ON A THEOREM OF C. POSSE
CONCERNING GAUSSIAN QUADRATURE OF CHEBYSHEV TYPE

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Abstract. We consider \((n, m)\) Chebyshev formulae of algebraic degree \(m\) using \(n\) nodes. The aim of this short note is to show that by a simple algebraic method C. Posse's theorem concerning Gaussian quadrature of Chebyshev type can be improved. Furthermore, we given an application of this method to Gauss-Kronrod quadrature of Chebyshev type.

1. Introduction and statement of the main result

Let \(L\) be a (bounded) linear functional on \(C[a, b]\). We say that \(L\) admits an \((n, m)\) Chebyshev formula if there are \(n\) nodes \(x^\nu, \nu \in \mathbb{R}\) such that

\[
L[p_\mu] = \frac{L[p_0]}{n} \sum_{\nu=1}^{n} (x^\nu, n)^\mu \quad \text{for } \mu = 0, 1, \ldots, m,
\]

where, here and in the following, \(p_\mu\) denotes the monomial \(p_\mu(x) = x^\mu\).

Chebyshev formulae have been investigated for more than a hundred years (see, e.g., [6] and the references cited therein). They were first considered by Chebyshev [3], who showed that the linear functionals \(T_{\alpha, \beta, \gamma}\),

\[
T_{\alpha, \beta, \gamma}[f] := \alpha \int_{\beta}^{\gamma} \frac{f(x)}{\sqrt{|x - \beta||x - \gamma|}} \, dx, \quad \alpha, \beta, \gamma \in \mathbb{R},
\]

admit \((n, 2n - 1)\) Chebyshev formulae for each \(n \in \mathbb{N}\), i.e., that each Gaussian quadrature formula is of Chebyshev type. Let us additionally note that the linear functionals \(S^\eta, \xi\),

\[
S^\eta, \xi[f] := \eta f(\xi), \quad \eta, \xi \in \mathbb{R},
\]

trivially admit \((n, m)\) Chebyshev formulae for all \(n, m \in \mathbb{N}\). By a result of Posse [11], \(T_{\alpha, \beta, \gamma}\) and \(S^\eta, \xi\) are the only linear functionals on \(C[a, b]\) admitting \((n, 2n - 1)\) Chebyshev formulae for each \(n \in \mathbb{N}\). Recently, using methods of complex analysis and Faber polynomials, Peherstorfer [8] has proved the surprising result, that \(T_{\alpha, \beta, \gamma}\) and \(S^\eta, \xi\) also are the only (positive) linear functionals on \(C[a, b]\), admitting \((1, 1)\) and \((n, n + 1)\) Chebyshev formulae for each \(n \in \mathbb{N}\\{1\}\). For other improvements of Posse's result see, e.g., [6, 5, 9].

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The aim of this note is to show that by a simple algebraic method, introduced by Radau [12] and extended, e.g., in [2, 1, 4], we can obtain more general results. This method is based on Newton's identities (see, e.g., [10, p. 150 ff]), which, for \( n \) arbitrary complex numbers \( z_\nu \), yield

\[
\begin{align*}
    s_1 + a_1 &= 0, \\
    s_2 + a_1 s_1 + 2a_2 &= 0, \\
    &
\end{align*}
\]

(4)

\[
\begin{align*}
    s_{n-1} + a_1 s_{n-2} + \cdots + a_{n-2}s_1 + (n-1)a_{n-1} &= 0, \\
    s_{\lambda+n} + a_1 s_{\lambda+n-1} + \cdots + a_{n-1}s_{\lambda+1} + a_n s_\lambda &= 0 \quad (\lambda = 0, 1, 2, \ldots),
\end{align*}
\]

where

\[
s_\mu = \sum_{\nu=1}^{n} z_\nu^\mu \quad (\mu = 0, 1, 2, \ldots),
\]

(5)

\[
F(t) = (t - z_1)(t - z_2)\cdots(t - z_n) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n.
\]

Theorem. Let \((n_1)_{i=1}^\infty\) be a strictly increasing sequence of natural numbers. Let \(L\) and \(H\) be linear functionals on \(C[a,b]\), both admitting \((n_i, n_{i+1})\) Chebyshev formulae for each \(i \in \mathbb{N}\). If

\[
\text{(6)} \quad L[p_\nu] = H[p_\nu] \quad \text{for} \quad \mu = 0, 1, \ldots, n_i,
\]

then the identity \(L = H\) follows.

