta.
\]
Therefore, we only need to prove that \( G_2 \) is bounded above. In addition, the results (2.3) and (2.4) hold for \( G_2 \).

Setting \( \eta = (1 + \rho \cos \theta, \rho \sin \theta) \), a simple calculation yields that
\[
\lim_{\rho \to 0} G_2(e_1, \eta) = \begin{cases} 
0 & \text{if } \delta > 0, \\
2^{p-2}(1 + (p-2)\cos^2 \theta)^{-1} & \text{if } \delta = 0.
\end{cases}
\]
Hence the desired result (2.1b). \( \square \)

The inequality (2.1a) was proved in Glowinski and Marrocco [5] for \( p \in (1, 2] \) with \( \delta = 2-p \), and for \( p \geq 2 \) with \( \delta = 0 \); similarly, (2.1b) was proved for \( p \in (1, 2] \) with \( \delta = 0 \), and for \( p \geq 2 \) with \( \delta = p-2 \).

For \( p \in (1, \infty) \) and \( \sigma \geq 0 \) we define for any \( v \in W^{1,p}(\Omega^h) \)
\[
|v|_{(p, \sigma)} \equiv \int_{\Omega^h} (|\nabla u| + |\nabla v|)^{p-\sigma}|\nabla v|^\sigma \, d\Omega^h,
\]
where \( u \) is the solution of \((\mathcal{P})\). We prove the following results for later use.

**Lemma 2.2.** For \( p \in (1, \sigma) \) we have
\[
|v|_{(p, \sigma)} \leq |v|_{W^{1,p}(\Omega^h)} \leq C[|u|_{W^{1,p}(\Omega^h)} + |v|_{W^{1,p}(\Omega^h)}]^{p-\sigma}|v|_{(p, \sigma)},
\]
and for \( p \in [\sigma, \infty) \),
\[
|v|_{W^{1,p}(\Omega^h)} \leq |v|_{(p, \sigma)} \leq C[|u|_{W^{1,p}(\Omega^h)} + |v|_{W^{1,p}(\Omega^h)}]^{p-\sigma}|v|_{W^{1,p}(\Omega^h)}.
\]
Hence, (2.9) is well defined for \( v \in W^{1,p}(\Omega^h) \).

**Proof.** Setting \( w \equiv (|\nabla u| + |\nabla v|)^{p-\sigma}, \) we first consider the case \( p \in (1, \sigma) \). The left inequality in (2.10a) follows immediately from noting that \( w \leq |\nabla v|^{p-\sigma} \).

Applying Hölder’s inequality, we have
\[
|v|_{W^{1,p}(\Omega^h)} \equiv \left\{ \int_{\Omega^h} w^{-p/\sigma}[w^{1/\sigma}|\nabla v|]^{p} \, d\Omega^h \right\}^{\sigma/p}
\leq \left\{ \int_{\Omega^h} w^{-p/\sigma(\sigma-p)} \, d\Omega^h \right\}^{(\sigma-p)/p} \left| \int_{\Omega^h} |\nabla v| \, d\Omega^h \right|^{\sigma-p/p} \left| v \right|_{(p, \sigma)}.
\]
The right inequality in (2.10a) follows by noting that for all \( \gamma \in [0, \infty) \) there exists \( C_\gamma > 0 \) such that \( |a + b|^\gamma \leq C_\gamma(|a|^\gamma + |b|^\gamma) \) for all \( a, b \in \mathbb{R} \).

The inequalities (2.10b) can be proved in a similar manner. \( \square \)

The next theorem is the natural generalization of the result in §7 of Chow [3]. We use the minimization property of \( u^h \) and Lemma 2.1, whereas Chow uses the Glowinski and Marrocco version of Lemma 2.1.
Theorem 2.1. Let $u$ and $u^h$ be the unique solutions of $(\mathcal{P}) \equiv (\mathcal{G})$ and $(\mathcal{P}^h) \equiv (\mathcal{G}^h)$, respectively. Then for any $\delta_1 \in [0, 2)$ and $\delta_2 \geq 0$, and any $v^h \in S_g^h$, it follows that

$$
(2.11) \quad |u - u^h|_{(\rho, 2+\delta_2)} \leq C|u - v^h|_{(\rho, 2-\delta_1)}.
$$

Proof. We have for any $v^h \in S_g^h$ that

$$
J_{\Omega^h}(v^h) - J_{\Omega^h}(u) = \int_0^1 J'_{\Omega^h}(u + s(v^h - u))(v^h - u) \, ds
$$

$$
= \int_0^1 \left[ J'_{\Omega^h}(u + s(v^h - u))(u + s(v^h - u)) - J'_{\Omega^h}(u)(u + s(v^h - u)) \right] \frac{ds}{s}
$$

$$
+ J'_{\Omega^h}(u)(v^h - u)
$$

$$
\equiv A(v^h) + J'_{\Omega^h}(u)(v^h - u),
$$

where from (1.3)

$$
(2.12b) \quad A(v^h) = \int_0^1 \left[ \int_{\Omega^h} \left\{ |\nabla(u + s(v^h - u))|^{p-2} \nabla(u + s(v^h - u)) - |\nabla u|^{p-2} \nabla u \right\} d\Omega^h \right] \, ds.
$$

