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ON THE SHARPNESS OF L^2-ERROR ESTIMATES OF H_0^1-PROJECTIONS ONTO SUBSPACES OF PIECEWISE, HIGH-ORDER POLYNOMIALS

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Dedicated to Professor R. Bruce Kellogg on the occasion of his 60th birthday [1]

ABSTRACT. In a plane polygonal domain, consider a Poisson problem $-\Delta u = f$ with homogeneous Dirichlet boundary condition and the p-version finite element solutions of this. We give various upper and lower bounds for the error measured in L^2. In the case of a single element (i.e., a convex domain), we reduce the question of sharpness of these estimates to the behavior of a certain inf-sup constant, which is numerically determined, and a likely sharp estimate is then conjectured. This is confirmed during a series of numerical experiments also for the case of a reentrant corner. For a one-dimensional analogue problem (of rotational symmetry), sharp L^2-error estimates are proven directly and via an extension of the classical duality argument. Here, we give sharp L^∞-error estimates in some weighted and unweighted norms also.

1. INTRODUCTION

The purpose of the paper is to study the influence of corner singularities on the accuracy of p-version finite element solutions, and the sharpness of L^2-norm error estimates. We note that in the context of the h-version finite element method for a corner problem, the sharpness of L^2-norm error estimates, and more generally of error estimates in negative-order Sobolev norms, has been fully studied in [17]. Some attempts have also been made regarding the p-version of the finite element method. In [13], error estimates in L^2 and H^{-s} $(s > 0)$ are derived, using the traditional duality technique. However, the question of the sharpness of L^2-norm error estimates is unanswered. This paper is a further attempt to answer the question. In §2, we give an L^2-norm error bound. In §3, we derive a lower bound for the L^2-norm error. By connecting to the optimality constant, the stability constant and inf-sup constants, we conjecture a likely optimal L^2-norm error estimate. The conjectured optimal L^2-norm error estimate is proved in §4 in a one-dimensional setting which mimics the two-dimensional corner singularities. The one-dimensional singular

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model problem we study is in some sense closer to two-dimensional corner problems than the one considered in [8]. In §5, we prove various sharp L^∞-norm error estimates for the one-dimensional model problem. In the last section, we present numerical results on corner domains with various internal angles to confirm the likely optimal L^2-norm error estimate.

Let Ω be a bounded, simply connected polygonal domain in the plane. Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with $f \in H^{s-1}(\Omega)$ for some $s \geq 0$. Let $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{M-1} \leq \omega_M$ denote the interior angles at the corners A_j of Ω, $1 \leq j \leq M$. Let

$$\alpha_j = \frac{\pi}{\omega_j} \in \left[\frac{1}{2}, \infty\right), \quad 1 \leq j \leq M. \tag{1.2}$$

The regularity of the solution u, in the presence of corners, is the subject of some classical treatments, cf. Dauge et al. [5], Grisvard [10], Kellogg [14] and Kondrat′ev [15]. Following Dauge et al., we introduce singular functions, written in terms of polar coordinates, centered at the vertex of a cone Γ. Within a neighborhood of a corner A_j, the domain Ω coincides with a cone $\Gamma_j = \{(r_j, \theta_j) : 0 < r_j < \infty, \ 0 < \theta_j < \omega_j\}$. Then, the singular functions are

$$S_{j,l} = \eta_j \sigma_{j,l}, \quad l \in Z_+,$$

where $\eta_j \in C^\infty$ is a cutoff function, $\eta_j = 1$ near A_j, and $\eta_j = 0$ outside some neighborhood of A_j,

$$\sigma_{j,l} = \begin{cases} r_j^{\lambda_{j,l}} \sin(\lambda_{j,l}\theta_j) & \text{if } \lambda_{j,l} \notin Z_+, \\ r_j^{\lambda_{j,l}} \left(\sin(\lambda_{j,l}\theta_j) \log r_j + \theta_j \cos(\lambda_{j,l}\theta_j)\right) & \text{if } \lambda_{j,l} \in Z_+ \end{cases}$$

and

$$\lambda_{j,l} = l\,\alpha_j = \frac{l\pi}{\omega_j}.$$

Then we can write the solution to the problem (1.1) as follows: supposing $f \in H^{s-1}(\Omega)$, $s \in R_+ \backslash Z$, we have

$$u = u_0 + \sum_{j=1}^{M} \sum_{\lambda_{j,l} < s} \gamma_{j,l} S_{j,l} \tag{1.3}$$

with

$$u_0 \in H^{s+1}(\Omega) \quad \text{and} \quad \|u_0\|_{s+1} + \sum_{j,l} |\gamma_{j,l}| \leq c\left(\|f\|_{s-1} + \|u\|_1\right). \tag{1.4}$$

