ON THE COMPUTATION OF BATTLE-LEMARIÉ'S WAVELETS

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Abstract. We propose a matrix approach to the computation of Battle-Lemarié's wavelets. The Fourier transform of the scaling function is the product of the inverse $F(x)$ of a square root of a positive trigonometric polynomial and the Fourier transform of a B-spline of order $m$. The polynomial is the symbol of a bi-infinite matrix $B$ associated with a B-spline of order $2m$. We approximate this bi-infinite matrix $B_{2m}$ by its finite section $A_N$, a square matrix of finite order. We use $A_N$ to compute an approximation $x_N$ of $x$ whose discrete Fourier transform is $F(x)$. We show that $x_N$ converges pointwise to $x$ exponentially fast. This gives a feasible method to compute the scaling function for any given tolerance. Similarly, this method can be used to compute the wavelets.

1. Introduction

Battle-Lemarié's wavelets [1, 3] may be constructed by using a multiresolution approximation built from polynomial splines of order $m > 0$. See, e.g., [4] or [2]. To be precise, let $V_0$ be the vector space of all functions of $L^2(\mathbb{R})$ which are $m - 2$ times continuously differentiable and equal to a polynomial of degree $m - 1$ on each interval $[n + m/2, n + 1 + m/2]$ for all $n \in \mathbb{Z}$. Define the other resolution space $V_k$ by

$$V_k := \{u(2^k t) : u \in V_0\}, \quad \forall k \in \mathbb{Z}.\$$

It is known that $\{V_k\}_{k \in \mathbb{Z}}$ provide a multiresolution approximation, and there exists a unique scaling function $\varphi$ such that

$$V_k = \text{span}_{L^2}\{2^{k/2}\varphi(2^k t - n) : n \in \mathbb{Z}\}$$

for all $k$, and the integer translates of $\varphi$ are orthonormal to each other. (See, e.g., [4].) Define a transfer function $H(\omega)$ by

$$H(\omega) = \frac{\hat{\varphi}(2\omega)}{\hat{\varphi}(\omega)},$$

where $\hat{\varphi}$ denotes the Fourier transform of $\varphi$. Then the wavelet $\psi$ associated with the scaling function $\varphi$ is given in terms of its Fourier transform by

$$\psi(\omega) = e^{-j\omega/2}H(\omega/2 + \pi)\hat{\varphi}(\omega/2).$$

Received by the editor May 14, 1993.
1991 Mathematics Subject Classification. Primary 41A15, 41A30, 42C15, 47B35.
Key words and phrases. B-spline, bi-infinite matrices, exponential decay, finite section, positive operator, Toeplitz matrix, wavelet.

Research supported by the National Science Foundation under Grant #DMS-9303121.
Here and throughout, \( j := \sqrt{-1} \). The scaling function \( \varphi \) associated with the multiresolution approximation may be given by

\[
\hat{\varphi}(\omega) = \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} |\hat{B}_m(\omega + 2k\pi)|^2}} \hat{B}_m(\omega),
\]

where \( B_m \) is the well-known central B-spline of order \( m \) whose Fourier transform is given by

\[
\hat{B}_m(\omega) = \left( \frac{\sin \omega/2}{\omega/2} \right)^m.
\]

By using Poisson's summation formula, we have

\[
\hat{\varphi}(\omega) = \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}} \hat{B}_m(\omega).
\]

Thus, the transfer function is

\[
H(\omega) = \sqrt{\frac{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}} (\cos \omega/2)^m.
\]

Then the wavelet \( \psi \) associated with \( \varphi \) is given by

\[
\psi(\omega) = e^{-j\omega/2}H(\omega/2 + \pi) \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega/2}}} \hat{B}_m(\omega/2).
\]

The above Fourier transforms of \( \varphi \), \( H \), and \( \psi \) suggest that the scaling function, transfer function, and wavelet have the following representations:

\[
\varphi(t) = \sum_{k \in \mathbb{Z}} \alpha_k B_m(t-k),
\]
\[
H(\omega) = \sum_{k \in \mathbb{Z}} \beta_k e^{-jk\omega},
\]
\[
\psi(t) = \sum_{k \in \mathbb{Z}} \gamma_k B_m(2t-k).
\]