From (2.12b) and (2.1a) we have that

$$
|A(v^h)| \leq C_1 \int_0^1 s^{1-\delta_1} \int_{\Omega^h} \left( |\nabla(u + s(v^h - u))| + |\nabla u| \right)^{p-2+\delta_1}
$$

$$
\times |\nabla(v^h - u)|^{2-\delta_1} \, d\Omega^h \, ds
$$

$$
\leq C|u - v^h|_{(\rho, 2-\delta_1)},
$$

where we have noted that for all $v_1$, $v_2$, and $s \in [0, 1]$

$$
(2.14) \quad \frac{1}{2}s(|\nabla v_1| + |\nabla v_2|) \leq |\nabla[v_1 + sv_2]| + |\nabla v_1| \leq 2(|\nabla v_1| + |\nabla v_2|).
$$

From (2.12b), (2.1b), and (2.14) we have that

$$
|A(v^h)| \geq C_2 \int_0^1 s^{1+\delta_2} \int_{\Omega^h} \left( |\nabla(u + s(v^h - u))| + |\nabla u| \right)^{p-2+\delta_2}
$$

$$
\times |\nabla(v^h - u)|^{2+\delta_2} \, d\Omega^h \, ds
$$

$$
\geq C|u - v^h|_{(\rho, 2+\delta_2)}.
$$

From (1.8) and (2.12) we have that for all $v^h \in S_g^h$

$$
(2.16) \quad A(u^h) + J'_{\Omega^h}(u)(u^h - u) \equiv J_{\Omega^h}(u^h) - J_{\Omega^h}(u) \leq J_{\Omega^h}(v^h) - J_{\Omega^h}(u)
$$

$$
\equiv A(v^h) + J'_{\Omega^h}(u)(v^h - u).
$$

Therefore, it follows from (2.16), (2.13), and (2.15) that

$$
(2.17) \quad |u - u^h|_{(\rho, 2+\delta_2)} \leq C|u - v^h|_{(\rho, 2-\delta_1)} + J'_{\Omega^h}(u)(v^h - u^h).
$$
As $\Omega^h$ is Lipschitz, $\Omega^h \subseteq \Omega$ and $\chi \equiv v^h - u^h \in S_0^h$, we can extend $\chi$ to be zero on $\Omega \setminus \Omega^h$. Denoting this extension by $\hat{\chi}$, we have that $\hat{\chi} \in W_0^{1,p}(\Omega)$ and hence from (P) that $J'_{\Omega^h}(u)(\hat{\chi}) \equiv J'_{\Omega}(u)(\hat{\chi}) = 0$. Therefore, the desired result (2.11) follows from (2.17). \[ \Box \]

3. Error bounds for $p \in (1, 2)$

Assuming that $u \in W^{2,1}(\Omega)$, which implies that $u \in C(\overline{\Omega})$, we can set $g^h \equiv \pi_h u$ in (1.7c). Choosing $\delta_2 = 0$ in (2.11) and noting (2.10a), (1.4), (1.9), and (1.6a), we have for all $\delta_1 \in [0, 2)$ and for all $v^h \in S_g^h$

\[
|u - u^h|^2_{W^{1,p}(\Omega^h)} \leq C|u - u^h|_{(p, 2)} \leq C|u - v^h|_{(p, 2 - \delta_1)}.
\]

Choosing $\delta_1 = 2 - p$ and noting (2.10a) yield that for all $v^h \in S_g^h$

\[
|u - u^h|^2_{W^{1,p}(\Omega^h)} \leq C|u - v^h|_{(p, p)} \leq C|u - v^h|^p_{W^{1,p}(\Omega^h)}.
\]

From a Poincaré inequality we have for all $q \in [1, \infty)$, for all $v \in W^{1,q}(\Omega)$, and for all $v^h, w^h \in S_g^h$ that

\[
\|v - w^h\|_{W^{1,q}(\Omega^h)} \leq C\|v - v^h\|_{W^{1,q}(\Omega^h)} + C\|v - w^h\|_{W^{1,q}(\Omega^h)}.
\]

Hence, from (3.2) with $v^h \equiv \pi_h u$, (3.3), and (1.6a) we have that

\[
\|u - u^h\|_{W^{1,p}(\Omega^h)} \leq C\|u - \pi_h u\|_{W^{1,p}(\Omega^h)} + C|u - \pi_h u|^p_{L^1(\Omega)} \leq Ch^{p/2} \quad \text{if } u \in W^{2,p}(\Omega),
\]

the generalization of the results (1.11a) and (1.12), for $p < 2$, of Chow [3] to the case of nonhomogeneous boundary data $g$ and $\Omega^h \subseteq \Omega$. Below we prove an optimal $W^{1,p}$ error bound for sufficiently regular $u$, based on choosing $\delta_1 = 0$ in (3.1).

