ON THE COMPUTATION OF BATTLE-LEMARIE'S WAVELETS

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Abstract. We propose a matrix approach to the computation of Battle-Lemarié's wavelets. The Fourier transform of the scaling function is the product of the inverse $F(x)$ of a square root of a positive trigonometric polynomial and the Fourier transform of a B-spline of order $m$. The polynomial is the symbol of a bi-infinite matrix $B$ associated with a B-spline of order $2m$. We approximate this bi-infinite matrix $B_{2m}$ by its finite section $A_N$, a square matrix of finite order. We use $A_N$ to compute an approximation $x_N$ of $x$ whose discrete Fourier transform is $F(x)$. We show that $x_N$ converges pointwise to $x$ exponentially fast. This gives a feasible method to compute the scaling function for any given tolerance. Similarly, this method can be used to compute the wavelets.

1. Introduction

Battle-Lemarié's wavelets [1, 3] may be constructed by using a multiresolution approximation built from polynomial splines of order $m > 0$. See, e.g., [4] or [2]. To be precise, let $V_0$ be the vector space of all functions of $L^2(\mathbb{R})$ which are $m - 2$ times continuously differentiable and equal to a polynomial of degree $m - 1$ on each interval $[n + m/2, n + 1 + m/2]$ for all $n \in \mathbb{Z}$. Define the other resolution space $V_k$ by

$$
V_k := \{u(2^k t) : u \in V_0\}, \quad \forall k \in \mathbb{Z}.
$$

It is known that $\{V_k\}_{k \in \mathbb{Z}}$ provide a multiresolution approximation, and there exists a unique scaling function $\varphi$ such that

$$
V_k = \text{span}_{L^2}(2^{k/2}\varphi(2^k t - n) : n \in \mathbb{Z})
$$

for all $k$, and the integer translates of $\varphi$ are orthonormal to each other. (See, e.g., [4].) Define a transfer function $H(\omega)$ by

$$
H(\omega) = \frac{\hat{\varphi}(2\omega)}{\hat{\varphi}(\omega)},
$$

where $\hat{\varphi}$ denotes the Fourier transform of $\varphi$. Then the wavelet $\psi$ associated with the scaling function $\varphi$ is given in terms of its Fourier transform by

$$
\psi(\omega) = e^{-j\omega/2}H(\omega/2 + \pi)\varphi(\omega/2).
$$

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Here and throughout, \( j := \sqrt{-1} \). The scaling function \( \varphi \) associated with the multiresolution approximation may be given by

\[
\varphi(\omega) = \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} |\hat{B}_m(\omega + 2k\pi)|^2}} \hat{B}_m(\omega),
\]

where \( B_m \) is the well-known central B-spline of order \( m \) whose Fourier transform is given by

\[
\hat{B}_m(\omega) = \left( \frac{\sin \omega / 2}{\omega / 2} \right)^m.
\]

By using Poisson's summation formula, we have

\[
\varphi(\omega) = \frac{1}{\sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega}} \hat{B}_m(\omega).
\]

Thus, the transfer function is

\[
H(\omega) = \sqrt{\frac{\sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega}}{\sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega}}} (\cos \omega / 2)^m.
\]

Then the wavelet \( \psi \) associated with \( \varphi \) is given by

\[
\psi(\omega) = e^{-j\omega / 2} H(\omega / 2 + \pi) \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega}}} \hat{B}_m(\omega / 2).
\]

The above Fourier transforms of \( \varphi, H, \) and \( \psi \) suggest that the scaling function, transfer function, and wavelet have the following representations:

\[
\varphi(t) = \sum_{k \in \mathbb{Z}} \alpha_k B_m(t - k),
\]

\[
H(\omega) = \sum_{k \in \mathbb{Z}} \beta_k e^{-jk\omega},
\]

\[
\psi(t) = \sum_{k \in \mathbb{Z}} \gamma_k B_m(2t - k).
\]