In this paper, we propose a matrix method to compute the \( \alpha_k \)'s, \( \beta_k \)'s, and \( \gamma_k \)'s. Let us use \( \varphi \) to illustrate our method as follows: view \( \sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega} \) as the symbol of a bi-infinite matrix \( B_{2m} = (b_{ik})_{i,k \in \mathbb{Z}} \) with \( b_{i,k} = b_{0,k-i} = B_{2m}(k-i) \) for all \( i, k \in \mathbb{Z} \). Similarly, \( \sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}} \) can be viewed as the symbol of another (unknown) bi-infinite matrix \( C_{2m} \). Then it is easy to see that

\[
C_{2m}^2 = B_{2m}.
\]

To find \( \sum_{k \in \mathbb{Z}} \alpha_k e^{-jk\omega} = 1 \) is equivalent to solving

\[
C_{2m}x = \delta
\]

with \( \delta = (\delta_i)_{i \in \mathbb{Z}}, \delta_0 = 1, \) and \( \delta_i = 0 \) for all \( i \in \mathbb{Z}\setminus\{0\} \), where \( x = (\alpha_k)_{k \in \mathbb{Z}} \). Our numerical method is to find an approximation to \( x \) within a given tolerance.
Let $A_N = (b_{ik})_{-N \leq i,k \leq N}$ be a finite section of $B_{2m}$. Note that $A_N$ is symmetric and totally positive. Thus, we can find $\widehat{P}_N$ such that

$$\widehat{P}_N^2 = A_N$$

by using, e.g., the singular value decomposition. Then we solve $\widehat{P}_N x_N = \delta_N$ with $\delta_N$ a vector of $2N+1$ components which are all zeros except for the middle one, which is 1. We can show that $x_N$ converges pointwise to $x$ exponentially fast. Similarly, we can use this idea to compute an approximation of $\{\beta_k\}_{k \in \mathbb{Z}}$ by (2) and $\{\gamma_k\}_{k \in \mathbb{Z}}$ by (3). Therefore, the discussion mentioned above furnishes a numerical method to compute Battle-Lemarié's wavelet.

To prove the convergence of $x_N$ to $x$, we place ourselves in a more general setting. We study a general bi-infinite matrix $A$. (For the case of Battle-Lemarié's wavelets, $A = B_{2m}$.) We look for certain conditions on $A$ such that the solution $x_N$ of $\widehat{P}_N x_N = \delta_N$ with $\widehat{P}_N^2 = A_N$ converges to the solution $x$ of $P x = \delta$ with $P^2 = A$, where $A_N$ is a finite section of $A$. This is discussed in the next section. In the last section, we show that the bi-infinite matrix $B_{2m}$ satisfies the conditions on $A$ obtained in §2. This will establish our numerical method for computing Battle-Lemarié's wavelets.

2. Main results

Let $\mathbb{Z}$ be the set of all integers. Let $l^2 := l^2(\mathbb{Z})$ be the space of all square summable sequences with indices in $\mathbb{Z}$. That is,

$$l^2(\mathbb{Z}) = \left\{ (\ldots, x_{-1}, x_0, x_1, \ldots) : \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty \right\}.$$

It is known that $l^2$ is a Hilbert space. We shall use $x$ to denote each vector in $l^2$ and use $A$ to denote a linear operator from $l^2$ to $l^2$. It is known that $A$ can be expressed as a bi-infinite matrix. Thus, we shall write $A = (a_{ik})_{i,k \in \mathbb{Z}}$.

In the following, we shall consider $A$ to be a banded and/or Toeplitz matrix. That is, $A$ is said to be banded if there exists a positive integer $b$ such that $a_{ik} = 0$ whenever $|i-k| > b$. The matrix $A$ is said to be Toeplitz if $a_{i+k,m+k} = a_{i,m}$ for all $i, k, m \in \mathbb{Z}$. Denote by $F(x)(\omega)$ the symbol of a vector $x \in l^2$, i.e.,

$$F(x)(\omega) = \sum_{i \in \mathbb{Z}} x_i e^{-j\omega i}.$$

Denote by $F(A)(\omega)$ the symbol of a Toeplitz matrix $A = (a_{ik})_{i,k \in \mathbb{Z}}$, i.e.,

$$F(A)(\omega) = \sum_{i \in \mathbb{Z}} a_{i,0} e^{-j\omega i}.$$

Suppose that $F(A)(\omega) \neq 0$ and $\sum_{i \in \mathbb{Z}} |a_{i,0}| < \infty$. It is known from the well-known Wiener's theorem that there exists a sequence $x$ such that