Lemma 3.1. Let $\alpha \in (-1, 0)$. If $v \in W^{2,1}(\Omega)$, then

\[
\int_{\Omega} |v|^\alpha |\nabla v|^2 \, d\Omega < \infty.
\]

Proof. We have that

\[
\int_{\Omega} |v|^\alpha |v_x|^2 \, dx \, dy \equiv \frac{1}{\alpha + 1} \int_{\Omega} (\text{sign}(v))|v|^{\alpha + 1} v_x \, dx \, dy
\]

\[
\equiv \frac{1}{\alpha + 1} \left\{ \int_{\partial \Omega} \text{sign}(v)|v|^{\alpha + 1} v_x \, dy - \int_{\Omega} \text{sign}(v)|v|^{\alpha + 1} v_{xx} \, dx \, dy \right\}
\]

and a similar identity with $v_x$ replaced by $v_y$. The desired result (3.5) then follows from the imbedding $W^{2,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ and the trace inequality $\| \cdot \|_{L^1(\partial \Omega)} \leq C\| \cdot \|_{W^{1,1}(\Omega)}$. \[ \Box \]

Theorem 3.1. If $u \in W^{3,1}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$, with $\alpha > 0$, then it follows that

\[
\|u - u^h\|^2_{W^{1,p}(\Omega)} \leq C[h^2 + h^{p(1+\alpha)}],
\]

and hence, if $u \in W^{3,1}(\Omega) \cap C^{2,2-p/p}(\overline{\Omega})$, then

\[
\|u - u^h\|_{W^{1,p}(\Omega^h)} \leq Ch.
\]
**Proof.** As \( u \in C^{2,\alpha}(\overline{\Omega}) \), we have from (1.6a) that for all \( \tau \in T^h \) and for all \((x,y) \in \mathbb{T}\)

\[
(3.7) \quad |\nabla (u - \pi_h u)(x,y)| \leq Ch|H[u]|_{L^\infty(\tau)} \leq Ch H[u](x,y) + Ch^{1+\alpha},
\]

where \( H[u] = |u_{xx}| + |u_{xy}| + |u_{yy}| \).

It is easy to check that the function \( q(t) \equiv (a + t)^{p-2}t^2 \) with \( a \geq 0 \) is increasing on \( \mathbb{R}^+ \) and hence that \( q(t_1 + t_2) \leq 2[q(t_1) + q(t_2)] \) for all \( t_1, t_2 \in \mathbb{R} \). Therefore, we have from (3.1) with \( \delta_1 = 0 \) and \( v^h = \pi_h u \), (3.7), and the above that

\[
|u - u^h|^2_{W^{1,p}(\Omega^h)} \leq C \int_{\Omega^h} (|\nabla u| + |\nabla (u - \pi_h u)|)^{p-2} |\nabla (u - \pi_h u)|^2 d\Omega^h \\
\leq Ch^2 \int_{\Omega^h} (|\nabla u| + Chf(u))^2 d\Omega^h \\
+ Ch^{2(1+\alpha)} \int_{\Omega^h} (|\nabla u| + Ch^{1+\alpha})^{p-2} d\Omega^h
\]

\[
(3.8)
\]

Setting \( v_1 \equiv u_x \) and \( v_2 \equiv u_y \), we have from (3.5), as \( v_1, v_2 \in W^{2,1}(\Omega) \), that

\[
(3.9) \quad \int_{\Omega^h} (|\nabla u|)^{p-2} (H[u])^2 d\Omega^h \leq C \int_{\Omega^h} (v_1^2 + v_2^2)^{(p-2)/2} (|\nabla v_1|^2 + |\nabla v_2|^2) d\Omega^h
\]

\[
\leq C \int_{\Omega^h} [v_1^{p-2} |\nabla v_1|^2 + v_2^{p-2} |\nabla v_2|^2] d\Omega^h < \infty.
\]

Combining (3.8) and (3.9) yields the result (3.6a) and hence (3.6b) with \( \| \cdot \|_{W^{1,p}(\Omega^h)} \) replaced by \( \| \cdot \|_{W^{2,1}(\Omega^h)} \). The results (3.6) then follow by noting (3.3), (1.6a), and that \( u \in W^{3,1}(\Omega) \) implies \( u \in W^{2,p}(\Omega) \). \( \Box \)

We note that one can prove (3.6b) under alternative regularity requirements on \( u \), e.g., \( u \in W^{3,p}(\Omega) \). However, we will not exploit this here. We now show that the regularity requirements on \( u \) in Theorem 3.1 hold for a model problem.

**Example 3.1.** We consider a radially symmetric version of problem \((\mathcal{P})\). Let \( \Omega \equiv \{ r: r < 1 \} \), \( f(x,y) \equiv F(r) \), \( f \in L^q(\Omega) \) for \( q > 2 \), and \( g \) be constant, where \( r \equiv (x^2 + y^2)^{1/2} \). Then

\[
(3.10a) \quad u(x,y) \equiv U(r) \equiv -\int_r^1 \text{sign}(Z(t))|Z(t)|^{1/(p-1)} \, dt + g,
\]

\[
(3.10b) \quad U'' \equiv (|Z|^{(2-p)/(p-1)}Z')/(p - 1)
\]

and

\[
(3.10c) \quad U''' \equiv C_1 \text{sign}(Z)|Z|^{(3-2p)/(p-1)}(Z')^2 + C_2 |Z|^{(2-p)/(p-1)}Z''
\]

for some constants \( C_i \), where

\[
(3.10d) \quad Z(x,y) \equiv Z(r) \equiv (|U'|^{p-2}U')(r) = -\frac{1}{r} \int_0^r tF(t) \, dt.
\]
It is a simple matter to deduce from (3.10d) that

\[(3.11a) \quad f \in C^{0, q}(\Omega) \Rightarrow z \in C^{1, q}(\Omega) \quad \forall q \in [0, 1]\]

and

\[(3.11b) \quad f \in W^{1, q}(\Omega) \quad \text{for} \quad q > 1 \Rightarrow z \in W^{2, 1}(\Omega) \Rightarrow z \in C(\Omega),\]