In this paper, we propose a matrix method to compute the \( \alpha_k \)'s, \( \beta_k \)'s, and \( \gamma_k \)'s. Let us use \( \varphi \) to illustrate our method as follows: view \( \sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega} \) as the symbol of a bi-infinite matrix \( B_{2m} = (b_{ik})_{i,k \in \mathbb{Z}} \) with \( b_{i,k} = b_0, k - i = B_{2m}(k - i) \) for all \( i, k \in \mathbb{Z} \). Similarly, \( \sqrt{\sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega}} \) can be viewed as the symbol of another (unknown) bi-infinite matrix \( C_{2m} \). Then it is easy to see that

\[
C_{2m}^2 = B_{2m}.
\]

To find

\[
\sum_{k \in \mathbb{Z}} \alpha_k e^{-jk\omega} = \frac{1}{\sqrt{\sum_{k \in \mathbb{Z}} B_2m(k)e^{-jk\omega}}}
\]

is equivalent to solving

\[
C_{2m} \mathbf{x} = \delta
\]

with \( \delta = (\delta_i)_{i \in \mathbb{Z}}, \delta_0 = 1, \) and \( \delta_i = 0 \) for all \( i \in \mathbb{Z} \setminus \{0\} \), where \( \mathbf{x} = (\alpha_k)_{k \in \mathbb{Z}} \). Our numerical method is to find an approximation to \( \mathbf{x} \) within a given tolerance.
Let $A_N = (b_{ik})_{-N \leq i, k \leq N}$ be a finite section of $B_{2m}$. Note that $A_N$ is symmetric and totally positive. Thus, we can find $\tilde{P}_N$ such that

$$\tilde{P}_N^2 = A_N$$

by using, e.g., the singular value decomposition. Then we solve $\tilde{P}_N x_N = \delta_N$ with $\delta_N$ a vector of $2N+1$ components which are all zeros except for the middle one, which is 1. We can show that $x_N$ converges pointwise to $x$ exponentially fast. Similarly, we can use this idea to compute an approximation of $\{\beta_k\}_{k \in \mathbb{Z}}$ by (2) and $\{\gamma_k\}_{k \in \mathbb{Z}}$ by (3). Therefore, the discussion mentioned above furnishes a numerical method to compute Battle-Lemarié's wavelet.

To prove the convergence of $x_N$ to $x$, we place ourselves in a more general setting. We study a general bi-infinite matrix $A$. (For the case of Battle-Lemarié's wavelets, $A = B_{2m}$.) We look for certain conditions on $A$ such that the solution $x_N$ of $\tilde{P}_N x_N = \delta_N$ with $\tilde{P}_N^2 = A_N$ converges to the solution $x$ of $P x = \delta$ with $P^2 = A$, where $A_N$ is a finite section of $A$. This is discussed in the next section. In the last section, we show that the bi-infinite matrix $B_{2m}$ satisfies the conditions on $A$ obtained in §2. This will establish our numerical method for computing Battle-Lemarié's wavelets.

### 2. Main Results

Let $\mathbb{Z}$ be the set of all integers. Let $l^2 := l^2(\mathbb{Z})$ be the space of all square summable sequences with indices in $\mathbb{Z}$. That is,

$$l^2(\mathbb{Z}) = \left\{ (\ldots, x_{-1}, x_0, x_1, \ldots)^t : \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty \right\}.$$  

It is known that $l^2$ is a Hilbert space. We shall use $x$ to denote each vector in $l^2$ and use $A$ to denote a linear operator from $l^2$ to $l^2$. It is known that $A$ can be expressed as a bi-infinite matrix. Thus, we shall write $A = (a_{ik})_{i, k \in \mathbb{Z}}$.