$$\frac{1}{F(A)(\omega)} = \sum_{k \in \mathbb{Z}} x_k e^{-jk\omega}$$

with $\sum_k |x_k| < \infty$. It is easy to see that to find this sequence $x$ is equivalent to solving the linear system of bi-infinite order:

$$Ax = \delta,$$
where $\delta = (\ldots, \delta_{-1}, \delta_0, \delta_1, \ldots)^t$ with $\delta_0 = 1$ and $\delta_i = 0$ for all $i \in \mathbb{Z}\setminus\{0\}$.

Furthermore, if the matrix $A$ is a positive operator, then there exists a unique positive square root $P$ of $A$. That is, $P^2 = A$. The symbol representation is $F(P)(\omega) = \sqrt{F(A)(\omega)}$. To find $F(P)(\omega)$ is equivalent to finding a matrix $P$ such that $P^2 = A$.

Certainly, we cannot solve a linear system of bi-infinite order. Neither can we decompose a matrix of bi-infinite order into two matrices of bi-infinite order. However, we can do this approximatively. Let $N$ be a positive integer, and let $A_N = (a_{ik})_{-N \leq i, k \leq N}$ be a finite section of $A$. Let $I_{N,\infty} = (0, I_{2N+1,2N+1}, 0)$ be a matrix of $2N + 1$ rows and bi-infinite columns with $I_{2N+1,2N+1}$ being the identity matrix of size $(2N + 1) \times (2N + 1)$ such that

$$A_N = I_{N,\infty} A I_{N,\infty}^t.$$

Denote $\delta_N = I_{N,\infty} \delta$ and $x_N = I_{N,\infty} x$. Then we shall solve the following linear system:

$$A_N \hat{x}_N = \delta_N.$$

We claim that $\hat{x}_N$ converges to $x$ exponentially fast as $N$ increases to $\infty$, under certain conditions on $A$. Furthermore, we shall solve $\hat{P}_N^2 = A_N$ for $\hat{P}_N$ by using the singular value decomposition. Once we have $\hat{P}_N$, we shall solve

$$\hat{P}_N \hat{y}_N = \delta_N.$$

We claim that $\hat{y}_N$ converges to $y$ exponentially fast as $N \to \infty$, provided $A$ satisfies certain conditions.

To check the conditions on $A$, we need the following definition.

**Definition.** A matrix $A = (a_{ik})_{i, k \in \mathbb{Z}}$ is said to be of exponential decay off its diagonal if

$$|a_{ik}| \leq K r^{|i-k|}$$

for some constant $K$ and $r \in (0, 1)$.

We begin with an elementary lemma.

**Lemma 1.** Suppose that $A$ is of exponential decay off its diagonal and has a bounded inverse. Suppose that $A_N^{-1} = (\hat{a}_{ik})_{-N \leq i, k \leq N}$ satisfies the property that

$$|\hat{a}_{i,k}(N)| \leq K r^{|i-k|}, \quad \forall -N \leq i, k \leq N,$$

for all $N > 0$. Then there exists $r_1 \in (0, 1)$ and a constant $K_1$ such that

$$\|I_{N,\infty} x - \hat{x}_N\|_2 \leq K_1 r_1^N,$$

where $x$ is the solution of $Ax = \delta$ and $x_N$ is the solution of $A_N x_N = \delta_N$.

**Proof.** From the assumption of the lemma, there exist $K$ and $r \in (0, 1)$ such that $A = (a_{ik})_{i, k \in \mathbb{Z}}$ and $A_N^{-1} = (\hat{a}_{ik}(N))_{-N \leq i, k \leq N}$ satisfy

$$|a_{ik}| \leq K r^{|i-k|} \quad \text{and} \quad |\hat{a}_{ik}| \leq K r^{|i-k|}.$$