The singular expansion coefficients $\gamma_{j,l}$ are exhibited as linear functionals of f in [5, §5]. The leading singularity in (1.3) is $S_{M,1} \in H^{1+\alpha_M-\varepsilon}(\Omega)$, $\forall \varepsilon > 0$. We denote

$$\alpha \stackrel{\text{def}}{=} \alpha_M.$$

2. Upper L^2-error bounds

Partition Ω into E elements $\bar{\Omega} = \bigcup_{i=1}^{E} \bar{\Omega}_i$, where Ω_i are parallelograms or triangles with $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$ and $\bar{\Omega}_i \cap \bar{\Omega}_j$ is either empty, a vertex or a common side of both Ω_i and Ω_j. Each vertex of Ω is assumed to be a vertex of some Ω_i. We denote by $\mathscr{P}$ the set of parallelograms and by $\mathscr{T}$ the set of triangles in the Euclidean plane.

Let $R = (-1, 1)^2$ and $T = \{(x, y) : -1 < y < x, |x| < 1\}$ denote the standard square and triangle, respectively. Let F_i be an affine, orientation preserving (i.e., $\det(DF_i) > 0$) mapping which maps Ω_i onto R if Ω_i is a parallelogram, and onto T if Ω_i is a triangle. Then we define the space of piecewise polynomials

$$S^p = \left\{ v \in H_0^1(\Omega) : v|_{\Omega_i} \circ F_i^{-1} \in \begin{cases} Q_p(R) & \text{if } \Omega_i \in \mathscr{P} \\ P_p(T) & \text{if } \Omega_i \in \mathscr{T} \end{cases} \right\}, \tag{2.1}$$

where Q_p and P_p denote the spaces of polynomials of separate degree, respectively total degree, less than or equal to p.

The Galerkin solution $u_p \in S^p$, characterized by varying p (and fixed partition), is then defined by

$$\int_\Omega \nabla u_p \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in S^p, \tag{2.2}$$

or we may write $u_p = \Pi_p u$, where Π_p is the H_0^1-projection onto S^p (defined by the Dirichlet form in (2.2)).

It is well known, cf. [3], that, denoting by $e_p = u - u_p$, one has

$$\|e_p\|_1 \leq c p^{-\min(2\alpha, s)} \tag{2.3}$$

and that, in general, one cannot expect a better convergence rate than that in the estimate (2.3). In this paper, we are interested in the L^2-norm error estimate.

Proposition 2.1 (as in [13, Theorem 2]). *Let $f \in H^{s-1}(\Omega)$, $s \in R_+ \backslash Z$. Assume $\alpha = \alpha_M \notin Z$. Let $u_p = \Pi_p u$ be defined as above. Then*

$$\|e_p\|_0 \leq c p^{-\min(s+1, 2\alpha+1)}. \tag{2.4}$$

Proof. Define an auxiliary function $w_p \in H_0^1(\Omega)$ by

$$\begin{cases} -\Delta w_p = e_p & \text{in } \Omega, \\ \quad w_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

In the case of a convex Ω (where the shift theorem $\|w_p\|_2 \leq c \|e_p\|_0$ holds),

$$\begin{aligned} \|e_p\|_0^2 &= (e_p, -\Delta w_p) = (\nabla e_p, \nabla w_p) \\ &= (\nabla e_p, \nabla(w_p - z_p)) \leq \|e_p\|_1 \|\nabla(w_p - z_p)\|_0 \\ &\leq c \|e_p\|_1 p^{-1} \|w_p\|_2 \leq c \|e_p\|_1 p^{-1} \|e_p\|_0 \end{aligned}$$

holds for some $z_p \in S^p$. So,

$$\|e_p\|_0 \leq c p^{-1} \|e_p\|_1$$

and (2.4) follows. (Cf. [12, Prop. 3.2] for a similar argument for higher-order problems.) In the case of a nonconvex Ω partitioned with multiple elements, we split w_p—as done in [13]—according to (cf. (1.3))

$$w_p = \bar{w}_p + \sum_{j=1}^{M} \sum_{\lambda_{j,l}<1} \gamma_{j,l} S_{j,l}$$

with $\bar{w}_p \in H^2(\Omega)$, and

$$\|\bar{w}_p\|_2 + \sum_{\lambda_{j,l}<1} |\gamma_{j,l}| \le c\,(\|e_p\|_0 + \|w_p\|_1) \le c\,\|e_p\|_0.$$