In the following, we shall consider $A$ to be a banded and/or Toeplitz matrix. That is, $A$ is said to be banded if there exists a positive integer $b$ such that $a_{ik} = 0$ whenever $|i-k| > b$. The matrix $A$ is said to be Toeplitz if $a_{i+k, m+k} = a_{i, m}$ for all $i, k, m \in \mathbb{Z}$. Denote by $F(x)(\omega)$ the symbol of a vector $x \in l^2$, i.e.,

$$F(x)(\omega) = \sum_{i \in \mathbb{Z}} x_i e^{-ji\omega}.$$  

Denote by $F(A)(\omega)$ the symbol of a Toeplitz matrix $A = (a_{ik})_{i, k \in \mathbb{Z}}$, i.e.,

$$F(A)(\omega) = \sum_{i \in \mathbb{Z}} a_{i, 0} e^{-ji\omega}.$$  

Suppose that $F(A)(\omega) \neq 0$ and $\sum_{i \in \mathbb{Z}} |a_{i, 0}| < \infty$. It is known from the well-known Wiener's theorem that there exists a sequence $x$ such that

$$\frac{1}{F(A)(\omega)} = \sum_{k \in \mathbb{Z}} x_k e^{-jk\omega}$$

with $\sum_k |x_k| < \infty$. It is easy to see that to find this sequence $x$ is equivalent to solving the linear system of bi-infinite order:

$$A x = \delta,$$
where \( \delta = (\ldots, \delta_{-1}, \delta_0, \delta_1, \ldots)' \) with \( \delta_0 = 1 \) and \( \delta_i = 0 \) for all \( i \in \mathbb{Z}\setminus\{0\} \).

Furthermore, if the matrix \( A \) is a positive operator, then there exists a unique positive square root \( P \) of \( A \). That is, \( P^2 = A \). The symbol representation is \( F(P)(\omega) = \sqrt{F(A)(\omega)} \). To find \( F(P)(\omega) \) is equivalent to finding a matrix \( P \) such that \( P^2 = A \).

Certainly, we cannot solve a linear system of bi-infinite order. Neither can we decompose a matrix of bi-infinite order into two matrices of bi-infinite order. However, we can do this approximatively. Let \( N \) be a positive integer, and let \( A_N = (a_{ik})_{-N \leq i, k \leq N} \) be a finite section of \( A \). Let \( I_{N,\infty} = (0, I_{2N+1,2N+1}, 0) \) be a matrix of \( 2N+1 \) rows and bi-infinite columns with \( I_{2N+1,2N+1} \) being the identity matrix of size \( (2N+1) \times (2N+1) \) such that

\[
A_N = I_{N,\infty} A I_{N,\infty}'.
\]

Denote \( \delta_N = I_{N,\infty} \delta \) and \( x_N = I_{N,\infty} x \). Then we shall solve the following linear system:

\[
A_N \hat{x}_N = \delta_N.
\]

We claim that \( \hat{x}_N \) converges to \( x \) exponentially fast as \( N \) increases to \( \infty \), under certain conditions on \( A \). Furthermore, we shall solve \( \hat{P}_N^2 = A_N \) for \( \hat{P}_N \) by using the singular value decomposition. Once we have \( \hat{P}_N \), we shall solve

\[
\hat{P}_N \hat{y}_N = \delta_N.
\]

We claim that \( \hat{y}_N \) converges to \( y \) exponentially fast as \( N \to \infty \), provided \( A \) satisfies certain conditions.

To check the conditions on \( A \), we need the following definition.

**Definition.** A matrix \( A = (a_{ik})_{i,k \in \mathbb{Z}} \) is said to be of exponential decay off its diagonal if

\[
|a_{ik}| \leq K r^{|i-k|}
\]

for some constant \( K \) and \( r \in (0, 1) \).

We begin with an elementary lemma.