It follows from (3.10b) and (3.11a) that

\[(3.12a) \quad p \in [1, \frac{1}{2}] \text{ and } f \in C^{0, \beta}(\Omega) \quad \text{for} \quad \beta \in [0, 1] \Rightarrow u \in C^{2, \beta}(\Omega),\]

\[(3.12b) \quad p \in [\frac{1}{2}, 2) \text{ and } f \in C^{0, \beta}(\Omega) \quad \text{for} \quad \beta \in [0, (2-p)/(p-1)] \Rightarrow u \in C^{2, \beta}(\Omega),\]

and from (3.10c), (3.11b), and Lemma 3.1 that

\[(3.12c) \quad p \in (1, 2) \text{ and } f \in W^{1, q}(\Omega) \quad \text{for} \quad q > 1 \Rightarrow u \in W^{3, 1}(\Omega).\]

4. Error bounds for \( p > 2 \)

Let \( g^h \equiv \pi_h u \) in (1.7c). From (2.11) with \( \delta_1 = 0 \) and \( \delta_2 = p - 2 \), (2.10b), (1.4), (1.9), and (1.6b) it follows that

\[(4.1) \quad |u - u^h|_{p, W^{1, p}(\Omega^h)} \leq |u - h^h|_{p, W^{1, p}(\Omega^h)} \leq C |u - \pi_h u|_{p, 2} \leq C |u - \pi_h u|_{p, W^{1, p}(\Omega^h)},\]

and hence it follows from (3.3) and (1.6a) that

\[(4.2a) \quad \|u - u^h\|_{W^{1, p}(\Omega^h)} \leq \|u - \pi_h u\|_{W^{1, p}(\Omega^h)} + C |u - \pi_h u|_{W^{1, p}(\Omega^h)}^{2/p},\]

\[(4.2b) \quad \leq Ch^{2/p} \quad \text{if} \quad u \in W^{2, p}(\Omega),\]

the generalization of the results (1.11b) and (1.12), for \( p > 2 \), of Chow [3] to the case of nonhomogeneous boundary data \( g \) and \( \Omega^h \subseteq \Omega \). Alternatively, assuming \( u \in W^{1, \infty}(\Omega) \), we have from (2.11) with \( \delta_1 = 2 - s \), \( s \in [1, 2] \), and \( \delta_2 = p - 2 \) and (1.6b) that

\[(4.3) \quad |u - u^h|_{p, W^{1, p}(\Omega^h)} \leq |u - h^h|_{p, W^{1, p}(\Omega^h)} \leq C |u - \pi_h u|_{p, s} \leq C |u - \pi_h u|_{W^{1, s}(\Omega^h)},\]

In addition, we note from (1.6b) that for \( u \in W^{1, \infty}(\Omega) \)

\[(4.4) \quad \|u - \pi_h u\|_{W^{1, s}(\Omega^h)} \leq C \|u - \pi_h u\|_{W^{1, s}(\Omega^h)}^q \quad \text{if} \quad q > s.\]

Hence, it follows from (4.3), (3.3), (4.4), and (1.6a) that if \( u \in W^{1, \infty}(\Omega) \cap W^{2, s}(\Omega) \), \( s \in [1, 2] \), then

\[(4.5a) \quad \|u - u^h\|_{W^{1, p}(\Omega^h)} \leq C \|u - \pi_h u\|_{W^{1, p}(\Omega^h)} + C |u - \pi_h u|_{W^{1, p}(\Omega^h)}^{s/p},\]

\[(4.5b) \quad \leq C \|u - \pi_h u\|_{W^{1, p}(\Omega^h)}^{s/p} \leq Ch^{2/p}.\]

Choosing \( f \equiv 1 \) and \( g \equiv 0 \) in Example 3.1 yields that \( u(x, y) \equiv C(1 - r)^{p/(p-1)} \), and so \( u \in W^{2, s}(\Omega) \) only if \( s < 2(p-1)/(p-2) \). Therefore, in general \( u \) rarely belongs to \( W^{2, p}(\Omega) \) in order for (4.2b) to guarantee that the error converges at least at the rate of \( h^{2/p} \) in \( W^{1, p} \). However, from (4.5b) we see that this rate is ensured under the far weaker regularity requirement of \( u \in W^{1, \infty}(\Omega) \cap W^{2, 2}(\Omega) \), and this is satisfied by the example above.