Thus, for some $z_p \in S^p$,

$$\|\nabla(w_p - z_p)\|_0 \le c\left(p^{-1}\|\bar{w}_p\|_2 + \sum_{\lambda_{j,l}<1} |\gamma_{j,l}|\,p^{-2l\alpha_j}\right) \le c\,p^{-1}\|e_p\|_0.$$

So, again, $\|e_p\|_0 \le c\,p^{-1}\|e_p\|_1$ and (2.4) follows. □

3. The L^2-distance of the H_0^1-projection from the exact solution bounded in terms of a stability constant

Let u_p be the Galerkin solution (i.e., H_0^1-projection onto S^p). We shall derive some lower bounds on the error $u - u_p$ measured in L^2, which are of lesser convergence rate than those of the L^2-projection. We recall that, cf. [3], for a problem like (1.1) with leading singularity $r^\alpha \sin\alpha\theta$,

$$\inf_{v\in S^p} \|u - v\|_0 \le c\,p^{-(2\alpha+2)}. \tag{3.1}$$

Following an idea of Wahlbin [17], one may easily prove

Proposition 3.1. *Let u and u_p be the exact and discrete solutions to* (1.1) *as defined before. Let Ω have reentrant corners. Then, if the coefficient of the leading term $r^\alpha \sin\alpha\theta$ in the singular expansion of u is nonzero, there exists a $c > 0$ such that*

$$\|u - u_p\|_0 \ge c\,p^{-4\alpha}. \tag{3.2}$$

Proof. A mere transcription of the main idea in [17]. Like in [17], it is enough to prove the result for a particular solution $u^0 = a_0 r^\alpha \sin\alpha\theta + \cdots$, $a_0 \neq 0$. Let K be an element in the triangulation of Ω with a vertex in common with a vertex of Ω (A_M, say) that supports the leading singularity. Let $x_0 \in \mathring{K}$ and let $B_0 \subset\subset B_1 \subset\subset K$ be disks centered at x_0 so that $\mathscr{A}_0 = B_1 \backslash B_0$ is an annulus. Let $\omega \in C^\infty(\bar{\Omega})$ be such that

$$\omega = \begin{cases} 1 & \text{outside } B_1, \\ 0 & \text{inside } B_0, \end{cases}$$

and $G(x; x_0)$ be the Green's function for (1.1). Now let

$$u^0(x) = \omega(x)\,G(x; x_0).$$

Then, u^0 has a singularity at the vertex of the type $r^\alpha \sin\alpha\theta$, as was shown in [17]. Let the H_0^1-projection of u^0 onto S^p be denoted by u_p^0, and let

$e_p^0 = u^0 - u_p^0$. Then, noticing that $\operatorname{supp}(\Delta u^0) \subset K$, we have

$$\begin{aligned} c p^{-4\alpha} \le \|e_p^0\|_1^2 &= \int_\Omega \nabla e_p^0 \nabla (u^0 - u_p^0)\, dx \\ &= \int_\Omega \nabla e_p^0 \nabla u^0\, dx = -\int_\Omega e_p^0 \Delta u^0\, dx \\ &\le \|e_p^0\|_{0,K} \|\Delta u^0\|_{0,K}, \end{aligned}$$

yielding (3.2). □

Remark 3.1. The estimate (3.2) yields a stricter lower bound than (3.1) for $\alpha < 1$, i.e., the case of a reentrant corner. Based on the lower bound (3.2), we see that the estimate (2.4) is optimal in rate for the case of a crack domain. For other corner domains, the estimate (2.4) is generally not sharp, as will be seen from the consideration below and numerical results later. □

We next look at the convex domain case and assume for simplicity that Ω is a single parallelogram. We will then look at the situation where there is only one element, namely, Ω itself. Let $S^p = Q_p(F(\Omega)) \cap H_0^1(\Omega)$, where F is affine, $\det(F) > 0$, $F(\Omega) = (-1, 1)^2$. Then we may integrate (2.2) by parts to get

$$-\int_\Omega u_p \Delta v\, dx = \int_\Omega f v\, dx, \quad \forall v \in S^p. \tag{3.3}$$

We may alternatively formulate (3.3) as our finite element method with an asymmetric bilinear form. In this setting, an L^2-norm error estimate is the "natural" one for e_p, and this is intimately connected with the inf-sup or stability constant for the bilinear form in (3.3).

