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A QUICK WAY OF OBTAINING AN APPROXIMATE SOLUTION TO A STURM-LIOUVILLE PROBLEM

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ABSTRACT. Use is made of the generalized consistency equation and the boundary conditions to obtain a quick approximate solution to a Sturm-Liouville problem.

1. INTRODUCTION

Consider the problem defined by the Sturm-Liouville equation

$$-\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\}+q(x)u=f(x), \qquad x_1 \le x \le x_2, \tag{1.1}$$

with $p(x_1)$, $p(x_2)$ nonzero, and boundary conditions of the form

$$-p(x_1)u'(x_1)+r_1u(x_1)=g_1, \qquad p(x_2)u'(x_2)+r_2u(2)=g_2. \tag{1.2}$$

Multiplying equation (1.1) by $u(x)$, integrating between x_1 and x_2, and using equations (1.2), one obtains

$$\begin{aligned}\int_{x_1}^{x_2}[p\{u'(x)\}^2+q\{u(x)\}^2]\,dx+r_1\{u(x_1)\}^2+r_2\{u(x_2)\}^2\\ =\int_{x_1}^{x_2}fu\,dx+g_1u(x_1)+g_2u(x_2).\end{aligned} \tag{1.3}$$

The condition that the problem defined by (1.1) and (1.2) has a unique solution is equivalent to u being identically zero over $x_1 \le x \le x_2$ when the excitations $f(x)$, g_1, and g_2 are zero. That is, the quadratic form on the left-hand side of (1.3) is positive definite.

The conditions are as follows: q, r_1, and r_2 are nonnegative, and p is positive but may be zero at isolated points in $x_1 < x < x_2$. If q, r_1, r_2 are all zero, however, the problem degenerates to that defined by

$$-\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\}=f(x), \qquad x_1 \le x \le x_2, \tag{1.4a}$$

$$-p(x_1)u'(x_1)=g_1, \qquad p(x_2)u'(x_2)=g_2. \tag{1.4b}$$

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Note that in this case a consistency condition

$$0 = \int_{x_1}^{x_2} f\,dx + g_1 + g_2 \tag{1.5}$$

must hold, and that u is indefinite by an arbitrary constant.

It is equation (1.5) which suggests the possibility of obtaining an approximate solution for the problem defined by (1.1) and (1.2) when a unique solution exists.

2. Generalized consistency equation

Integrating the differential equation (1.1) over $x_1 \le x \le x_2$ and substituting the boundary conditions (1.2), one finds

$$\int_{x_1}^{x_2} qu\,dx + r_1 u(x_1) + r_2 u(x_2) = \int_{x_1}^{x_2} f\,dx + g_1 + g_2 . \tag{2.1}$$

This equation, by virtue of its connection with equation (1.5), will be termed the generalized consistency equation, and a mean value w_0 of u obeying

$$w_0 \left[\int_{x_1}^{x_2} q\,dx + r_1 + r_2 \right] = \int_{x_1}^{x_2} f\,dx + g_1 + g_2 \tag{2.2}$$

may be defined, the quantity in square brackets being intrinsically positive. Thus, the locality of the solution can be estimated by means of two simple integrations. Equation (2.1) can also be used to obtain an approximate quadratic solution for u:

(2.3) Let w be an approximation to u of the form $w = \alpha + \beta x + \gamma x^2$.

Then the set of three equations

$$-p(x_1)w'(x_1) + r_1 w(x_1) = g_1 , \qquad p(x_2)w'(x_2) + r_2 w(x_2) = g_2 , \tag{2.4a}$$

$$\int_{x_1}^{x_2} qw\,dx + r_1 w(x_1) + r_2 w(x_2) = \int_{x_1}^{x_2} f\,dx + g_1 + g_2 \tag{2.4b}$$

makes it possible to determine α, β, and γ.

3. An example

Consider the example

$$-\frac{d^2u}{dx^2} + u = 1 , \qquad 0 \le x \le 1 , \tag{3.1a}$$

$$u'(0) = 0 , \qquad u'(1) + u(1) = 0 . \tag{3.1b}$$

The uniqueness conditions are satisfied and the generalized consistency condition is

$$\int_0^1 u\,dx + u(1) = 1 . \tag{3.1c}$$

The exact solution of the problem defined by the relations (3.1) is given by

$$u = 1 - e^{-1} \cosh x . \tag{3.2}$$

The mean value w_0 of the solution is given by

$$w_0\left[\int_0^1 1\,dx + 1\right] = 1\,, \quad \text{whence } w_0 = .5\,. \tag{3.3}$$

The approximate solution

$$w = \alpha + \beta x + \gamma x^2 \tag{3.4a}$$

must obey

$$w'(0) = 0\,, \quad w'(1) + w(1) = 0\,, \quad \int_0^1 w\,dx + w(1) = 1\,, \tag{3.4b}$$

whence

$$w = \frac{1}{14}(9 - 3x^2)\,. \tag{3.5c}$$

A comparison follows:

x	$u(x)$	$w(x)$
0	.6321	.6429
.5	.5832	.5893
1.0	.4323	.4286

4. Comments

Clearly, it is not easy, and may well be impossible to make an error estimate for the approximate solution obtained by this method. Nevertheless, it is a method which produces an idea of the behavior of the solution and of its approximate values with remarkably little effort. All that is required is evaluations of the integrals $\int_{x_1}^{x_2} qw\,dx$ and $\int_{x_1}^{x_2} f\,dx$.

In particular, an approximate solution can be produced when q and f are only known numerically. The method will presumably be particularly effective when p, q, and f are monotonically increasing or decreasing.

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