Write

$$AI_{N,\infty}^t = \begin{bmatrix} B \\ A_N \\ C \end{bmatrix} \quad \text{and} \quad d = BA_N^{-1} \delta_N \quad \text{with} \quad d = (\ldots, d_{-N-1}, d_{-N})^t.$$
Then we have, for each $i = -\infty, \ldots, -N - 1, -N$,

$$|d_i| = \left| \sum_{k=-N}^{N} a_{ik} \delta_{k,0}(N) \right| \leq K^2 \sum_{k=-N}^{N} r^{|i-k|} \left| p^{|k|} \right|$$

$$= K^2 \left( r^{|i|} \sum_{k=0}^{N} r^{2k} + Nr^{-i} \right) \leq C \lambda^{-i}$$

for some constant $C$ and $\lambda \in (0, 1)$. Thus, $\|BA^{-1}\delta_N\|_2 \leq C'\lambda^N$. Similarly, $\|CA^{-1}_N\delta_N\|_2 \leq C'\lambda^N$. Hence,

$$\|I_N,_{\infty}\delta_N\|_2 \leq \|\delta - A^{-1}_{\infty}\|_2 \leq \|A^{-1}_{\infty}\|_2 \|\delta - A^{-1}_{\infty}\|_2 \leq 2C'\lambda^N,$$

hence the assertion with $K_1 = 2C'\|A^{-1}\|_2$ and $r_1 = \lambda$. This establishes the lemma. $\square$

Next, we consider approximating the square root of a positive operator.

**Lemma 2.** Let $P$ be the unique square root of a positive operator $A$. Suppose that $A$ is banded and $\|A - I\|_2 \leq r < 1$, where $I$ is the identity operator from $l^2$ to $l^2$. Then $P = (p_{ik})_{i,k \in \mathbb{Z}}$ is of exponential decay off its diagonal.

**Proof.** The uniqueness of $P$ and the convergence of the series

$$\sum_{i=0}^{\infty} (-1)^i \frac{(2i-3)!!}{(2i)!!} (A - I)^i$$

imply that

$$P = \sqrt{A} = \sqrt{I + (A - I)} = \sum_{i=0}^{\infty} (-1)^i \frac{(2i-3)!!}{(2i)!!} (A - I)^i.$$ 

The matrix $A$ is banded and so is $A - I$. If $A - I$ has bandwidth $b$, then $(A - I)^i$ is also banded with bandwidth $ib$. Thus, $|p_{ik}| \leq Kr_{i-k}/b$ for some constant $K$. This finishes the proof. $\square$

**Lemma 3.** Let $P$ be the unique square root of a positive operator $A$. Suppose that $A$ is banded and $\|A - I\|_2 \leq r < 1$, where $I$ is the identity operator from $l^2$ to $l^2$. Then $P^{-1} = (\hat{p}_{ik})_{i,k \in \mathbb{Z}}$ is of exponential decay off its diagonal.

**Proof.** The uniqueness of $P^{-1}$ and the convergence of the series

$$\sum_{i=0}^{\infty} (-1)^i \frac{(2i-1)!!}{(2i)!!} (A - I)^i$$

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imply that
\[ P^{-1} = (A)^{-1/2} = (I + (A - I))^{-1/2} = \sum_{i=0}^{\infty} (-1)^i \frac{(2i - 1)!!}{(2i)!!} (A - I)^i. \]

Now we use the same argument as in the lemma above to conclude that \( P^{-1} \) is of exponential decay off its diagonal. \( \Box \)

Let \( \tilde{P}_N \) be the square root of \( A_N \). That is, \( \tilde{P}_N^2 = A_N \). Denote \( P_N = I_N, \infty P_{N}^t, \infty \). We need to estimate \( P_N \tilde{P}_N - \tilde{P}_N P_N \). We have

**Lemma 4.** Let \( R = (r_{ik})_{-N \leq i, k \leq N} := P_N \tilde{P}_N - \tilde{P}_N P_N \). Then \( r_{ik} = O(r^{N/(4b)}) \) for \( k = -N/4 + 1, \ldots, N/4 - 1 \) and \( i = -N, \ldots, N \), where \( b \) is the bandwidth of \( A \) and \( r \) is as defined in Lemma 3.

**Proof.** It is known that \( P \) and \( A \) commute. Let us write

\[
P = \begin{bmatrix} \alpha_1 & B & \alpha_2 \\ B^t & P_N & C^t \\ \alpha_3 & C & \alpha_4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \beta_1 & a & \beta_2 \\ a^t & A_N & c \\ \beta_3 & c & \beta_4 \end{bmatrix}.
\]

We have \( B'a + P_N A_N + C^t c = a'B + A_N P_N + c'C \). Thus, \( P_N A_N - A_N P_N = a'B - B'a + c'C - C^t c \). Let \( E = a'B - B'a + c'C - C^t c \) and \( I_N := I_{2N+1, 2N+1} \).