Proof. First, let \(L[p_0] \neq 0\). Applying Newton's identities to (1), we get for \(\nu = 1, 2, \ldots\)

\[
\text{(7)} \quad \sum_{\nu=1}^{n_i} (x_\nu^C, n_i)^\mu = \frac{n_i}{L[p_0]} L[p_\mu] \quad \text{for} \quad \mu = 0, 1, \ldots, n_{i+1}.
\]

For given \(L[p_0], L[p_1], \ldots, L[p_{n_i}]\) we directly obtain that the values of \(L[p_\mu]\) are uniquely determined for each \(\mu \in \{0, 1, 2, \ldots, n_{i+1}\}\). Using (6), we have \(L[p_\mu] = H[p_\mu]\) for each \(\mu \in \mathbb{N}_0\). Since \(L\) and \(H\) are bounded, by the approximation theorem of Weierstrass the result follows. If \(L[p_0] = 0\), then with (1) we have \(L = H \equiv 0\). \(\square\)

The following corollary extends the results on Chebyshev formulae mentioned in the introduction.

**Corollary 1.** Let \((n_1)_{i=1}^\infty\) be a strictly increasing sequence of natural numbers. Let \(L\) be a linear functional on \(C[a, b]\) admitting \((n_i, n_{i+1})\) Chebyshev formulae for each \(i \in \mathbb{N}\). Then there exist \(\eta, \xi \in \mathbb{R}\) or \(\alpha, \beta, \gamma \in \mathbb{R}\), such that, using the notations in (2) and (3),

(i) \(L = S_\eta, \xi\) if \(n_1 = 1\),

(ii) \(L = S_\eta, \xi\) or \(L = T_\alpha, \beta, \gamma\) if \(n_1 = 2\) and \(n_{i+1} < 2n_i\) for each \(i \in \mathbb{N}\).

Proof. As in the proof of the theorem, we can assume that \(L[p_0] \neq 0\). First, let \(L[p_2] = L[p_0]\). We define

\[
\text{(8)} \quad \eta = L[p_0], \quad \xi = L[p_1]/L[p_0].
\]
We have $L[p_\mu] = H[p_\mu]$ for $\mu = 0,1,2$; therefore, the result follows from the theorem. Now let $L[p_2]L[p_0] > L^2[p_1]$. We define

$$
\begin{align*}
\alpha &= \frac{L[p_0]}{\pi}, \\
\beta &= \frac{L[p_1] - \delta}{L[p_0]}, \\
\gamma &= \frac{L[p_1] + \delta}{L[p_0]}, \\
\delta &= \frac{1}{2}(L[p_2]L[p_0] - L^2[p_1])^{1/2}.
\end{align*}
$$

A short calculation, using

$$
H[f] = \alpha \int_0^1 f \left( \frac{\gamma - \beta}{2} y + \frac{\gamma + \beta}{2} \right) (1 - y^2)^{-1/2} dy,
$$

gives $L[p_\mu] = H[p_\mu]$ for $\mu = 0,1,2$. Since $H$ admits $(n, 2n-1)$ Chebyshev formulae for each $n \in \mathbb{N}$, the result follows from the theorem. Finally, let $L[p_2]L[p_0] < L^2[p_1]$. This inequality is equivalent to the inequality

$$
(x_{1,2}^C)^2 + (x_{2,2}^C)^2 < (x_{1,2}^C + x_{2,2}^C)^2/2,
$$

which is impossible for real numbers. □

Remarks. 1. We say that a linear functional $L$ on $C[a, b]$ admits extended $(n, m)$ Chebyshev formulae if $x_{\nu,n}^C \in \mathbb{C}$, $x_{\nu,n}^C$ real or complex conjugate, and (1) holds. Newton's identities (4) are valid for $z_{\nu} \in \mathbb{C}$, and therefore the theorem is also valid for extended Chebyshev formulae.

