ON FABER POLYNOMIALS GENERATED BY AN \textit{m}-STAR

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Abstract. In this paper, we study the Faber polynomials $F_n(z)$ generated by a regular $m$-star ($m = 3, 4, \ldots$)

$$S_m = \{x \omega^k; 0 \leq x \leq 4^{1/m}, \ k = 0, 1, \ldots, m - 1, \ \omega^m = 1\}.$$  

An explicit and precise expression for $F_n(z)$ is obtained by computing the coefficients via a Cauchy integral formula. The location and limiting distribution of zeros of $F_n(z)$ are explored. We also find a class of second-order hypergeometric differential equations satisfied by $F_n(z)$. Our results extend some classical results of Chebyshev polynomials for a segment $[-2, 2]$ in the case when $m = 2$.

1. Introduction

Let $E$ be a compact set (not a single point) whose complement $\mathbb{C}^* \setminus E$ with respect to the extended plane is simply connected. The Riemann mapping theorem asserts that there exists a conformal mapping $w = \Phi(z)$ of $\mathbb{C}^* \setminus E$ onto the exterior of a circle $|w| = \rho_E$ in the $w$-plane. For a unique choice of $\rho_E$, we can insist that

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) = 1,$$

so that, in a neighborhood of infinity,

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots.$$  

The polynomial part of $\{\Phi(z)\}^n$, denoted by $F_n(z) = z^n + \cdots$, is called the Faber polynomial of degree $n$ generated by the set $E$.

Let

$$\Psi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots$$

be the inverse function of $w = \Phi(z)$. Thus, $\Psi(w)$ maps the domain $|w| > \rho_E$ conformally onto $\mathbb{C}^* \setminus E$. Faber [2] proved that

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad |w| > \rho_E, \ z \in E.$$  

The Faber polynomials play an important role in approximation theory and geometric function theory. It can be shown that, under suitable conditions, a
function analytic in the inner domain of a Jordan curve \( J \) can be expanded into a series of Faber polynomials that come from the mapping function of the outer domain of \( J \) (cf. [1]).

The explicit construction of the Faber polynomials of a given set \( E \) depends essentially on the knowledge of the mapping function \( \Phi(z) \). For \( E = [-2, 2] \), we know that \( \Phi(z) = (z + \sqrt{z^2 - 4})/2 \) with inverse \( \Phi(w) = w + 1/w \). For \( n \geq 1 \), the polynomial part of \( \{\Phi(z)\}_n \) is the same as the polynomial part of

\[
\Phi^n(z) + \Phi^{-n}(z) = w^n + w^{-n},
\]

which reduces to \( 2 \cos n\theta \), when \( w = e^{i\theta} \). Thus the Faber polynomials are (apart from a multiplicative constant) the same as the classical Chebyshev polynomials \( T_n(x) \) for \([-2, 2]\).

The following properties for \( T_n(x) \) are well known [7]:

(i) For \( n \geq 1 \),

\[
T_n(x) = n \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \frac{(n-j-1)!}{(n-2j)!j!} x^{n-2j},
\]

where

\[
\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} n/2 & \text{if } n \text{ is even}, \\ n-1/2 & \text{if } n \text{ is odd}. \end{cases}
\]

(ii) \( T_n(x) \) satisfies the following differential equation:

\[
(4-x^2)y'' - xy' + n^2y = 0.
\]

(iii) The zeros of \( T_n(x) \) are located on \((-2, 2)\) for every \( n \geq 1 \).

(iv) The asymptotic behavior of the zeros of \( T_n(x) \) is given by the arcsine distribution

\[
d\mu(t) = \frac{1}{\pi} \frac{1}{\sqrt{4-t^2}} dt.
\]

In this paper we shall study the Faber polynomials generated by a regular \( m \)-star \((m = 2, 3, \ldots)\)

\[
S_m = \{x\omega^k ; 0 \leq x \leq 4^{1/m}, \ k = 0, 1, \ldots, m-1, \ \omega^m = 1\}.
\]