Below we prove error bounds in weaker norms, \( \| \cdot \|_{W^{1, q}(\Omega^h)} \) with \( q \in [1, p) \).
Lemma 4.1. For all $t \in [2, p]$ and $q \in [1, t]$ for which

\begin{equation}
\int_{\Omega} |\nabla u|^{-(p-t)q/(t-q)} d\Omega < \infty \quad \text{if } q \in [1, t),
\end{equation}

and

\begin{equation}
|\nabla u|^{-(p-t)} \in L^\infty(\Omega) \quad \text{if } q = t,
\end{equation}

we have for $u \in W^{1, \infty}(\Omega) \cap W^{2,s}(\Omega)$ with $s \in [1, 2]$ that

\begin{equation}
||u - u^h||_{W^{1,s}(\Omega^h)} \leq Ch^{s/t}.
\end{equation}

Proof. Choosing $\delta_1 = 2 - s$ and $\delta_2 = t - 2$ in (2.11), noting (4.3) and (4.6), and applying a Hölder inequality, we obtain that

\begin{equation}
|u - u^h|_{W^{1,s}(\Omega^h)} \leq C|u - u^h|_{(p,t)} \leq C|u - \pi_h u|_{(p,s)} \leq C|u - \pi_h u|_{W^{1,s}(\Omega^h)}.
\end{equation}

The desired result (4.7) then follows from (4.8), (1.6a), (3.3), and (4.4). \qed

To improve on the $h^{t/p}$ convergence rate for the error in (4.5b), we wish to take $t \in [2, p)$, which gives rise to the restrictions (4.6) on $u$; that is, we require $\{(x, y) \in \Omega: |\nabla u(x, y)| = 0\}$ to have zero measure and a growth condition on $|\nabla u|^{-1}$. From inspection we see that the weakest growth restriction on $|\nabla u|^{-1}$ for a fixed $t$ is needed when $q = 1$. We now look for sufficient conditions on $u$ and the data $f$ in order for these restrictions to hold.

Lemma 4.2. If $u \in W^{1, \infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, \infty)$, then there exists an $M \in L^s(\Omega)$ such that

\begin{equation}
|f| \leq M|\nabla u|^{p-2} \quad \text{a.e. in } \Omega.
\end{equation}

Proof. Let $\nabla u \equiv (v_1, v_2) \in [W^{1,s}(\Omega)]^2$ and $v \equiv (v_1^2 + v_2^2)^{1/2} \equiv |\nabla u| \in L^\infty(\Omega)$. As $|v_1/v| + |v_2/v|$ is bounded and $\nabla v \equiv (v_1 v_1 + v_2 v_2)/v$, it follows that $v \in W^{1,s}(\Omega)$. In addition, we have that

\begin{equation}
f = -\div(u^{p-2}v_1, u^{p-2}v_2)
= -u^{p-2}[|(v_1)_x + (v_2)_y| + (p-2)[v_1 v_x + v_2 v_y]/v].
\end{equation}

Hence the desired result (4.9). \qed

Under the assumption that $\{(x, y) \in \Omega: f(x, y) = 0\}$ has zero measure, the inequality (4.9), for example, yields for $t \geq 2$ and $1 \leq q < t < p$ that

\begin{equation}
\int_{\Omega} |\nabla u|^{-(p-t)q/(t-q)} d\Omega \leq \int_{\Omega} [M/|f|]^{(p-t)q/(p-2)(t-q)} d\Omega.
\end{equation}

Therefore, with $M \in L^s(\Omega)$, for a given $s \in [1, \infty)$, and imposing a growth condition on $|f|^{-1}$, one can choose appropriate $t$ and $q$ so that (4.6a) and hence (4.7) hold. Below we give an example of such a result.

Theorem 4.1. Let $u \in W^{1, \infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, \infty]$. If $|f|^{-\gamma} \in L^1(\Omega)$ for some $\gamma \in (0, \infty)$, or if $|f|^{-1} \in L^\infty(\Omega)$ we set $\gamma = \infty$, then we have for $q \in [1, p)$ that

\begin{equation}
\|u - u^h\|_{W^{1,s}(\Omega^h)} \leq \begin{cases} 
Ch^{2/t} & \text{if } s \geq 2, \\
Ch^{s/t} & \text{if } s \in [1, 2),
\end{cases}
\end{equation}

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where

\[(4.12b) \quad t = \max\{2, q[(s + \gamma)p + (p - 2)\gamma s]/[(s + \gamma)q + (p - 2)\gamma s]\}.\]

**Proof.** First a simple calculation yields that \( t \) satisfying \((4.12b)\) is such that \( t \in [2, p) \) and \( t > q \). Setting

\[\eta = q(p - t)/[(p - 2)(t - q)],\]

we conclude that \( \eta \leq \gamma s/(s + \gamma) \) and hence \( s\eta \leq \gamma(s - \eta) \), and if \( \gamma \) is finite then \( \eta < s \). Therefore, from \((4.11)\), the assumptions on \( f \) and Hölder’s inequality we have

\[
\int_{\Omega} |\nabla u|^{-(p-\eta)q/(t-\eta)} d\Omega \leq \int_{\Omega} (M|f|^{-1})^\eta d\Omega \\
\leq \left( \int_{\Omega} M^s d\Omega \right)^{\eta/s} \left( \int_{\Omega} |f|^{-\eta s/(s-\eta)} d\Omega \right)^{(s-\eta)/s} \leq C.
\]

Similarly, \((4.13)\) holds if \( \gamma \) is infinite, as \( \eta \leq s \). The desired result \((4.12a)\) then follows from \((4.6a)\) and \((4.7)\). \(\square\)

We note that for fixed \( q, \gamma, \) and \( s \) the right-hand side of \((4.12b)\) tends to \( \max\{2, q[(s + \gamma) + \gamma s]/\gamma s\} \) as \( p \to \infty \). Therefore, the error bound \((4.12a)\) does not degenerate as \( p \to \infty \), unlike \((4.2b)\) and \((4.5b)\).