Denote, for convenience, the bilinear form by

$$B(u, v) = -\int_\Omega u \Delta v\, dx, \quad u \in L^2,\ v \in H^2 \cap H_0^1. \tag{3.4}$$

We introduce two constants,

$$\begin{aligned} C(u) &= \frac{\|u - u_p\|_0}{\inf_{v \in S^p} \|u - v\|_0}, \quad \text{the optimality constant at } u, \\ D(u) &= \sup_{v \in S^p} \frac{\|(u+v)_p\|_0}{\|u + v\|_0}, \quad \text{the stability constant at } u, \end{aligned} \tag{3.5}$$

where u is given by (1.1), u_p by (3.3) and $(u+v)_p \in S^p$ by

$$B((u+v)_p, w) = B(u + v, w), \quad \forall w \in S^p,$$

i.e., the projection of $u + v$ onto S^p given by the inner product defined in (3.4). It was shown in [2] that C and D are interconnected,

$$D(u) - 1 \le C(u) \le D(u) + 1. \tag{3.6}$$

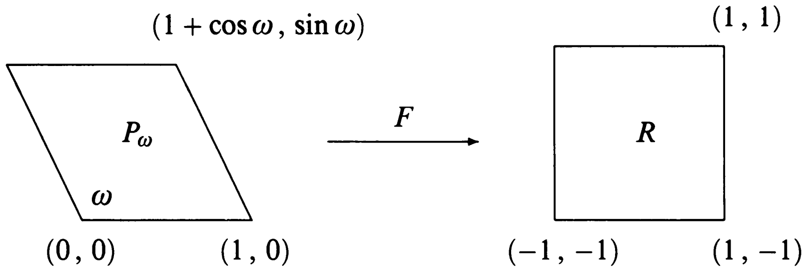

FIGURE 3.1. Affine map F with pre-image and image depicted

The constant D turns out to be connected to the inf-sup constant. We introduce two new constants,

$$\begin{aligned} \mu &= \inf_{u\in L^2} \sup_{v\in H^2\cap H_0^1} \frac{|B(u, v)|}{\|u\|_0\|v\|_2}, \\ \mu_p &= \inf_{u\in S^p} \sup_{v\in S^p} \frac{|B(u, v)|}{\|u\|_0\|v\|_2}. \end{aligned} \tag{3.7}$$

From the Buniakowsky-Cauchy-Schwarz inequality, we get

$$|B(u, v)| \le \sqrt{2}\,\|u\|_0\|v\|_2, \tag{3.8}$$

so that, via [2], the following relations hold,

$$\frac{\mu}{\mu_p} \le \sup_u D(u) \le \frac{\sqrt{2}}{\mu_p}. \tag{3.9}$$

We note that for any $u \in L^2(\Omega)$ there is a unique $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\Delta v = u$, and since Ω is convex, we have the estimate $\|v\|_2 \le c_\Omega\|u\|_0$. With (3.8), we easily get

$$\frac{1}{c_\Omega} \le \mu \le \sqrt{2}. \tag{3.10}$$

Remark 3.2. Since Ω is convex, $\|v\|_2$ is equivalent to $\|\Delta v\|_0$ on $H^2(\Omega) \cap H_0^1(\Omega)$. If we replace $\|v\|_2$ by $\|\Delta v\|_0$ in the definition of μ, then we have $\mu = 1$. $\square$

Clearly, the existence of quasi-optimal error estimates is linked with the inf-sup constant μ_p. We will show that in some sense to be made more clear in the next lemma, it is enough to consider μ_p for a square.

Let P_ω be the parallelogram in Figure 3.1.

The affine mapping which maps P_ω onto $R = (-1, 1)^2$ is

$$F(x) = 2\begin{pmatrix} 1 & -\cot\omega \\ 0 & \csc\omega \end{pmatrix} x - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $S_\omega^p = \{\tilde{v} = v \circ F : v \in S^p(R)\}$ with

$$S^p(R) = \overset{\circ}{Q}_p(R) = (1 - x_1^2)(1 - x_2^2)\,Q_{p-2}(R).$$

Define, as in (3.7),

$$\mu_{p,\omega} = \inf_{\tilde{u}\in S_\omega^p} \sup_{\tilde{v}\in S_\omega^p} \frac{|B(\tilde{u}, \tilde{v})|}{\|\tilde{u}\|_{0,P_\omega}\|\tilde{v}\|_{2,P_\omega}}. \tag{3.11}$$

