POLAR GENERATION OF RANDOM VARIATES WITH
THE $\tau$-DISTRIBUTION

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Abstract. The "polar" method of Box and Muller uses two independent uniform variates in order to generate two independent normal variates. It can be adapted so that two variates from Student's $\tau$-distribution with parameter $\nu$ are generated, though the two variates are now not independent. An algorithm based on the polar method is exact, inexpensive, and valid for all $\nu > 0$.

Box and Muller's [1] polar method for generating random normal variates relies on two convenient properties of the normal distribution, which we may formulate as follows:

(i) Let $X \sim N(0, 1)$. Then $X$ can be regarded as the real part of a complex random variable $Z$ which has a radial distribution (the contours of the density function of $Z$ form circles centered at the origin);

(ii) Write $Z = X + iY = Re^{i\theta}$. Then the distribution function $F_R$ ($\equiv 1 - G_R$) of $R$ is a simple algebraic expression, so simple that it is invertible; that is, given $G \equiv G_R(r)$, we can write down a closed expression for $r$ in terms of $G$.

The aim of this article is to show that properties (i) and (ii) are shared by the Student $t$-distribution with parameter $\nu$ (the $\tau_\nu$-distribution) defined by the density

$$f_T(x) = B(\nu/2, 1/2)^{-1} \cdot \nu^{-1/2} \cdot (1 + x^2/\nu)^{-(\nu+1)/2}.$$\hspace{1cm}(1)

Thus, we are asserting that the $\tau_\nu$-distribution, like the normal, has a tractable radial parent. If $T$ has the density (1), we shall write "$T \sim \tau_\nu$".

Many methods have been proposed for the generation of $\tau_\nu$-variates. The most important ones are described in Devroye [2, pp. 445-450], whose masterly survey we shall not attempt to emulate. The faster algorithms may require either a comparatively great programming effort, or the expensive recalculation of certain quantities required by the algorithm, whenever $\nu$ is changed. As Devroye notes, problems arise when $\nu$ is small and the departure from normality is greatest, particularly in the region $0 < \nu < 1$, where many of the algorithms fail to work at all.

So one would be interested in a theoretically simple and practically effective generator, valid for any $\nu > 0$. We now show how the polar method may be applied to the $\tau_\nu$-distribution to yield such an algorithm.

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Definition. Let \( Z = R e^{i\Theta} \) be a complex random variable such that

(i) \( R \) and \( \Theta \) are independently distributed;
(ii) \( \Theta \) is uniformly distributed on \([0, 2\pi)\);
(iii) the probability that \( R (= |Z|) \) is greater than \( r \) is

\[
GR(r) = (1 + r^2/\nu)^{-\nu/2}, \quad r > 0, \ \nu > 0.
\]

Then we shall say that \( Z \) has the radial \( t_\nu \)-distribution.

Theorem 1. If \( Z = X + Yi \) has the radial \( t_\nu \)-distribution, then the marginal distributions of \( X \) and \( Y \) are given by \( X \sim t_\nu \) and \( Y \sim t_\nu \). The variates \( X \) and \( Y \) are not independent.

Proof. By assumption, \( \Theta \) has density \( 1/2\pi \), and \( R \) has, independently, the density \( f_R(r) = -dGR/dr = (r/\nu) \cdot (1 + r^2/\nu)^{-\nu/2-1} \). The transformation \( x = r \cos \Theta, \ y = r \sin \Theta \) (which has Jacobian \( r \)) shows that the joint density of \( X, Y \) is

\[
f_{X,Y}(x, y) = (2\pi)^{-1} \cdot (1 + (x^2 + y^2)/\nu)^{-\nu/2-1}.
\]

Now integrate out \( y \) (note that the integrand is an even function of \( y \) and let \( u = (1 + y^2/(\nu + x^2))^{-1} \), expressing the integral as a multiple of a beta integral) to confirm that the marginal distribution of \( f_X(x) \) has indeed the required form (1). The result for \( Y \) follows by symmetry.

The only complex radial distribution for which \( X \) and \( Y \) are independent with continuous marginal densities is (Mathai and Pederzoli [5, pp. 9-12]) the zero-mean, equal-variance, zero-covariance bivariate normal. Hence \( X \) and \( Y \) in Theorem 1 are not independent. \( \square \)

Theorem 1 leads to our main result, the polar method for generating \( t_\nu \)-variates:

Theorem 2. Let \( G, H \) be iid variates, uniformly distributed on \([0, 1]\). Let \( \Theta = 2\pi \cdot H \), let \( R = (\nu(G^{-2/\nu} - 1))^{1/2} \ (\nu > 0) \), let \( X = R \cos \Theta \), and let \( Y = R \sin \Theta \). Then \( X \sim t_\nu \) and \( Y \sim t_\nu \). The variates \( X \) and \( Y \) are not independent.

Proof. Let \( Z \) have the radial \( t_\nu \)-distribution. If we let \( G = G_R(R) \), where \( G_R \) is given by (2), then we know that \( G \) is uniformly distributed on \([0, 1]\). Conversely, if \( G \) is uniformly distributed on \([0, 1]\) and we take the inverse transformation \( R = (\nu(G^{-2/\nu} - 1))^{1/2} \), then we know that \( Z \equiv R \exp(2\pi i H) = Re^{i\Theta} \) has the radial \( t_\nu \)-distribution. Now apply Theorem 1. \( \square \)

Theorem 2 is conceptually the simplest formulation of the polar method for the \( t_\nu \)-distribution. However, further improvements are possible. One can (see for instance Marsaglia and Bray [4]) avoid expensive calculation of the cosine...
by using the fact that if \( U + Vi \) is uniformly distributed on the unit disk, 
\( W \equiv U^2 + V^2 \), and \( C \equiv U/\sqrt{W} \), then \( W \) and \( C \) are independent, \( W \) has a uniform \([0, 1]\) distribution, and \( C \) has the same density as \( \cos \Theta \), that is, 
\[
f_C(x) = \frac{1}{\pi \sqrt{1 - x^2}}, \quad -1 < x < 1.
\]
We incorporate this modification into our proposed algorithm, and note that \( Y \) in Theorem 2 is discarded, because of dependence on \( X \).

**Polar algorithm for generating \( t_\nu \)-variates.**

(a) Generate iid uniform \([0, 1]\) variates \( U \) and \( V \). Replace \( U \) by \( 2U - 1 \), \( V \) by \( 2V - 1 \).

(b) Define \( W \equiv U^2 + V^2 \). If \( W > 1 \) return to (a).

(c) Let \( C = \frac{U}{VW} \), \( R = \left( \nu(W^{-2/\nu} - 1) \right)^{1/2} \), \( X = RC \).

Then \( X \sim t_\nu \).

In order to sidestep one of the square-root calculations, we can rewrite (c) as (c') Let \( C^2 = \frac{U^2}{W} \), \( R^2 = \nu(W^{-2/\nu} - 1) \), \( X = \sqrt{(R^2C^2)} \).

Thus, the only expensive steps in the polar algorithm are to calculate \( W^{-2/\nu} \) and one square root, whose sign should be chosen at random.

As Neave [6] pointed out in connection with the original Box-Muller algorithm, care must be taken if the uniform random numbers required by the method (\( G \) and \( H \) in Theorem 2) are in fact pseudorandom numbers generated by the congruential method, as is currently almost invariably the case in practice. Possible cures are surveyed in Golder and Settle [3]. In particular, one can simply use two congruential generators, to different moduli (the "two-sequence method"), to generate \( G \) and \( H \). Golder and Settle show this to be an effective cure.

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**Bibliography**


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