We have \( P_N (A_N - I_N) = (A_N - I_N) P_N + E \) and

\[
P_N (A_N - I_N)^n = (A_N - I_N)^n P_N + \sum_{k=0}^{n-1} (A_N - I_N)^k E (A_N - I_N)^{n-k-1}
\]

by using induction. Then, we have

\[
P_N \tilde{P}_N = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} P_N (A_N - I_N)^n \\
= \sum_{n=0}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} (A_N - I_N)^n P_N \\
+ \sum_{n=0}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} \sum_{k=0}^{n-1} (A_N - I_N)^k E (A_N - I_N)^{n-k-1} \\
= \tilde{P}_N P_N + \sum_{n=0}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} \sum_{k=0}^{n-1} (A_N - I_N)^k E (A_N - I_N)^{n-k-1}.
\]

To estimate \( R = P_N \tilde{P}_N - \tilde{P}_N P_N \) which is the summation above, we break \( R \) into two parts and estimate the first by

\[
\left\| \sum_{n=N1+1}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} \sum_{k=0}^{n-1} (A_N - I_N)^k E (A_N - I_N)^{n-k-1} \right\|_2 \\
\leq \sum_{n=N1+1}^{\infty} \frac{(2n-3)!!}{(2n)!!} n \|E\|_2 \|A_N - I_N\|_2^n \leq K_1 \|A_N - I_N\|_2^{N1}.
\]

Thus, this part has the desired property if we choose \( N1 \) appropriately. Next, we note that \( A_N - I_N \) is banded and its bandwidth is \( b \). Thus, for \( 0 \leq n \leq N1 \), \( (A_N - I_N)^n \) is also banded and has bandwidth \( nb \leq bN1 \).
Note also $E = (e_{ik})_{-N \leq i, k \leq N}$ has the following property:

$$e_{ik} = \begin{cases} 
0 & \text{for } -N + b < k < N - b, \ -N + b < i < N - b, \\
O(r^{N-|k|}) & \text{for } -N \leq i \leq -N + b \text{ and } N - b \leq i \leq N, \ -N \leq k \leq N.
\end{cases}$$

It follows that $(A_N - I_N)^l E$ has a similar property as $E$:

$$((A_N - I_N)^l E)_{ik} = \begin{cases} 
0 & \text{for } -N + b < k < N - b, \\
O(r^{N-|k|}) & \text{for } -N \leq i \leq -N + lb + b, \\
O(r^{N-|k|}) & \text{for } N - lb - b \leq i \leq N, \ -N \leq k \leq N.
\end{cases}$$

Choose $N_1$ such that $N/(4b) \leq N_1 < N/(4b) + 1$. Then $(A_N - I)^{N_1}$ has bandwidth $bN_1 < N/4 + b$ and hence

$$((A_N - I)^l (A_N - I)^{N_1 - l})_{ik} = \begin{cases} 
O(r^{N/4 - b - |k|}) & \text{if } |k| \leq N/4 \text{ and } -N \leq i \leq N, \\
O(1) & \text{otherwise}
\end{cases}$$

for $l = 1, \ldots, N_1$. Putting these two parts together, we have established that $R$ has the desired property. □

We are now ready to prove the following.

**Theorem 1.** Suppose that $A$ is a positive operator and $\|A - I\|_2 < 1$. Suppose that $A$ is a banded matrix. Let $P$ be the unique square root of $A$ and $\gamma$ the solution of $P\gamma = \delta$. Let $\hat{P}_N$ be a square root matrix such that $\hat{P}_N = A_N$ and $\hat{y}_N$ the solution of $\hat{P}_N \hat{y}_N = \delta_N$. Then

$$\|I_N, \infty \gamma - \hat{y}_N\|_2 \leq K_1N$$

for some $r \in (0, 1)$ and a constant $K > 0$.