**Lemma 1.** Suppose that \( A \) is of exponential decay off its diagonal and has a bounded inverse. Suppose that \( A_N^{-1} = (\hat{a}_{ik})_{-N \leq i, k \leq N} \) satisfies the property that

\[
|\hat{a}_{i,k}(N)| \leq K r^{|i-k|}, \quad \forall -N \leq i, k \leq N,
\]

for all \( N > 0 \). Then there exists \( r_1 \in (0, 1) \) and a constant \( K_1 \) such that

\[
\|I_{N,\infty} x - \hat{x}_N\|_2 \leq K_1 r_1^N,
\]

where \( x \) is the solution of \( Ax = \delta \) and \( x_N \) is the solution of \( A_N x_N = \delta_N \).

**Proof.** From the assumption of the lemma, there exist \( K \) and \( r \in (0, 1) \) such that \( A = (a_{ik})_{i,k \in \mathbb{Z}} \) and \( A_N^{-1} = (\hat{a}_{i,k}(N))_{-N \leq i, k \leq N} \) satisfy

\[
|a_{ik}| \leq K r^{|i-k|} \quad \text{and} \quad |\hat{a}_{ik}| \leq K r^{|i-k|}.
\]

Write

\[
A I_{N,\infty}' = \begin{bmatrix} B & C \\ A_N & 0 \end{bmatrix} \quad \text{and} \quad d = B A_N^{-1} \delta_N \quad \text{with} \quad d = (\ldots, d_{-N-1}, d_{-N})'.
\]
Then we have, for each \( i = -\infty, \ldots, -N - 1, -N \),

\[
|d_i| = \left| \sum_{k=-N}^{N} a_{ik} \delta_{k,0}(N) \right| \leq K^2 \sum_{k=-N}^{N} r^{|i-k|} r^{|k|} \\
= K^2 \left( r^{-i} \sum_{k=0}^{N} r^{2k} + N r^{-i} \right) \leq C \lambda^{-i}
\]

for some constant \( C \) and \( \lambda \in (0, 1) \). Thus, \( \|BA^{-1}\delta_N\|_2 \leq C' \lambda^N \). Similarly, \( \|CA_N^{-1}\delta_N\|_2 \leq C' \lambda^N \). Hence,

\[
\|I_{N,\infty}x - x_N\|_2 \leq \|x - I_{N,\infty} x_N\|_2 \leq \|A^{-1}\|_2 \|\delta - AI_N^{-1}\delta_N\|_2 \\
\leq \|A^{-1}\|_2 \left\| \begin{bmatrix} B \\ A_N^{-1} \end{bmatrix} \delta_N \right\|_2 \\
\leq \|A^{-1}\|_2 \left( \|BA_N^{-1}\delta_N\|_2 + \|CA_N^{-1}\delta_N\|_2 \right) \leq \|A^{-1}\|_2 2 C' \lambda^N,
\]

hence the assertion with \( K_1 = 2 C' \|A^{-1}\|_2 \) and \( r_1 = \lambda \). This establishes the lemma. \( \square \)

Next, we consider approximating the square root of a positive operator.

**Lemma 2.** Let \( P \) be the unique square root of a positive operator \( A \). Suppose that \( A \) is banded and \( \|A - I\|_2 \leq r < 1 \), where \( I \) is the identity operator from \( l^2 \) to \( l^2 \). Then \( P = (P_{ik})_{i,k \in \mathbb{Z}} \) is of exponential decay off its diagonal.

**Proof.** The uniqueness of \( P \) and the convergence of the series

\[
\sum_{i=0}^{\infty} (-1)^i \frac{(2i-3)!!}{(2i)!!} (A - I)^i
\]

imply that

\[
P = \sqrt{A} = \sqrt{I + (A - I)} = \sum_{i=0}^{\infty} (-1)^i \frac{(2i-3)!!}{(2i)!!} (A - I)^i.
\]

The matrix \( A \) is banded and so is \( A - I \). If \( A - I \) has bandwidth \( b \), then \( (A - I)^i \) is also banded with bandwidth \( ib \). Thus, \( |p_{ik}| \leq K r^{|i-k|/b} \) for some constant \( K \). This finishes the proof. \( \square \)

**Lemma 3.** Let \( P \) be the unique square root of a positive operator \( A \). Suppose that \( A \) is banded and \( \|A - I\|_2 \leq r < 1 \), where \( I \) is the identity operator from \( l^2 \) to \( l^2 \). Then \( P^{-1} = (P_{ik})_{i,k \in \mathbb{Z}} \) is of exponential decay off its diagonal.