Clearly, \( S_m \) becomes \([-2, 2]\) when \( m = 2 \). It is natural to ask what can be said about Faber polynomials when \( m = 3, 4, 5, \ldots \). By using the properties and characteristics of the Chebyshev polynomials as a motivation, we are able to compute an explicit representation for \( F_n(z) \). In addition, the \( F_n(z) \) are found to be a family of solutions to a class of second-order hypergeometric differential equations when \( n \equiv 0 \pmod{m} \). We also found that the zeros of \( F_n(z) \) when \( n \equiv 0 \pmod{m} \), or \( n \equiv \frac{m}{2} \pmod{m} \) if \( m \) is even, are located on \( S_m \). The asymptotic behavior of the zeros of \( F_n(z) \) is also determined explicitly. Numerical results for the zeros of \( F_n(z) \) will be given in \( \S 3 \).

2. Main results and proofs

As we shall see, the Faber polynomials for \( S_m \) with \( m > 2 \) enjoy certain properties that are similar to those satisfied by Chebyshev polynomials. Our main results in this direction are the following:
Theorem. Let \( F_n(z) \) be the Faber polynomials of \( S_m \) of degree \( n \). Then

(i) For \( n \geq 1 \),

\[
F_n(z) = \frac{2n}{m} \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} (-1)^j \frac{\Gamma(\frac{2n-j}{m})}{\Gamma(\frac{2n}{m} - 2j + 1)} z^{n-mj},
\]

where

\[
\left\lfloor \frac{n}{m} \right\rfloor = \begin{cases} \frac{n}{m} & \text{if } n \equiv 0 \pmod{m}, \\ \frac{n-k}{m} & \text{if } n \equiv k \pmod{m}, \quad k = 0, 1, 2, \ldots, m-1. \end{cases}
\]

(ii) If \( n \equiv 0 \pmod{m} \), then \( F_n(z) \) satisfies the following differential equation:

\[
(4 - z^m)z^{n} - [z^m + 2m - 4]y' + n^2 z^{m-1} y = 0.
\]

(iii) If \( n \equiv 0 \pmod{m} \), or \( n \equiv \frac{m}{2} \pmod{m} \) when \( m \) is even, then the zeros of \( F_n(z) \) are located on \( S_m \).

(iv) For \( n \geq 1 \), the zero distribution of \( F_n(z) \) for \( S_m \) is given by

\[
d\mu(t) = \frac{1}{\pi \sqrt{4 - t^m}} dt, \quad t \in S_m.
\]

One can see that the above theorem generalizes (1.4), (1.5), and (1.6) when \( m = 2 \). We now proceed to prove our theorem.

Proof. (i). It is known [3, p. 395] that for \( m = 2, 3, \ldots, \)

\[
z = \Psi(w) = w \left( 1 + \frac{1}{w^m} \right)^{2/m}
\]

maps \( |w| > 1 \) conformally onto \( \mathbb{C} \setminus S_m \). Let \( \Phi(z) \) be its inverse. It follows from the definition of Faber polynomials that \( F_n(z) \) generated by \( S_m \) is given by

\[
F_n(z) = \sum_{j=0}^{n} c_{n-j} z^{n-j},
\]

where \( c_{n-j} \) is the Laurent coefficient in the expansion of \( \{\Phi(z)\}^n \), that is, for \( j = 0, 2, 3, \ldots, n, \)

\[
c_{n-j} = \int_{|z|=R} \frac{\{\Phi(z)\}^n}{z^{n-j+1}} dz,
\]

with \( R \) chosen sufficiently large so that \( S_m \) is contained in the interior of the region bounded by the circle \( |z| = R \). Alternatively, using the substitution \( z = \Psi(w) \), we obtain for \( j = 0, 2, 3, \ldots, n, \)

\[
c_{n-j} = \int_{|w|=\rho} \frac{w^n \Psi'(w)}{\{\Psi(w)\}^{n-j+1}} dw.
\]