**Corollary 4.1.** Let \( u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega), s \in [1, \infty]. \) Suppose that there exists a constant \( \rho > 0 \) such that \( |f| \geq \rho \) a.e. in \( \Omega \); then for \( q \in [1, p) \) we have that

\[
(4.14a) \quad \|u - u^h\|_{W^{1,s}(\Omega^h)} \leq \begin{cases} Ch^{2/t} & \text{if } s \geq 2, \\
Ch^{s/t} & \text{if } s \in [1, 2), \end{cases}
\]

where

\[
(4.14b) \quad t = \max\{2, q[p + (p - 2)s]/[q + (p - 2)s]\}.
\]

Hence, we have that for \( q = 2s/(1 + s) \)

\[
(4.14c) \quad \|u - u^h\|_{W^{1,s}(\Omega^h)} \leq \begin{cases} Ch & \text{if } s \geq 2, \\
Ch^{s/2} & \text{if } s \in [1, 2). \end{cases}
\]

**Proof.** The result \((4.14a,b)\) follows directly from setting \( \gamma = \infty \) in \((4.12)\). The result \((4.14c)\) then follows from \((4.14a,b)\) by noting that \( t = 2 \) if \( q = 2s/(1 + s) \). \(\square\)

5. **Numerical examples**

The standard Galerkin method analyzed in the previous sections requires the term \( \int_{\Omega^h} f v^h d\Omega^h \) for all \( v^h \in S_0^h \) to be integrated exactly. This is difficult in practice, and it is computationally more convenient to consider a scheme where numerical integration is applied to this term. With \( \Omega^h \equiv \bigcup_{\tau \in T^h} \tau \) and \( \{a_i\}_{i=1}^3 \) being the vertices of a triangle \( \tau \), we define the quadrature rule

\[
(5.1) \quad Q_\tau(v) \equiv \frac{1}{3} \text{meas}(\tau) \sum_{i=1}^3 v(a_i) \equiv \int_\tau \pi_h v \, d\tau
\]
approximating \( \int v \, d\tau \) for \( v \in C(\bar{\Omega}) \). Then, for \( v, w \in C(\bar{\Omega}^h) \), we set

\[
(v, w)^h = \sum_{\tau \in T^h} Q_h(vw) = \int_{\Omega^h} \pi_h(vw) \, d\Omega^h
\]

as an approximation to \( \int_{\Omega^h} vw \, d\Omega^h \).

The fully practical finite element approximation of \((\mathcal{P})\) that we wish to consider is:

(\(\mathcal{B}^h\)) Find \( \hat{u}^h \in S_g^h \) such that

\[
\int_{\Omega^h} |\nabla \hat{u}^h|^p (\nabla \hat{u}^h, \nabla v^h)_{\mathbb{R}^2} \, d\Omega^h = (f, v^h)^h \quad \forall v^h \in S_g^h.
\]

The corresponding minimization problem is:

(\(\mathcal{A}^h\)) Find \( \hat{u}^h \in S_g^h \) such that

\[
\bar{J}^h_{\Omega^h}(\hat{u}^h) \leq \bar{J}^h_{\Omega^h}(v^h) \quad \forall v^h \in S_g^h,
\]

where

\[
\bar{J}^h_{\Omega^h}(v^h) = \frac{1}{p} \int_{\Omega^h} |\nabla v^h|^p \, d\Omega^h - (f, v^h)^h.
\]

The well-posedness of \((\mathcal{B}^h) \equiv (\mathcal{A}^h)\) follows in an analogous way to that of \((\mathcal{P})\) and \((\mathcal{A})\), and

\[
\|\hat{u}^h\|_{W^{1,p}(\Omega^h)} \leq C[\|f\|_{L^1(\Omega^h)}^{1/p - 1} + \|g\|_{W^{1,p}(\Omega^h)}].
\]

We now bound the error \( u^h - u \). First we have the analogue of Theorem 2.1.

**Theorem 5.1.** Let \( u \) and \( \hat{u}^h \) be the unique solutions of \((\mathcal{P}) \equiv (\mathcal{A})\) and \((\mathcal{B}^h) \equiv (\mathcal{A}^h)\), respectively. Let \( f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \). Then for any \( \delta_1 \in [0, 2) \) and \( \delta_2 \geq 0 \), and any \( v^h \in S_g^h \), it follows that

\[
\|u - \hat{u}^h\|_{(\rho, 2 + \delta_2)} \leq C\|u - v^h\|_{(\rho, 2 - \delta_1)} + Ch^2\|v^h - \hat{u}^h\|_{W^{1,1}(\Omega^h)}.
\]