**Proof.** The uniqueness of \( P^{-1} \) and the convergence of the series

\[
\sum_{i=0}^{\infty} (-1)^i \frac{(2i-1)!!}{(2i)!!} (A - I)^i
\]
imply that
\[ P^{-1} = (A)^{-1/2} = (I + (A - I))^{-1/2} = \sum_{i=0}^{\infty} \frac{(-1)^i (2i - 1)!!}{(2i)!!} (A - I)^i. \]

Now we use the same argument as in the lemma above to conclude that \( P^{-1} \) is of exponential decay off its diagonal. \( \square \)

Let \( \hat{P}_N \) be the square root of \( A_N \). That is, \( \hat{P}_N^2 = A_N \). Denote \( P_N = I_N, \infty P \hat{P}_N, \infty \). We need to estimate \( P_N \hat{P}_N - \hat{P}_N P_N \). We have

**Lemma 4.** Let \( R = (r_{ik})_{-N \leq i, k \leq N} = P_N \hat{P}_N - \hat{P}_N P_N \). Then \( r_{ik} = O(r^{N/(4b)}) \) for \( k = -N/4 + 1, \ldots, N/4 - 1 \) and \( i = -N, \ldots, N \), where \( b \) is the bandwidth of \( A \) and \( r \) is as defined in Lemma 3.

**Proof.** It is known that \( P \) and \( A \) commute. Let us write
\[
\begin{bmatrix}
\alpha_1 & B & \alpha_2 \\
B' & P_N & C' \\
\alpha_3 & C & \alpha_4
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\beta_1 & a & \beta_2 \\
a' & A_N & c' \\
\beta_3 & c & \beta_4
\end{bmatrix}.
\]

We have \( B'a + P_N A_N + C'c = a'B + A_N P_N + c'C \). Thus, \( P_N A_N - A_N P_N = a'B - B'a + c'C - C'c \). Let \( E = a'B - B'a + c'C - C'c \) and \( I_N := I_{2N+1, 2N+1} \). We have \( P_N(A_N - I_N) = (A_N - I_N)P_N + E \) and
\[
P_N(A_N - I_N)^n = (A_N - I_N)^n P_N + \sum_{k=0}^{n-1} (A_N - I_N)^k E(A_N - I_N)^{n-k-1}
\]
by using induction. Then, we have
\[
P_N \hat{P}_N = \sum_{n=0}^{\infty} (-1)^n \frac{(2n - 3)!!}{(2n)!!} P_N(A_N - I_N)^n
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \frac{(2n - 3)!!}{(2n)!!} (A_N - I_N)^n P_N
\]
\[
+ \sum_{n=0}^{\infty} (-1)^n \frac{(2n - 3)!!}{(2n)!!} \sum_{k=0}^{n-1} (A_N - I_N)^k E(A_N - I_N)^{n-k-1}
\]
\[
= \hat{P}_N P_N + \sum_{n=0}^{\infty} (-1)^n \frac{(2n - 3)!!}{(2n)!!} \sum_{k=0}^{n-1} (A_N - I_N)^k E(A_N - I_N)^{n-k-1}.
\]