2. In the proof of Corollary 1, we see that (11) is possible if and only if $x_{1,2}^C, x_{2,2}^C \in \mathbb{C}$, $x_{1,2}^C = u + iv$, $x_{2,2}^C = u - iv$, and $v \neq 0$. We obtain $L[p_1] = uL[p_0]$, $L[p_2] = (u^2 - v^2)L[p_0]$. Defining $\hat{f} := (p_1 - up_0)^2 + v^2p_0/2$, we have

$$
L[\hat{f}] = -\frac{1}{2}v^2L[p_0], \quad \hat{f} > 0,
$$

which is impossible if $L[p_0]L$ is a positive functional. Therefore, for positive functionals $L$, Corollary 1 also is valid for extended Chebyshev formulae. For $n_\mu = i + 1$ this has been proved in [8].

3. In Corollary 1(ii) the assumption $n_{i+1} < 2n_i$ cannot be omitted: consider the functional $S_{n_1,n_2}$ defined by $S_{n_1,n_2}[f] = \frac{1}{2}[f(\xi_1) + f(\xi_2)]$, which admits $(2n, m)$ Chebyshev formulae for all $n, m \in \mathbb{N}$.

4. In Corollary 1, we only have considered $n_1 = 1$ or $n_1 = 2$. For $n_1 > 2$, using the method described, we are also able to investigate linear functionals admitting (extended) Chebyshev formulae. The author intends to state such results and further applications in a forthcoming paper.

2. Application to Chebyshev formulae having preassigned nodes

The above method is also helpful if some of the nodes $x_{\nu,n}^C$ of the Chebyshev formulae are preassigned. As an example, we assume that the linear functional $L$ also admits a Gaussian formula of order $k$; i.e., there exist $k$ nodes $x_{\nu,k}^G$ and $k$ weights $a_{\nu,k}^G \in \mathbb{R}$ such that

$$
L[p_\mu] = \sum_{\nu=1}^k a_{\nu,k}^G (x_{\nu,k}^G)\mu \quad \text{for } \mu = 0, 1, \ldots, 2k - 1.
$$
If \( L[p_0] \neq 0 \), then a Gaussian formula of order 1 trivially always exists and is uniquely determined by

\[
(14) \quad x_{1,1}^G = L[p_1]/L[p_0].
\]

If there exists also a \((3,2)\) Chebyshev formula such that \( x_{1,1}^G \) is one of its nodes, then a short calculation using (13) and (1) shows that

\[
\begin{align*}
x_{1,3}^C & := x_{1,1}^G, \quad x_{2,3}^C = x_{1,3}^C - \sqrt{3/2} \psi, \quad x_{3,3}^C = x_{1,3}^C + \sqrt{3/2} \psi, \\
\psi & = \frac{1}{L[p_0]} \sqrt{L[p_0]L[p_2] - L^2[p_1]}.
\end{align*}
\]

Therefore, it is necessary and sufficient for the existence of such a \((3,2)\) Chebyshev formula that

\[
(15) \quad L[p_0]L[p_2] > L^2[p_1].
\]

Furthermore, if this \((3,2)\) Chebyshev formula is also a \((3,3)\) Chebyshev formula, then it follows that

\[
(16) \quad L[p_3] = \frac{L[p_1]}{L^2[p_0]} (3L[p_0]L[p_2] - 2L^2[p_1]).
\]

Inequality (16) is equivalent to the existence of a \((2,2)\) Chebyshev formula,

\[
(17) \quad x_{1,2}^C = x_{1,1}^G - \psi, \quad x_{2,2}^C = x_{1,1}^G + \psi,
\]

where \( \psi \) is defined in (15). A further short calculation shows that this \((2,2)\) Chebyshev formula is also a \((2,3)\) Chebyshev formula, i.e., a Gaussian formula of order 2, if and only if (17) additionally holds. Therefore, the existence of a \((3,3)\) Chebyshev formula using the node \( x_{1,1}^G \) is equivalent to the existence of a \((2,3)\) Chebyshev formula.