Lemma 3.2. *Let $\mu_p = \mu_{p,\pi/2}$ be as defined in* (3.11) *with respect to $P_{\pi/2} = R$ and $\mu_{p,\omega}$ as in* (3.11). *Then, given a $\omega_0 \in (0, \pi/2)$, there exist c and C, depending on ω_0, so that*

$$c\,\mu_p \le \mu_{p,\omega} \le C\,\mu_p, \quad \forall\,\omega \in [\omega_0, \pi - \omega_0]. \tag{3.12}$$

Proof. We prove the equivalence between the respective inner products in (3.11). Let $x \approx y$ denote an equivalence in the sense that there exist positive constants c and C independent of x and y, so that $cx \le y \le Cx$. First we see that

$$\int_{P_\omega} \tilde{u}\tilde{v}\,dx = \int_R uv\,|J|^{-1}dx \approx \int_R uv\,dx,$$

where $J = DF$ is the Jacobian of F and $|J| = 4\csc\omega$, provided we require $\omega_0 \le \omega \le \pi - \omega_0$ for some $\omega_0 \in (0, \pi/2)$. Secondly, one may prove that

$$\int_{P_\omega} \nabla_{\tilde{x}}\tilde{u}\nabla_{\tilde{x}}\tilde{v}\,dx = \int_R \nabla u\,J^TJ\,\nabla v\,|J|^{-1}dx \approx \int_R \nabla u\nabla v\,dx$$

by verifying that the eigenvalues of J^TJ lie in a positive interval for $\omega_0 \le \omega \le \pi - \omega_0$. This gives the equivalence of the bilinear forms. Thirdly,

$$|\tilde{u}|^2_{H^2(P_\omega)} = \int_R (D^2u)^T H\,D^2u\,|J|^{-1}dx \approx |u|^2_{H^2(R)},$$

where $H = \sum_{i,j=1}^2 f_{ij}^T f_{ij}$ with $f_{ij} = (J_{1i}J_{1j}, J_{1i}J_{2j} + J_{2i}J_{1j}, J_{2i}J_{2j})$ and $D^2u = (u_{11}, u_{12}, u_{22})$. Through a lengthy calculation, it can be shown that the eigenvalues of H are in some positive interval for $\omega_0 \le \omega \le \pi - \omega_0$. We proved $|\tilde{u}|^2_{H^k(P_\omega)} \approx |u|^2_{H^k(R)}$ for $k = 0, 1$ earlier. Thus, $\|\tilde{u}\|^2_{H^2(P_\omega)} \approx \|u\|^2_{H^2(R)}$ and (3.12) follows from the definition (3.11). □

Remark 3.3. We may equally well handle a reduction from an (oblique) triangle to a right-angled one.

Remark 3.4. If $\Omega \in \mathscr{P}$, then $\mu_p \ge cp^{-1}$ for some constant c independent of p, by the equivalence in the above lemma and the lower bound in (3.9). For certain special cases involving C^1-elements, one will not be able to obtain a better inequality, $\mu_p \ge cp^{-\nu}$, for $\nu \in (0, 1)$. Specifically, consider solving $-\Delta u = f$ in the V-shaped domain of Figure 6.1, with smooth f and zero Dirichlet boundary conditions, using p-version C^1-elements. Collecting several earlier results, we have

$$c(\alpha)\,p^{-4\alpha} \le \|u - u_p\|_0 \le C(\alpha)\left(\frac{\sqrt{2}}{\mu_p} + 1\right)p^{-2\alpha-2}.$$

Now select $\alpha = (1+\varepsilon)/2$ for small $\varepsilon > 0$; then isolating μ_p and contracting the above inequalities, one finds

$$\mu_p \le \bar{c}(\varepsilon)\,p^{-1+\varepsilon}.$$

Remark 3.5. Lemma 3.2 is significant in the sense that in the guaranteed upper estimate $C(u) \le \sqrt{2}/\mu_{p,\omega} + 1 \le c/\mu_{p,\pi/2} + 1$, one will observe the same degree of suboptimality for all angles in any compact subinterval of $(0, \pi)$. □

The inf-sup constant $\mu_{p,\pi/2}$, which by Lemma 3.2 characterizes the behavior of all convex, single-element cases, can be computed numerically for a given choice of finite element space, via a generalized eigenvalue problem. This was done for $\mathring{Q}_p$ in [13] with an emerging behavior of

$$c p^{-1/2} \le \mu_{p,\pi/2} \le C p^{-1/2}, \tag{3.13}$$

indicating the loss of a half power of p relative to the L^2-distance (3.1). It is an open problem to prove (3.13) theoretically. If one were able to show (3.13), it would follow that $\|e_p\|_0 \le c p^{-2\alpha-3/2}$. We shall further justify this for a purely radial version of the problem in the next section and further demonstrate that the classical duality argument can be extended to obtain the optimal convergence estimate $\|e_p\|_0 \le c p^{-2\alpha-3/2}$, in contrast to (2.4).