To estimate \( R = P_N \hat{P}_N - \hat{P}_N P_N \) which is the summation above, we break \( R \) into two parts and estimate the first by
\[
\left\| \sum_{n=N1+1}^{\infty} (-1)^n \frac{(2n - 3)!!}{(2n)!!} \sum_{k=0}^{n-1} (A_N - I_N)^k E(A_N - I_N)^{n-k-1} \right\|_2
\]
\[
\leq \sum_{n=N1+1}^{\infty} \frac{(2n - 3)!!}{(2n)!!} \|E\|_2 \|A_N - I_N\|_2^n \leq K_1 \|A_N - I_N\|_2^{N1}.
\]
Thus, this part has the desired property if we choose \( N1 \) appropriately. Next, we note that \( A_N - I_N \) is banded and its bandwidth is \( b \). Thus, for \( 0 \leq n \leq N1 \), \( (A_N - I_N)^n \) is also banded and has bandwidth \( nb \leq bN1 \).
Note also \( E = (e_{ik})_{-N \leq i, k \leq N} \) has the following property:

\[
e_{ik} = \begin{cases} 
0 & \text{for } -N + b < k < N - b, \ -N + b < i < N - b, \\
O(r^{N-|k|}) & \text{for } -N \leq i \leq -N + b \text{ and } N - b \leq i \leq N, \ -N \leq k \leq N.
\end{cases}
\]

It follows that \( (A_N - I_N)^l E \) has a similar property as \( E \):

\[
((A_N - I_N)^l E)_{ik} = \begin{cases} 
0 & \text{for } -N + b < k < N - b, \\
- N + kb + b < i < N - lb - b, \\
O(r^{N-|k|}) & \text{for } -N \leq i \leq -N + lb + b, \\
N - lb - b \leq i \leq N, \text{ and } -N \leq k \leq N.
\end{cases}
\]

Choose \( N_1 \) such that \( N/(4b) \leq N_1 < N/(4b) + 1 \). Then \( (A_N - I)^{N_1} \) has bandwidth \( bN_1 < N/4 + b \) and hence

\[
((A_N - I)^l E(A_N - I)^{n-l-1})_{ik} = \begin{cases} 
O(r^{3N/4-b-|k|}) & \text{if } |k| \leq N/4 \text{ and } -N \leq i \leq N, \\
O(1) & \text{otherwise}
\end{cases}
\]

for \( l = 1, \ldots, N_1 \). Putting these two parts together, we have established that \( R \) has the desired property. \( \Box \)

We are now ready to prove the following.

**Theorem 1.** Suppose that \( A \) is a positive operator and \( \|A - I\|_2 < 1 \). Suppose that \( A \) is a banded matrix. Let \( P \) be the unique square root of \( A \) and \( y \) the solution of \( Py = \delta \). Let \( \hat{P}_N \) be a square root matrix such that \( \hat{P}_N^2 = A_N \) and \( \hat{y}_N \) the solution of \( \hat{P}_N \hat{y}_N = \delta_N \). Then

\[
\|I_{N, \infty} y - \hat{y}_N\|_2 \leq K r^N
\]

for some \( \lambda \in (0, 1) \) and a constant \( K > 0 \).

**Proof.** Let \( P = (p_{ik})_{i, k \in \mathbb{Z}} \) and \( P_N = (p_{ik})_{-N \leq i, k \leq N} \). By Lemma 2, the matrix \( P \) is of exponential decay off its diagonal. By Lemma 3, we know that \( P_N \) is of exponential decay off its diagonal uniformly with respect to \( N \) because of \( \|A_N - I_{2N+1, 2N+1}\|_2 < 1 \), which follows from \( \|A - I\|_2 < 1 \). The invertibility of \( A \) implies that \( P \) is invertible. From \( \|A - I\|_2 < 1 \) it follows that the inverse of \( P \) is bounded. Let \( \hat{y}_N \) be the solution of \( P_N \hat{y}_N = \delta_N \). Thus, we apply Lemma 1 to conclude that

\[
\|I_{N, \infty} y - \hat{y}_N\|_2 \leq K r^N
\]

for some \( r \in (0, 1) \).

We now proceed to estimate \( \|\hat{y}_N - \hat{y}_N\|_2 \).