Now, one may ask, e.g., if there exists an \((n,m)\) Chebyshev formula having some of the nodes \( x_{k,n}^G \) of a Gaussian formula of order \( k \). In this situation, using methods from the theory of orthogonal polynomials, Notaris [7], for positive \( L \), has recently proved the following interesting result: If, for each \( n \in \mathbb{N} \), there exists a \((2n + 1, 3n + 1)\) Chebyshev formula with \( \{x_{n,\nu}^C, \nu = 1, 2, \ldots, n\} \subseteq \{x_{n,\nu}^G, \nu = 1, 2, \ldots, 2n + 1\} \)—i.e., these Chebyshev formulae are so-called Gauss-Kronrod formulae—then \( L \) is of type \( S_{\eta,\xi} \) being defined in (3). The following Corollary 2 extends this result.

**Corollary 2.** Let \( L \) be a linear functional admitting a \((3,4)\) Chebyshev formula and \((2n + 1, 2n + 3)\) Chebyshev formulae for each \( n \in \mathbb{N} \setminus \{1\} \). For \( L[p_0] \neq 0 \), let the node \( x_{1,1}^G \) of the Gaussian formula of order 1 be a node of the \((3,4)\) Chebyshev formula, and let the two nodes \( x_{1,2}^G, x_{2,2}^G \) of the Gaussian formula of order 2 be nodes of the \((5,7)\) Chebyshev formula. Then, there exist \( \eta, \xi \in \mathbb{R} \) such that \( L = S_{\eta,\xi} \).

**Proof.** Let \( L[p_0] \neq 0 \). The existence of a \((3,3)\) Chebyshev formula using the node \( x_{1,1}^G \) implies that the Gaussian formula of order 2 is a \((2,3)\) Chebyshev formula; see (17) and (18) above. If there exists a \((5,3)\) Chebyshev formula having nodes \( x_{1,5}^C := x_{1,2}^G = x_{1,1}^G \) and \( x_{2,5}^C := x_{2,2}^G = x_{2,2}^C \), then a short
calculation using

\[ L[p_\mu] = \frac{L[p_0]}{5} \left\{ \sum_{\nu=1}^{2} (x_{\nu,2}^C)^\mu + \sum_{\nu=3}^{5} (x_{\nu,5}^C)^\mu \right\} \]

(19)

\[ = \frac{L[p_0]}{5} \left\{ 2 \frac{L[p_\mu]}{L[p_0]} + \sum_{\nu=3}^{5} (x_{\nu,5}^C)^\mu \right\} \quad \text{for } \mu = 0, 1, 2, 3 \]

shows that \{x_{3,5}^C, x_{4,5}^C, x_{5,5}^C\} = \{x_{1,3}^C, x_{2,3}^C, x_{3,3}^C\}, and therefore, since this (5, 3) Chebyshev formula is also a (5, 4) Chebyshev formula, it follows that

(20)

\[ L[p_4] = \frac{L[p_0]}{5} \left\{ \sum_{\nu=1}^{3} (x_{\nu,3}^C)^4 + \sum_{\nu=1}^{2} (x_{\nu,2}^C)^4 \right\} . \]

On the other hand, the existence of a (3, 4) Chebyshev formula having a node \( x_{1,3}^C = x_{1,1}^G \) yields the identity

(21)

\[ L[p_4] = \frac{\sum_{\nu=1}^{3} (x_{\nu,3}^C)^4}{3} . \]

By comparing (20) and (21), it follows from (15) and (18) that

(22)

\[ L_2[p_0] = L[p_0]L[p_2] - L^2[p_1] = 0. \]

Therefore, all nodes \( x_{1,3}^C \) and \( x_{4,5}^C \) are equal to \( x_{1,1}^G \). Since the (5, 4) Chebyshev formula is also a (5, 7) Chebyshev formula, we have \( L[p_\mu] = S_\eta,\xi[p_\mu] \) for \( \mu = 0, 1, \ldots, 7 \), where \( \eta \) and \( \xi \) are given in (8). The result now follows from the theorem. If \( L[p_0] = 0 \), then with (1) we have \( L \equiv 0 \), which is fulfilled for \( S_\eta,\xi \) with \( \eta = 0 \).