**Proof.** The proof follows exactly that of Theorem 2.1 with \( \hat{u}^h \) and \( J_{\Omega^h} \) instead of \( u^h \) and \( J_{\Omega^h} \) in (2.16). However, whereas \( J'_{\Omega^h}(u)(v^h - u^h) = 0 \), we now have for all \( v^h \in S_g^h \)

\[
J'_{\Omega^h}(u)(v^h - \hat{u}^h) \equiv J'_{\Omega^h}(u)(v^h - \hat{u}^h)
\]

\[
- \int_{\Omega^h} f(v^h - \hat{u}^h) \, d\Omega^h + (f, v^h - \hat{u}^h)^h
\]

\[
\equiv - \int_{\Omega^h} f(v^h - \hat{u}^h) \, d\Omega^h + (f, v^h - \hat{u}^h)^h
\]

and

\[
\left| \int_{\Omega^h} f(v^h - \hat{u}^h) \, d\Omega^h - (f, v^h - \hat{u}^h)^h \right| \leq C\|I - \pi_h\|_{L^1(\Omega^h)} \|f(v^h - \hat{u}^h)\|_{L^1(\Omega^h)}
\]

\[
\leq C h^2 |f(v^h - \hat{u}^h)|_{W^{2,1}(\Omega^h)} \leq C h^2 \|v^h - \hat{u}^h\|_{W^{1,1}(\Omega^h)},
\]

provided \( f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \). Hence, we obtain the desired result (5.6).

\[\Box\]
In particular, assuming $u \in W^{2,1}(\Omega)$ if $p < 2$, we have for $v^h \equiv g^h \equiv \pi_h u$ that for any $\delta_1 \in [0, 2)$ and $\delta_2 \geq 0$

\begin{equation}
|u - \tilde{u}^h|_{(p, 2+\delta_2)} \leq C|u - \pi_h u|_{(p, 2-\delta_1)} + Ch^2.
\end{equation}

Hence, it is a simple matter to check that the results of the previous sections hold for $\tilde{u}^h$ as well as $u^h$ if $f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. We note that this constraint on $f$ can be weakened and is imposed here for ease of exposition only.

We now report on some numerical results with the fully practical approximation (5.3). For computational ease we took $\Omega$ to be the square $[0, 1] \times [0, 1]$. This was partitioned into uniform right-angled triangles by dividing it first into equal squares of sides of length $1/N$ and then into triangles by inserting the SW-NE diagonals. We imposed homogeneous Neumann data on the sides $x = 0$ and $y = 0$ and Dirichlet data on the sides $x = 1$ and $y = 1$. Therefore, the problem can be viewed as a Dirichlet problem over $[-1, 1] \times [-1, 1]$, and so our error analysis applies directly.

We computed our approximation (5.3) by solving the equivalent minimization problem (5.4). We used a Polak-Ribière conjugate gradient method, which worked reasonably well for the values of $p$ reported here. We did not experiment with the augmented Lagrangian approach advocated by Glowinski and Marrocco [5], but this conjugate gradient approach was far superior to the gradient method suggested by Wei [10].

For our test problems we consider solutions of the radially symmetric problem, Example 3.1, extended to the unit square. In the first three examples we took for various values of $p$ and $\gamma$

\begin{equation}
f = F(r) \equiv r^\alpha \quad \text{and} \quad u = U(r) \equiv (p - 1)[1/(\sigma + 2)]^{1/(p-1)}[1 - r^{(\sigma+p)/(p-1)}]/(\sigma + p).
\end{equation}

In all the examples, $f$ is sufficiently smooth, so that the error bounds for $u^h$ in the previous sections hold for $\tilde{u}^h$ as well.

**Example 5.1.** This is (5.9) with $\sigma = 0$ and $p = 1.5$. It follows from (3.12) that $u \in W^{3,1}(\Omega) \cap C^{2,1}(\overline{\Omega})$, and so from Theorem 3.1 we expect $O(h)$ convergence in $W^{1,1.5}(\Omega)$. This is certainly achieved by inspecting Table 5.1, where we adopt the notation $0.8233(-3) = 0.8233 \times 10^{-3}$. In fact, $\tilde{u}^h$ is converging to $u$ at the rate $O(h^2)$ in $L^\infty(\Omega)$, and there is a superconvergence for $\pi_h u - \tilde{u}^h$ in $W^{1,1}(\Omega)$ and $W^{1,p}(\Omega)$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|\pi_h u - \tilde{u}^h|_{W^{1,1}(\Omega)}$</th>
<th>$|\pi_h u - \tilde{u}^h|_{W^{1,p}(\Omega)}$</th>
<th>$|\pi_h u - \tilde{u}^h|_{L^\infty(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.8233(-3)</td>
<td>0.4823(-3)</td>
<td>0.8150(-3)</td>
</tr>
<tr>
<td>20</td>
<td>0.2061(-3)</td>
<td>0.1207(-3)</td>
<td>0.2034(-3)</td>
</tr>
<tr>
<td>40</td>
<td>0.5196(-4)</td>
<td>0.3043(-4)</td>
<td>0.5109(-4)</td>
</tr>
<tr>
<td>80</td>
<td>0.1235(-4)</td>
<td>0.723(-5)</td>
<td>0.1263(-4)</td>
</tr>
</tbody>
</table>
Example 5.2. This is (5.9) with $\sigma = 0$ and $p = 4$. It follows from Example 3.1 and §4 that $u \in W^{2,1}(\Omega)$, with $s < 3$, and from (4.14c) we expect $O(h)$ convergence in $W^{1,1}(\Omega)$. From Table 5.2 we see this is achieved. In fact, $\pi_h u - \hat{u}^h$ exhibits superconvergence in $W^{1,1}(\Omega)$. □

