NUMERICAL EVALUATION OF SOME TRIGONOMETRIC SERIES

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Abstract. We present a method for the accurate numerical evaluation of a family of trigonometric series arising in the design of special-purpose quadrature rules for boundary element methods. The series converge rather slowly, but can be expressed in terms of Fourier-Chebyshev series that converge rapidly.

Let \( r \) be a positive integer, and define the functions \( G_r \) and \( H_r \) by

\[
G_r(t) = 2 \sum_{m=1}^{\infty} \frac{1}{mr} \cos 2\pi mt,
\]
\[
H_r(t) = 2 \sum_{m=1}^{\infty} \frac{1}{mr} \sin 2\pi mt.
\]

This note describes an efficient method for evaluating \( G_r \) and \( H_r \) to high accuracy. When \( r \) is small, direct evaluation of the trigonometric series is impractical because they converge too slowly.

The author's interest in \( G_r \) and \( H_r \) stems from their role in the analysis and design of numerical integration techniques for boundary element methods. That application leads to systems of nonlinear equations involving \( G_r \) and \( H_r \) (for small values of \( r \)), the solutions of which yield the weights and integration points of nonstandard, Gauss-like rules. See Chandler and Sloan [4, §5] or the survey article [11, §7] for a particularly simple example, where the integration points are the solutions of just a single equation, \( G_r(t) = 0 \). Brown et al. [3] discuss some important analytical properties of \( G_r \) and \( H_r \).

Other authors have considered the numerical evaluation of certain closely related trigonometric series, arising from plate contact problems. For some methods quite different from the one presented here, see Boersma and Dempsey [2] and papers cited therein.

To evaluate \( G_r \) and \( H_r \), it suffices to evaluate

\[
C_r(t) = -2 \sum_{m=1}^{\infty} \frac{1}{mr} \cos(2\pi mt - r\pi/2),
\]
\[
S_r(t) = -2 \sum_{m=1}^{\infty} \frac{1}{mr} \sin(2\pi mt - r\pi/2),
\]

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because for $k \geq 1$,
\[
G_{2k-1}(t) = (-1)^{k+1}S_{2k-1}(t), \quad G_{2k}(t) = (-1)^{k+1}C_{2k}(t), \\
H_{2k-1}(t) = (-1)^kC_{2k-1}(t), \quad H_{2k}(t) = (-1)^kS_{2k}(t).
\]

We shall work with $C_r$ and $S_r$ instead of with $G_r$ and $H_r$ because the relations
\[
C'_r(t) = 2\pi C_{r-1}(t) \quad \text{and} \quad S'_r(t) = 2\pi S_{r-1}(t)
\]
are more convenient than the corresponding ones involving $G'_r$, $H'_r$, $G_{r-1}$, and $H_{r-1}$.

An elementary calculation reveals that
\[
C_1(t) = 2\pi(t - \frac{1}{2}) \quad \text{for} \quad 0 < t < 1,
\]
so for all $r \geq 1$ the restriction of $C_r$ to the unit interval is just a polynomial of degree $r$. In fact,
\[
C_r(t) = \frac{(2\pi)^r}{r!} B_r(t) \quad \text{for} \quad 0 < t < 1,
\]
where $B_r$ is the Bernoulli polynomial of degree $r$; cf. [1, p. 805]. Thus, numerical evaluation of $H_1$, $G_2$, $H_3$, $G_4$, ... presents no difficulties, but dealing with the remaining cases $G_1$, $H_2$, $G_3$, $H_4$, ... requires an efficient method of computing $S_r$. When $r = 1$ there is a convenient, closed form,
\[
S_1(t) = -2 \log |2 \sin \pi t|;
\]
however, in general, $S_r$ is a rescaled version of the Clausen function of order $r$:
\[
S_r(t) = (-1)^{k+1}2 \text{Cl}_r(2t) \quad \text{for} \quad r = 2k - 1 \text{ or } 2k,
\]
following the notation of Lewin [9]. Our strategy for computing $S_2$, $S_3$, ... uses (2) and the second formula in (1).