**Proof.** Let $P = (p_{ik})_{i,k \in \mathbb{Z}}$ and $P_N = (p_{ik})_{-N \leq i, k \leq N}$. By Lemma 2, the matrix $P$ is of exponential decay off its diagonal. By Lemma 3, we know that $P_N$ is of exponential decay off its diagonal uniformly with respect to $N$ because of $\|A_N - I_{2N+1,2N+1}\|_2 < 1$, which follows from $\|A - I\|_2 < 1$. The invertibility of $A$ implies that $P$ is invertible. From $\|A - I\|_2 < 1$ it follows that the inverse of $P$ is bounded. Let $\hat{y}_N$ be the solution of $P_N \hat{y}_N = \delta_N$. Thus, we apply Lemma 1 to conclude that

$$\|I_N, \infty \gamma - \hat{y}_N\|_2 \leq K_1N$$

for some $r \in (0, 1)$.

We now proceed to estimate $\|\hat{y}_N - \hat{y}_N\|_2$.

Note that $P^2 = A$ implies $A_N = P_N^2 + B^iB + C^iC$ or $\hat{P}_N^2 - \hat{P}_N^2 = B^iB + C^iC$. Thus, we have

$$(P_N + \hat{P}_N)(\hat{P}_N - P_N) = \hat{P}_N^2 - P_N^2 + P_N \hat{P}_N - \hat{P}_N P_N = B^iB + C^iC + R,$$

where $R$ was defined in Lemma 4. Hence,

$$\hat{P}_N - P_N = (P_N + \hat{P}_N)^{-1}(B^iB + C^iC + R).$$

Note that the entries of $B^iB + C^iC$ have the exponential decay property: $(B^iB + C^iC)_{ik} = O(r^{N-|k|})$. By Lemma 4, we know that each entry of the middle section ($N/2$ columns) of the columns of $B^iB + C^iC + R$ has exponential
decay $O(r^{N(4b)})$. Both $P_N$ and $\hat{P}_N$ are positive and $\|(P_N + \hat{P}_N)^{-1}\|_2 \leq \|\hat{P}_N^{-1}\|_2$ is bounded. Recall that $\hat{P}_N^{-1}$ is of exponential decay off its diagonal. We have

$$\|\hat{y}_N - y_N\|_2 \leq \|\hat{P}_N^{-1}\|_2 \|\delta_N - \hat{P}_N P_N^{-1} \delta_N\|_2$$

$$\leq \|\hat{P}_N^{-1}\|_2 \|(P_N - \hat{P}_N)(P_N^{-1} \delta_N)\|_2$$

$$\leq \|\hat{P}_N^{-1}\|_2 \|(P_N + \hat{P}_N)^{-1}\|_2 \|(B'B + C'C + R)\| P_N^{-1} \delta_N\|_2$$

$$\leq K\lambda^N$$

for some $\lambda \in (r, 1)$. This completes the proof. \[ \square \]

In the proof above, an essential step is to show that each entry of the middle section of the columns of $\hat{P}_N - P_N$ is of exponential decay. This indeed follows from $(\hat{P}_N - P_N) = (P_N + \hat{P}_N)^{-1}(B'B + C'C + R)$, the boundedness of $(P_N + \hat{P}_N)^{-1}$, and the fact that each entry of the middle section of the columns of $B'B + C'C + R$ is of exponential decay. This has its own interest. Thus, we have the following

**Theorem 2.** Suppose that $A$ is a positive operator and $\|A - I\|_2 < 1$. Suppose that $A$ is a banded matrix. Let $P$ be the unique square root of $A$ and $P_N = I_N,\infty P(I_N,\infty)^t$. Let $\hat{P}_N$ be a square root matrix such that $\hat{P}_N^2 = A_N$. Then

$$\|P_N \delta_N - \hat{P}_N \delta_N\|_2 \leq K\lambda^N$$

for some $\lambda \in (0, 1)$ and a constant $K$.

Finally, we remark that if $\|A - I\|_2 = 1$, then each entry of the middle section of the columns of $R$ is convergent to 0 with speed $\frac{1}{N}$. The exponential decay in the above has to be replaced by

$$\|P_N \delta_N - \hat{P}_N \delta_N\|_2 \leq \frac{K}{N}.$$

### 3. Computation of Battle-Lemarié's wavelets

Fix a positive integer $m$. Let $A = B_{2m}$ be the bi-infinite matrix whose symbol is $\sum_{k \in \mathbb{Z}} B_{2m}(k) e^{-jk\omega}$. Clearly, $A$ is a banded Toeplitz matrix. To see that $A$ is a positive operator on $l^2$, we show that $A \geq cI$ for some $c > 0$ as follows: For any $x \in l^2$, we have