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|\pi_h u - \hat{u}^h|_{W^{1,1}(\Omega)}$</th>
<th>$|\pi_h u - \hat{u}^h|_{W^{1,p}(\Omega)}$</th>
<th>$|\pi_h u - \hat{u}^h|_{L^{\infty}(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.1789(-2)</td>
<td>0.4486(-2)</td>
<td>0.3790(-2)</td>
</tr>
<tr>
<td>20</td>
<td>0.5049(-3)</td>
<td>0.2519(-2)</td>
<td>0.1585(-2)</td>
</tr>
<tr>
<td>40</td>
<td>0.1376(-3)</td>
<td>0.1414(-2)</td>
<td>0.6493(-3)</td>
</tr>
<tr>
<td>80</td>
<td>0.3659(-4)</td>
<td>0.7936(-3)</td>
<td>0.2625(-3)</td>
</tr>
</tbody>
</table>

Example 5.3. Here we take (5.9) with $\sigma = 7$ and $p = 4$. It follows from Example 3.1 that $u \in W^{2,\infty}(\Omega)$. From (4.12), as $s = \infty$ and $\gamma < 2/7$, we have with $q = 1$ that $t > 32/11$. Therefore, for all $\epsilon > 0$ we have that

\[
(5.10) \quad \|u - \hat{u}^h\|_{W^{1,1}(\Omega)} \leq C h^{(11-\epsilon)/16}.
\]

We note that a sharper bound, $h^{(14-\epsilon)/19}$, can be obtained by noting that for this model problem $|U'(r)| \geq C r^{8/3}$ and applying (4.6) and (4.7) directly. From Table 5.3 we see that the above bounds are pessimistic. In fact, we have $O(h^2)$ convergence in $L^{\infty}(\Omega)$ and $\pi_h u - \hat{u}^h$ exhibits superconvergence in $W^{1,1}(\Omega)$. □

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|\pi_h u - \hat{u}^h|_{W^{1,1}(\Omega)}$</th>
<th>$|\pi_h u - \hat{u}^h|_{W^{1,p}(\Omega)}$</th>
<th>$|\pi_h u - \hat{u}^h|_{L^{\infty}(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.8153(-3)</td>
<td>0.5988(-2)</td>
<td>0.5014(-2)</td>
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<tr>
<td>20</td>
<td>0.2164(-2)</td>
<td>0.1931(-2)</td>
<td>0.1235(-2)</td>
</tr>
<tr>
<td>40</td>
<td>0.5918(-3)</td>
<td>0.8893(-3)</td>
<td>0.2989(-3)</td>
</tr>
<tr>
<td>80</td>
<td>0.1429(-3)</td>
<td>0.1952(-3)</td>
<td>0.6449(-4)</td>
</tr>
</tbody>
</table>

Finally we consider an example for $p > 2$, where $\{(x, y) \in \Omega: f(x, y) = 0\}$ does not have zero measure.

Example 5.4. We take

\[
(5.11a) \quad F(r) \equiv \begin{cases} 0 & \text{for } r \leq a, \\ 4p-1(r-a)^{(3p-4)[2+(a/r)-3p]} & \text{for } r \geq a \end{cases}
\]

and

\[
(5.11b) \quad U(r) \equiv \begin{cases} 0 & \text{for } r \leq a, \\ (r-a)^4 & \text{for } r \geq a \end{cases}
\]
with \( a = 0.3 \) and \( p = 4 \). At present the only global error estimate we have for this case is the result (4.5b). Clearly, this is pessimistic from inspecting Table 5.4, where once again we see \( O(h^2) \) convergence in \( L^\infty(\Omega) \) and \( \pi_h u - \hat{u}^h \) is superconvergent in \( W^{1,1}(\Omega) \). We note that the maximum error did not occur in the disc \( \{ r : r \leq 0.3 \} \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( | \pi_h u - \hat{u}^h |_{W^{1,1}(\Omega)} )</th>
<th>( | \pi_h u - \hat{u}^h |_{W^{1,p}(\Omega)} )</th>
<th>( | \pi_h u - \hat{u}^h |_{L^\infty(\Omega)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.5879(-1)</td>
<td>0.4653(-1)</td>
<td>0.3080(-1)</td>
</tr>
<tr>
<td>20</td>
<td>0.1553(-1)</td>
<td>0.1182(-1)</td>
<td>0.7930(-2)</td>
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<tr>
<td>40</td>
<td>0.4332(-2)</td>
<td>0.3118(-2)</td>
<td>0.1992(-2)</td>
</tr>
<tr>
<td>80</td>
<td>0.1139(-2)</td>
<td>0.1706(-2)</td>
<td>0.4923(-3)</td>
</tr>
</tbody>
</table>

**BIBLIOGRAPHY**

1. C. Atkinson and C. R. Champion, *Some boundary-value problems for the equation \( \nabla \cdot (|\nabla \phi|^p \nabla \phi) = 0 \),* Quart. J. Mech. Appl. Math. 37 (1984), 401–419.

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