Note that \( P^2 = A \) implies \( A_N = P_N^2 + B^t B + C^t C \) or \( \hat{P}_N^2 - P_N^2 = B^t B + C^t C \).

Thus, we have

\[
(P_N + \hat{P}_N)(\hat{P}_N - P_N) = \hat{P}_N^2 - P_N^2 + P_N \hat{P}_N - \hat{P}_N P_N = B^t B + C^t C + R,
\]

where \( R \) was defined in Lemma 4. Hence,

\[
(\hat{P}_N - P_N) = (P_N + \hat{P}_N)^{-1}(B^t B + C^t C + R).
\]

Note that the entries of \( B^t B + C^t C \) have the exponential decay property:

\[
(B^t B + C^t C)_{ik} = O(r^{N-|k|}) \].

By Lemma 4, we know that each entry of the middle section \((N/2)\) columns of the columns of \( B^t B + C^t C + R \) has exponential
decay $O(r^{-N/(4b)})$. Both $p_n$ and $\tilde{p}_n$ are positive and $\|(p_n + \tilde{p}_n)^{-1}\|_2 \leq \|\tilde{p}_n^{-1}\|_2$ is bounded. Recall that $p_n^{-1}$ is of exponential decay off its diagonal. We have

$$
\|\tilde{y}_n - y_n\|_2 \leq \|\tilde{p}_n^{-1}\|_2 \|\delta_n - p_n^{-1} \delta_n\|_2
\leq \|\tilde{p}_n^{-1}\|_2 \|(p_n - \tilde{p}_n)(p_n^{-1} \delta_n)\|_2
\leq \|\tilde{p}_n^{-1}\|_2 \|(p_n + \tilde{p}_n)^{-1}\|_2 \|(B' B + C' C + R)p_n^{-1} \delta_n\|_2
\leq K \lambda^N
$$

for some $\lambda \in (r, 1)$. This completes the proof. \qed

In the proof above, an essential step is to show that each entry of the middle section of the columns of $\tilde{p}_n - p_n$ is of exponential decay. This indeed follows from $\tilde{p}_n - p_n = (p_n + \tilde{p}_n)^{-1}(B' B + C' C + R)$, the boundedness of $(p_n + \tilde{p}_n)^{-1}$, and the fact that each entry of the middle section of the columns of $B' B + C' C + R$ is of exponential decay. This has its own interest. Thus, we have the following

**Theorem 2.** Suppose that $A$ is a positive operator and $\|A - I\|_2 < 1$. Suppose that $A$ is a banded matrix. Let $P$ be the unique square root of $A$ and $p_n = I_{n, \infty}p(I_{n, \infty})^t$. Let $\tilde{p}_n$ be a square root matrix such that $\tilde{p}_n^2 = A_n$. Then

$$
\|p_n \delta_n - \tilde{p}_n \delta_n\|_2 \leq K \lambda^N
$$

for some $\lambda \in (0, 1)$ and a constant $K$.

Finally, we remark that if $\|A - I\|_2 = 1$, then each entry of the middle section of the columns of $R$ is convergent to 0 with speed $\frac{1}{N}$. The exponential decay in the above has to be replaced by

$$
\|p_n \delta_n - \tilde{p}_n \delta_n\|_2 \leq \frac{K}{N}.
$$

### 3. Computation of Battle-Lemarié’s wavelets

Fix a positive integer $m$. Let $A = B_{2m}$ be the bi-infinite matrix whose symbol is $\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jkw}$. Clearly, $A$ is a banded Toeplitz matrix. To see that $A$ is a positive operator on $l^2$, we show that $A \geq cI$ for some $c > 0$ as follows: For any $x \in l^2$, we have

$$
x^t Ax = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)(\omega) F(A)(\omega) F(x)(\omega) d\omega
= F(A)(\xi) \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(x)(\omega)|^2 d\omega
\geq \min_{\omega} F(A)(\omega) \|x\|_2^2.
$$