$$x^t A x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\omega)} f(A)(\omega) d\omega$$

$$= F(A) (\xi \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\omega)|^2 d\omega$$

$$\geq \min_{\omega} F(A)(\omega) ||x||_2^2.$$

With $c = \min_{\omega} F(A)(\omega) > 0$, we have $A \geq cI$. Similarly, we can show that
\[ \|A - I\|_2 < 1. \] Indeed,
\[
\| (A - I)x \|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(A - I)(\omega)|^2 |F(x)(\omega)|^2 \, d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - F(A)(\omega)|^2 |F(x)(\omega)|^2 \, d\omega
\]
\[
\leq \max_{\omega} |1 - F(A)(\omega)| \|x\|_2^2 \leq \left( 1 - \min_{\omega} F(A)(\omega) \right)^2 \|x\|_2^2.
\]
Thus, we have
\[
\| (A - I)x \|_2 \leq \left( 1 - \min_{\omega} F(A)(\omega) \right) \|x\|_2
\]
and hence, \( \|A - I\|_2 < 1. \) Thus, \( B_{2m} \) satisfies all the conditions of Theorem 1.

By (1), we have
\[
\phi(\omega) = \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}} \left( \frac{\sin \omega/2}{\omega/2} \right)^m.
\]
Thus, \( \phi(t) = \sum_k \alpha_k B_m(t - k) \) with \( x = (\alpha_k)_{k \in \mathbb{Z}} \) satisfying
\[
C_{2m}x = \delta \quad \text{and} \quad C_{2m}^2 = B_{2m}.
\]
Using our Theorem 1, we conclude that our numerical method is valid to compute the \( \alpha_k \)'s.

By (2), the transfer function is
\[
H(\omega) = \frac{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}}{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}} \cos^m(\omega/2).
\]
Note that when \( m \) is even, then \( \cos^m(\omega/2) = (1 - (e^{j\omega} + e^{-j\omega})/2)^{m/2} \), which is a finite series. However, when \( m \) is odd, \( \cos^m(\omega/2) \) is no longer a finite series. In order to compute \( H(\omega) \), let \( S_m \) be the Toeplitz matrix whose symbol is \( \cos^2m(\omega/2) = (1 - (e^{j\omega} + e^{-j\omega})/2)^{m} \). Let \( Z \) be a zero insertion operator on \( l^2 \) defined by
\[
Zx = Z(x_i)_{i \in \mathbb{Z}} = (z_i)_{i \in \mathbb{Z}} \quad \text{with} \quad z_i = \begin{cases} x_i/2 & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}
\]
Thus, \( H(\omega) = \sum_{k \in \mathbb{Z}} \beta_k e^{-jk\omega} \) with \( x = (\beta_k)_{k \in \mathbb{Z}} \) satisfying
\[
x = w * y * z,
\]
where \( * \) denotes the convolution operator of two vectors in \( l^2 \) and
\[
y = C_m \delta, \quad z = ZC_m^{-1} \delta, \quad w = T \delta
\]
with \( C_m^2 = B_{2m} \), \( T_m^2 = S_m \). Using our Theorems 1 and 2, we know that our numerical method gives a good approximation to \( y \) and \( z \). For \( m \) even, our numerical method produces an \( x_N \) which converges pointwise to \( x \) exponentially. When \( m \) is odd, the remark after Theorem 2 has to be applied, and the \( w_N \) produced by this procedure does no longer converge to \( w \) exponentially.
By (3), the wavelet $\psi$ associated with $\varphi$ is given by
\[
\hat{\psi}(2\omega) = e^{-j\omega H(\omega + \pi)}\hat{\varphi}(\omega).
\]
Once $\{\alpha_k\}_{k \in \mathbb{Z}}$ and $\{\beta_k\}_{k \in \mathbb{Z}}$ are computed, $\{\gamma_k\}_{k \in \mathbb{Z}}$ can be obtained by convolution.

We have implemented this method to compute Battle-Lemarié’s wavelets in MATLAB. The graphs of Battle-Lemarié’s wavelets are shown in the following figures.

![Graphs of Battle-Lemarié's wavelets](image-url)
ON THE COMPUTATION OF BATTLE-LEMARIÉ'S WAVELETS

BIBLIOGRAPHY


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