By the symmetry of \( S_m \), we see that \( \Psi(w) \) is an \( m \)-fold symmetric mapping function (i.e., \( \Psi(\exp(2k\pi \frac{m}{m} w)) = \exp(2k\pi \frac{m}{m}) \Psi(w), \quad k = 0, 1, \ldots, m-1 \)). It is easy to see that \( c_{n-j} = 0 \) if \( j \not\equiv 0 \pmod{m} \). Thus, \( F_n(z) \) has the following form:

\[
F_n(z) = \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} c_{n-mj} z^{n-mj}.
\]
By Cauchy's Theorem, we see that the coefficients $c_{n-mj}$ are the same as those of $\frac{1}{w}$ in the expansion of $w^n\Psi'(w)/[\Psi(w)]^{n-mj+1}$. Using (2.4), we get

$$\Psi'(w) = \left(1 - \frac{1}{w^m}\right) \left(1 + \frac{1}{w^m}\right)^{(2-m)/m}.$$ 

Thus,

$$\frac{w^n\Psi'(w)}{[\Psi(w)]^{n-mj+1}} = w^{mj-1} \left(1 - \frac{1}{w^m}\right) \left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m}$$

$$= w^{mj-1} \left(1 + \frac{1}{w^m}\right)^{-1} (1 + \frac{1}{w^m})^{-(m+2n-2mj)/m}$$

Noticing that $|w| > 1$, we have

$$\left(1 + \frac{1}{w^m}\right)^{-(m+2n-2mj)/m} = \sum_{i=0}^{\infty} (-1)^i \left(\frac{m+2n-2mj}{m}\right)^i \frac{w^{-mi}}{i!},$$

where

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1), \quad (\alpha)_0 = 1, \quad (\alpha)_1 = 0.$$ 

Thus, the coefficient of $\frac{1}{w}$ is given by

$$c_{n-mj} = (-1)^j \frac{(m+2n-2mj)_j}{j!} \frac{(m+2n-2mj)_{j-1}}{(j-1)!}$$

$$= (-1)^j \frac{2n}{n} \frac{\Gamma(\frac{2n}{m} - j)}{\Gamma(\frac{2n}{m} - 2j + 1)j!}.$$ 

Proof of (ii). Let $x = zm$ in (2.2). Then we have the following hypergeometric differential equation:

$$x(4 - x)y'' + (2 - x)y' + n^2y = 0.$$ 

Replacing $n$ by $nm$ in (2.1) and substituting $z^m$ for $x$, we get

$$y := F_{nm}(z) = 2n \sum_{j=0}^{n} (-1)^j \frac{\Gamma(2n-j)}{\Gamma(2n-2j+1)j!} x^{n-j}.$$ 

It is easy to verify that $y$ is the polynomial solution of (2.5). Therefore, $F_{nm}(z)$ is the polynomial solution of (2.2). □

Proof of (iii). If $n \equiv 0 \pmod{m}$, or $n \equiv \frac{m}{2} \pmod{m}$ ($m$ is even), then by (1.4)

$$F_{nm}(z) = 2n \sum_{j=0}^{n} (-1)^j \frac{\Gamma(2n-j)}{\Gamma(2n-2j+1)j!} z^{nm-mj},$$

$$F_{nm+m/2}(z) = (2n+1) z^{m/2} \sum_{j=0}^{n} (-1)^j \frac{\Gamma(2n+1-j)}{\Gamma(2n+1-2j+1)j!} z^{nm-mj}. $$
Comparing $F_{mn}(z)$ and $F_{mn+m/2}(z)$ with (1.4) yields
\begin{equation}
F_{mn}(z) = T_{2n}(z^{m/2}), \quad F_{mn+m/2}(z) = T_{2n+1}(z^{m/2}).
\end{equation}
Since the zeros of $T_{2n}(x)$ and $T_{2n+1}(x)$ are located on $[-2, 2]$, it follows from (2.6) that the zeros of $F_{mn}(z)$ and $F_{mn+m/2}(z)$ are located on $S_m$. □

**Remark.** Using (2.4), we can derive the relations
\[ \Phi(z) = \{\phi(z^{m/2})\}^{2/m} \quad \text{and} \quad \Psi(z) = \{\psi(z^{m/2})\}^{2/m}, \]
where $\psi(w) = w + \frac{1}{w}$ and $\phi(z)$ is the inverse mapping function of $\psi(w)$. By the above equations and the definition of the Faber polynomials, one can also obtain (2.6).