With $c = \min_{\omega} F(A)(\omega) > 0$, we have $A \geq cI$. Similarly, we can show that
\[ \|A - I\|_2 < 1. \] Indeed,
\[
\|(A - I)x\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(A - I)(\omega)|^2 |F(x)(\omega)|^2 \, d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - F(A)(\omega)|^2 |F(x)(\omega)|^2 \, d\omega
\]
\[
\leq \max_{\omega} |1 - F(A)(\omega)|^2 \|x\|_2^2 \leq \left(1 - \min_{\omega} F(A)(\omega)\right)^2 \|x\|_2^2.
\]
Thus, we have
\[
\|(A - I)x\|_2 \leq \left(1 - \min_{\omega} F(A)(\omega)\right)\|x\|_2
\]
and hence, \(\|A - I\|_2 < 1\). Thus, \(B_{2m}\) satisfies all the conditions of Theorem 1.

By (1), we have
\[
\varphi(t) = \sum_k \alpha_k B_m(t - k) \quad \text{with} \quad x = (\alpha_k)_{k \in \mathbb{Z}} \quad \text{satisfying}
\]
\[
C_{2m} x = \delta \quad \text{and} \quad C_{2m}^2 = B_{2m}.
\]

Using our Theorem 1, we conclude that our numerical method is valid to compute the \(\alpha_k\)'s.

By (2), the transfer function is
\[
H(\omega) = \frac{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}}{\sqrt{\sum_{k \in \mathbb{Z}} B_{2m}(k)e^{-jk\omega}}} \cos^m(\omega/2).
\]

Note that when \(m\) is even, then \(\cos^m(\omega/2) = (1 - (e^{j\omega} + e^{-j\omega})/2)^m\), which is a finite series. However, when \(m\) is odd, \(\cos^m(\omega/2)\) is no longer a finite series. In order to compute \(H(\omega)\), let \(S_m\) be the Toeplitz matrix whose symbol is \(\cos^2m(\omega/2) = (1 - (e^{j\omega} + e^{-j\omega})/2)^m\). Let \(Z\) be a zero insertion operator on \(l^2\) defined by
\[
Zx = Z(x_i)_{i \in \mathbb{Z}} = (z_i)_{i \in \mathbb{Z}} \quad \text{with} \quad z_i = \begin{cases} x_{i/2} & \text{if } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd}. \end{cases}
\]
Thus, \(H(\omega) = \sum_{k \in \mathbb{Z}} \beta_k e^{-jk\omega}\) with \(x = (\beta_k)_{k \in \mathbb{Z}}\) satisfying
\[
x = w * y * z,
\]
where \(*\) denotes the convolution operator of two vectors in \(l^2\) and
\[
y = C_m \delta, \quad z = ZC_m^{-1} \delta, \quad w = T \delta
\]
with \(C_m^2 = B_{2m}\), \(T_m^2 = S_m\). Using our Theorems 1 and 2, we know that our numerical method gives a good approximation to \(y\) and \(z\). For \(m\) even, our numerical method produces an \(x_N\) which converges pointwise to \(x\) exponentially. When \(m\) is odd, the remark after Theorem 2 has to be applied, and the \(w_N\) produced by this procedure does no longer converge to \(w\) exponentially.
By (3), the wavelet $\psi$ associated with $\phi$ is given by
\[
\psi(2\omega) = e^{-j\omega H(\omega + \pi)}\phi(\omega).
\]
Once $\{\alpha_k\}_{k}\in\mathbb{Z}$ and $\{\beta_k\}_{k}\in\mathbb{Z}$ are computed, $\{\gamma_k\}_{k}\in\mathbb{Z}$ can be obtained by convolution.

We have implemented this method to compute Battle-Lemarié's wavelets in MATLAB. The graphs of Battle-Lemarié's wavelets are shown in the following figures.
ON THE COMPUTATION OF BATTLE-LEMARIÉ'S WAVELETS

BIBLIOGRAPHY


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