By Corollary 1 it follows that, if there exists a (2, 3) Chebyshev formula and \( (2n + 1, 2n + 3) \) Chebyshev formulae for each \( n \in \mathbb{N} \), then \( L \) is of type \( S_\eta,\xi \) or \( T_{\alpha,\beta,\gamma} \). For \( (2n + 1, 2n + 2) \) Chebyshev formulae, using the above method, we have the following result.

**Corollary 3.** Let \( L \) be a linear functional admitting \( (2n + 1, 2n + 2) \) Chebyshev formulae for each \( n \in \mathbb{N} \). For \( L[p_0] \neq 0 \) let \( x_{1,1}^G = L[p_1]/L[p_0] \) be a node of each of these \( (2n + 1, 2n + 2) \) Chebyshev formulae. Then there exist \( \eta, \xi \in \mathbb{R} \) or \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( L = S_\eta,\xi \) or \( L = T_{\alpha,\beta,\gamma} \).

**Proof.** Let \( L[p_0] \neq 0 \). Since \( L \) admits a (3, 2) Chebyshev formula having the node \( x_{1,3}^C := x_{1,1}^G \), the relations (15) imply that \( L[p_0]L[p_2] \geq L^2[p_1] \). Therefore, there exists a functional \( H \) of type \( S_\eta,\xi \) or \( T_{\alpha,\beta,\gamma} \) such that \( H[p_\mu] = L[p_\mu] \) for \( \mu = 0, 1, 2 \)—see the proof of Corollary 1. Now assume that, for given \( n \in \mathbb{N} \),

(23)

\[ H[p_\mu] = L[p_\mu] \quad \text{for } \mu = 0, 1, \ldots, 2n. \]

In the following, by \( x_{\nu,n}^CH \) we denote the nodes of the \( (n, 2n - 1) \) Chebyshev formula of \( H \). Using (1), we have

(24)

\[ \sum_{\nu=1}^{2n+1} (x_{\nu,2n+1}^C)^\mu = (2n + 1) \frac{L[p_\mu]}{L[p_0]} \quad \text{for } \mu = 0, 1, \ldots, 2n + 2. \]
From (23) it follows that the nodes $x_{1,2n+1}^C, x_{2,2n+1}^C, \ldots, x_{2n+1,2n+1}^C$ are also the nodes of a $(2n+1, 2n)$ Chebyshev formula of $H$. Let $F_{2n+1}^C$ and $F_{2n+1}^{CH}$ be the polynomials

$$F_{2n+1}^C(t) = \prod_{\nu=1}^{2n+1} (t - x_{\nu,2n+1}^C), \quad F_{2n+1}^{CH}(t) = \prod_{\nu=1}^{2n+1} (t - x_{\nu,2n+1}^{CH}).$$

Newton's identities (4) and (5) now show that $F_{2n+1}^C(t) - F_{2n+1}^{CH}(t)$ is equal to a fixed constant $c$ for all $t \in \mathbb{R}$. Since $x_{1,1}^G$ is a zero of $F_{2n+1}^{CH}$ and since $x_{1,1}^G$ is also a zero of $F_{2n+1}^C$, it follows that $F_{2n+1}^C \equiv F_{2n+1}^{CH}$. Therefore, (24) yields $H[p_\mu] = L[p_\mu]$ for $\mu = 0, 1, 2, \ldots, 2n+2$. Now, by induction, we have $H[p_\mu] = L[p_\mu]$ for each $\mu \in \mathbb{N}$, which, using the approximation theorem of Weierstrass, yields $L = H$. □

**Bibliography**


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