In order to split $S_r$ into a sum of singular and regular terms, we define a function $\Phi_r$ on the interval $[-1, 1]$, by writing
\[
S_r(t) = -2 \frac{(2\pi)^{r-1}}{(r-1)!}[t^{r-1}\log t + (1-t)^{r-1}\log(1-t)] + \Phi_r(2t-1)
\]
for $0 < t < 1$. The problem now reduces to evaluating $\Phi_r$. When $r = 1$, the closed form (2) implies that
\[
\Phi_1(x) = \begin{cases} -2 \log \frac{8 \cos(\pi x/2)}{1 - x^2} & \text{if } -1 < x < 1, \\ -2 \log 2\pi & \text{if } x = \pm 1, \end{cases}
\]
and by differentiating (3) and using (1), we find that
\[
\Phi'_r(x) = \pi[\Phi_{r-1}(x) + Q_{r-2}(x)] \quad \text{for} \quad r \geq 2,
\]
where
\[
Q_r(x) = \frac{\pi^r}{(r+1)!}[(1+x)^r + (-1)^r(1-x)^r] \quad \text{for} \quad r \geq 0.
\]
We generate $\Phi_2$, $\Phi_3$, $\Phi_4$, ... by repeated integration of $\Phi_1$, using the formula (5). The constants of integration follow from the relation
\[
\int_0^1 S_r(t) \, dt = 0 \quad \text{for} \quad r \geq 1.
\]
and the numerical calculations are easily performed using the method of Clenshaw and Curtis [6], as we now demonstrate.

Let \( T_k \) denote the first-kind Chebyshev polynomial of degree \( k \), i.e.,
\[
T_k(\cos \theta) = \cos k\theta,
\]
and put
\[
a_{rk} = \frac{2}{\pi} \int_{-1}^{1} \phi_r(x) T_k(x) \frac{dx}{\sqrt{1-x^2}} \quad \text{for } k \geq 0,
\]
so that
\[
\Phi_r(x) = \sum_{k=0}^{\infty} a_{rk} T_k(x) = \frac{a_{r0}}{2} T_0(x) + \sum_{k=1}^{\infty} a_{rk} T_k(x).
\]
(The prime on the summation sign indicates that the coefficient of \( T_0(x) \) is multiplied by \( 1/2 \).) We see from (4) that \( \phi_1 \), and hence \( \Phi_r \) for all \( r \geq 1 \), has an analytic continuation to the strip \(-3 < \text{Re} \ z < 3\). In particular, if
\[
1 < \rho < 3 + 2\sqrt{2},
\]
then \( \Phi_r \) is analytic inside and on the ellipse \( \{ (z + z^{-1})/2 : |z| = \rho \} \), and therefore
\[
|a_{rk}| \leq \frac{\text{const}_r \rho^k}{\rho^k} \quad \text{for } k \geq 0;
\]
see Rivlin [10, p. 143]. This estimate shows that the series (7) converges rapidly, and so is suitable for numerical evaluation of \( \Phi_r \). Another attractive feature of the representation (7) is that, because of the parity properties
\[
S_r(1-t) = (-1)^{r+1} S_r(t), \quad \Phi_r(-x) = (-1)^{r+1} \Phi_r(x), \quad Q_r(-x) = (-1)^r Q_r(x),
\]
half of the coefficients vanish:
\[
a_{rk} = 0 \quad \text{if } r \text{ is even} \quad \text{and } k \text{ is odd}.
\]
Thus, \( \Phi_r \) can be evaluated using standard methods for the summation of even and odd Fourier-Chebyshev series; see, e.g., Clenshaw [5, pp. 9–10]. We shall also require the expansion
\[
Q_r(x) = \sum_{k=0}^{\infty} b_{rk} T_k(x) = \sum_{k=0}^{r} b_{rk} T_k(x),
\]
in which \( b_{rk} = 0 \) for \( k > r \) because \( Q_r \) is a polynomial of degree \( r \).