**Proof of (iv).** In order to prove (iv), we first develop some notation. The term “capacity” means inner logarithmic capacity (cf. [8, p. 55]). For any set $E \subset \mathbb{C}$, the capacity of $E$ will be denoted by $C(E)$. If $E$ is a compact set with positive capacity, then $\mu_E$ shall denote the unique unit equilibrium measure on $E$ with the property that
\[ \int_E \log |x-t| \, d\mu_E(t) = \log C(E) \]
quasi-everywhere (q.e.) on $E$ (cf. [8, p. 60]). A property is said to hold q.e. on a set $A$ if the subset $E$ of $A$ where it does not hold satisfies $C(E) = 0$.

To each $p_n(z) = \prod_{k=1}^{n}(z - z_k)$, we associate the normalized zero distribution measure $\nu(p_n)$ defined by
\[ \nu(p_n) := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}, \]
where $\delta_{z_k}$ is the point distribution with total mass 1 at $z_k$.

We now prove the following lemma.

**Lemma.** Let $E$ be a compact subset (not a single point) whose complement $\mathbb{C}^* \setminus E$ is simply connected. Assume that $E$ has empty interior. Then the asymptotic distribution of zeros of Faber polynomials $F_n(z)$ coincides with the equilibrium measure of the set $E$. More precisely, the normalized zero distribution measures $\nu(F_n)$ converge in the weak-star topology to $\mu_E$.

**Proof.** By Theorem 2.3 in [6], we need only to prove that Faber polynomials are an asymptotically minimal norm sequence of monic polynomials, that is,
\begin{equation}
\limsup_{n \to \infty} \|F_n\|_{\text{supp}(\mu_E)}^{1/n} \leq \text{cap}(E) = 1,
\end{equation}
where $\mu_E$ is the equilibrium distribution of $E$. It is known from [5, p. 108] that for $r > 1$,
\[ \frac{1}{2} |\Phi(z)| \leq |F_n(z)| \leq \frac{1}{2} |\Phi(z)|, \]
where $z$ is on or exterior to $C_r = \{|\Phi(z)| = r\}$. For $n$ large enough we know that
\begin{equation}
\|F_n\|_E \leq \|F_n\|_C \leq \frac{3}{2} r^n.
\end{equation}
Taking the $n$th roots in (2.8), we get $\|F_n\|_E^{1/n} \leq (\frac{3}{2})^{1/n} r$, so we have
\[ \limsup_{n \to \infty} \|F_n\|_E^{1/n} \leq r. \]
Letting $r \to 1$, we get (2.7). □
We now go back to prove (iv). Clearly, the $m$-star has empty interior. According to our lemma, we know that the limiting distribution of the zeros of Faber polynomials associated with $S_m$ coincides with equilibrium measure $d\mu^*(t)$ for $S_m$. We now determine $d\mu^*(t)$. Define

\[
M_k := \int_{S_m} t^k d\mu^*(t), \quad k = 0, 1, \ldots ,
\]

where $M_k$ is the so-called $k$th moment of the equilibrium distribution on $S_m$. On the other hand, $M_k$ can also be expressed as (cf. [4, p. 345])

\[
M_k = \frac{1}{\pi i} \int_{|s| = r} |\Psi(s)|^k \frac{ds}{s}
\]

for $r > 1$, where $z = \Psi(w) = w(1 + w^{-m})^{2/m}$ maps the exterior of $|w| = 1$ onto the exterior of $S_m$ (cf. [3, p. 395]).

We now compute the $k$th moment $M_k$ from (2.10). Let

\[
t = \Psi(s) = s \left( 1 + \frac{1}{s^m} \right)^{2/m} = (s^{m/2} + s^{-m/2})^{2/m}
\]

in (2.10). Then

\[
dt = (s^{m/2} + s^{-m/2})^{2/m-1} \left( \frac{m}{2} - m s^{-2/m} - m^{-1} s^{-m/2} \right) ds
\]

and so

\[
M_k = \frac{1}{\pi i} \int_{|s| = r} |\Psi(s)|^k \frac{ds}{s} = \frac{1}{\pi} \int_{C_r} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}} \, dt ,
\]

where $C_r$ is the image of $|w| = r$ under the mapping $\Psi(w)$. Letting $r \to 1$, we get

\[
M_k = \frac{1}{\pi} \int_{S_m} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}} \, dt .
\]

By (2.9) we have

\[
\int_{S_m} t^k d\mu^*(t) = M_k = \frac{1}{\pi} \int_{S_m} t^k \sqrt{\frac{t^{m-2}}{4 - t^m}} \, dt , \quad k = 0, 1, \ldots .
\]

Thus, $d\mu^*(t) = d\mu(t)$, where $d\mu(t)$ is defined in (2.3). □

We have completed the proof of the theorem.