To compute \( a_{rk} \) when \( r = 1 \), we apply Gauss-Chebyshev quadrature, defining
\[
a_{1k}^{(N)} = \frac{2}{N} \sum_{n=1}^{N} \phi_1(x_n^{(N)}) T_k(x_n^{(N)}) \quad \text{where } x_n^{(N)} = \cos \left( \frac{2(n-1)\pi}{2N} \right),
\]
and using the explicit formula (4) for \( \phi_1 \). By [10, equation (3.58)],
\[
a_{1k}^{(N)} = a_{1k} + \sum_{p=1}^{\infty} (-1)^p (a_1, 2p, N-k + a_1, 2p, N+k) \quad \text{for } 0 \leq k < N,
\]
so the estimate (8) implies that

\[ |a_{1k}^{(N)} - a_{1k}| \leq \frac{\text{const}_\rho}{\rho^{2N-k}} \text{ for } N > k \geq 0. \]

It follows from (5) that the Chebyshev coefficients of \( \Phi_r, \Phi_{r-1}, \) and \( Q_{r-2} \) satisfy

\[ a_{rk} = \frac{\pi}{2k} [a_{r-1,k-1} + b_{r-2,k-1} - a_{r-1,k+1} - b_{r-2,k+1}] \text{ for } k \geq 1 \text{ and } r \geq 2; \]

cf. [5] or [6] or [7, p. 59]. The case \( k = 0 \) is handled by observing that (3) and (6) imply

\[ a_{r0} = \begin{cases} 0 & \text{if } r \text{ is even,} \\ -\frac{8}{r} \frac{(2\pi)^{r-1}}{r!} + 2 \sum_{k=1}^{\infty} \frac{a_{r2k}}{4k^2-1} & \text{if } r \text{ is odd.} \end{cases} \]

The coefficients \( b_{rk} \) can be evaluated in closed form: after using the parity property \( T_k(-x) = (-1)^k T_k(x) \) and the substitution \( x = \cos \theta \), we find with the help of Gradshteyn and Ryzhik [8, p. 372, formula 9] that

\[ b_{rk} = \pi^r \frac{1 + (-1)^{r+k}}{2^{r-2}} \frac{(2r)!}{(r+1)!(r+k)!(r-k)!} \text{ for } 0 \leq k \leq r. \]

**Table 1.** Chebyshev coefficients of \( \Phi_r \) for \( 1 \leq r \leq 6 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a_{1k} )</th>
<th>( a_{3k} )</th>
<th>( a_{5k} )</th>
</tr>
</thead>
<tbody>
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<td>-14.38120 33723 33972</td>
<td>-15.40706 70937 28069</td>
</tr>
<tr>
<td>2</td>
<td>0.24152 87859 01736</td>
<td>4.73732 65271 83045</td>
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<tr>
<td>4</td>
<td>0.00203 43769 84711</td>
<td>0.04899 74836 10452</td>
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</tr>
<tr>
<td>6</td>
<td>0.00003 56715 82236</td>
<td>0.00016 23361 08813</td>
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<tr>
<td>8</td>
<td>0.00000 07625 83158</td>
<td>0.00000 15125 91966</td>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
<td>18</td>
<td>0.00000 00000 00007</td>
<td>0.00000 00000 00002</td>
<td>0.00000 00000 00001</td>
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</table>

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a_{2k} )</th>
<th>( a_{4k} )</th>
<th>( a_{6k} )</th>
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<tbody>
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<td>1</td>
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<td>-4.19275 26396 29885</td>
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<tr>
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<td>0.12539 89792 71593</td>
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<td>0.01534 20140 62813</td>
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<td>9</td>
<td>0.00000 01299 93449</td>
<td>0.0000 02605 01145</td>
<td>0.0000 12207 48844</td>
</tr>
<tr>
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<td>0.0000 00000 00000</td>
<td>0.0000 00000 00000</td>
</tr>
</tbody>
</table>
Table 1 lists the nonzero coefficients $a_{rk}$ for $1 \leq r \leq 6$. The calculations were performed as described above in 25-digit, decimal arithmetic using MAPLE, and the results then rounded to 15 decimal places. In the case $r = 1$, the coefficients were obtained by computing $a_{1k}^{(N)}$ with $N = 24$. From the behavior of $a_{1k}^{(N)}$ for $N$ in the range $k < N \leq 30$, our values of $a_{1k}$ appeared correct to about the 22nd decimal place before being rounded.

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Bibliography


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