The situation for obtaining the locations of the zeros of $F_n(z)$ for every $n$ is not so favorable. However, supported by (iii) in our theorem, and numerical experiments, we formulate the following.

**Conjecture.** The zeros of Faber polynomials generated by the regular $m$-star $S_m$ ($m = 3, 4, \ldots$) are located on $S_m$ for every $n \geq 1$.

For a sequence of polynomials $F_{nm}(z)$, and $F_{nm+m/2}(z)$ when $m$ is even, our theorem confirmed this conjecture. For other sequences of Faber polynomials,
we provide in §3 numerical results produced by Matlab in the cases when \( m = 3, 4, \) and \( 5. \)

3. Numerical results on the zeros of Faber polynomials

We first briefly state our algorithm for computing the zeros of Faber polynomials generated by a set \( E. \)

Let

\[
\Psi(z) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots
\]

be the mapping function as in (1.2) associated with a set \( E. \) Then

\[
\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad |w| > \rho_E, \ z \in E.
\]

It follows from (3.2) that \( F_n(z) \) satisfies the recurrence relation

\[
F_{n+1}(z) = (z - b_0)F_n(z) - \sum_{k=1}^{n-1} b_k F_{n-k}(z) - (n+1)b_n,
\]

with initial condition \( F_0(z) = 1. \) This difference equation for Faber polynomials plays a very important role in computing the zeros of \( F_n(z) \). Our algorithm consists of four steps:

Step 1. Determine the coefficients \( b_k \)'s of the mapping function \( \Psi(w) \) as in (3.1).

Step 2. Compute the coefficients of \( F_n(z) \) by using (3.3).

Step 3. Find the zeros of \( F_n(z) \).

Step 4. Plot the zeros in the complex plane.

We now apply the algorithm to compute the zeros of \( F_n(z) \) for \( S_m \). The results are shown in Figures 1–6 (see pp. 284–286).

For \( m = 2, 3, \ldots, \) we normalize the mapping function such that

\[
\Psi(w) = w \left( 1 + \frac{1}{w^m} \right)^{2/m}.
\]

Then

\[
\Psi(w) = w + \sum_{k=1}^{\infty} \frac{b_{mk-1}}{w^{mk-1}},
\]

where

\[
b_{mk-1} = (-1)^{k-1} \frac{2(m-2)(2m-2) \cdots ((k-1)m-2)}{m^k k!}.
\]

By the recurrence relation (3.3), we find, in particular for \( m = 3, \)

\[
F_0(z) = 1, \quad F_1(z) = z, \quad F_2(z) = z^2, \\
F_3(z) = z^3 - 2, \quad F_4(z) = z^4 - \frac{8}{3}z, \quad F_5(z) = z^5 - \frac{10}{3}z^2, \\
F_6(z) = z^6 - 4z^3 + 2, \quad F_7(z) = z^7 - \frac{14}{3}z^4 + \frac{35}{9}z, \quad F_8(z) = z^8 - \frac{16}{3}z^5 + \frac{56}{9}z^2, \\
F_9(z) = z^9 - 6z^6 + z^3 - 2, \quad F_{10}(z) = z^{10} - \frac{20}{3}z^7 + \frac{110}{9}z^4 - \frac{400}{81}z.
\]
Figure 1. Zeros of $F_n(z)$ for 3-star when $n = 31$

Figure 2. Zeros of $F_n(z)$ for 3-star when $n = 32$
Figure 3. Zeros of $F_n(z)$ for 4-star when $n = 31$

Figure 4. Zeros of $F_n(z)$ for 4-star when $n = 33$
Figure 5. Zeros of $F_n(z)$ for 5-star when $n = 31$

Figure 6. Zeros of $F_n(z)$ for 5-star when $n = 32$
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