4. A one-dimensional analogue

It is well known that the singularities of (1.1) supported at the vertices of Ω are essentially radial and of the form $r^\alpha g(r, \theta)$ in local polar coordinates. We shall investigate a purely radial model problem with such a singularity and $g \equiv 1$ for simplicity.

The purely radial singular function $u = r^\alpha$, $\alpha > 0$, is the unique solution to

$$\begin{cases} \frac{1}{r}(r u')' = \alpha^2 r^{\alpha-2} & \text{in } (0, 1), \\ |u(0)| < \infty, \quad u(1) = 1. \end{cases} \tag{4.1}$$

Let

$$\mathscr{P}_N^0 = \{v : v \text{ is a polynomial of degree} \le N,\ v(1) = 0\}. \tag{4.2}$$

Then, for $N \ge 1$, the Galerkin solution $u_N \in r + \mathscr{P}_N^0$ satisfies

$$\int_0^1 (u_N' - u')\, v' r\, dr = 0, \quad \forall v \in \mathscr{P}_N^0, \tag{4.3}$$

which is then our simple one-dimensional problem.

Remark 4.1. An indication of how (4.1), then (4.3), could arise naturally follows. Let Ω be the intersection of a cone with opening π/α ($\alpha \notin Z_+$), $\alpha \ge 1/2$, and the open unit ball centered at the apex of the cone whose right leg is assumed to coincide with the positive x-axis. Let $\tilde{u} = r^\alpha \sin \alpha\theta$ be the exact solution to

$$\begin{cases} -\Delta \tilde{u} = 0 & \text{in } \Omega = \{(r, \theta) : r \in (0, 1),\ \theta \in (0, \pi/\alpha)\}, \\ \tilde{u} = 0 & \text{on } \partial\Omega \cap \{(r, \theta) : \theta = 0 \text{ or } \theta = \pi/\alpha\}, \\ \tilde{u} = \sin \alpha\theta & \text{on } \partial\Omega \cap \{(r, \theta) : r = 1\}. \end{cases} \tag{4.4}$$

The purely radial part of $\tilde{u}$ is u. Suppose we used finite elements of the type $u_N(r) \sin \alpha\theta$, where $u_N(r)$ were a polynomial of degree at most N, equal to 1 at $r = 1$ (vanishing at 1 for the test functions). Then the Galerkin equation would read

$$
\begin{aligned}
0 &= \int_\Omega \nabla[(u_N - u)\sin\alpha\theta]\nabla[v\sin\alpha\theta]\, r\,dr\,d\theta \\
&= \frac{\pi}{2\alpha}\int_0^1 \left\{(u_N' - u')v' + \frac{\alpha^2}{r^2}(u_N - u)v\right\} r\,dr.
\end{aligned}
\tag{4.5}
$$

If we now drop the zeroth-order term, we get (4.3). □

Let us define the weighted inner product

$$
(u, v) = \int_0^1 uvr\,dr \tag{4.6}
$$

for $u, v \in L^2_{01}(0, 1)$, subscripts corresponding to our weight $(1-r)^0 r^1$ at the endpoints of $(0, 1)$. Then (4.3) implies:

Lemma 4.1. *The derivative u_N' is the $L^2_{01}(0, 1)$ projection of u' onto $\mathscr{P}_{N-1}$.*

Proof. A constant is the derivative of a function in $\mathscr{P}_N^0$: $1 = d(r-1)/dr$. The remaining monomials of $\mathscr{P}_{N-1}$ also belong to $d(\mathscr{P}_N^0)/dr$. □

Note that we may translate all statements to the interval $(-1, 1)$ by the affine transformation

$$
(0, 1) \ni r \mapsto x = t(r) = 2r - 1 \in (-1, 1). \tag{4.7}
$$

We recall that on the standard interval $[-1, 1]$ the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n \geq 0$, are orthogonal with respect to the weight $w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha > -1$, $\beta > -1$. The value of $P_n^{(\alpha,\beta)}(x)$ at $x = 1$ is $\binom{n+\alpha}{n}$, and

$$
\int_{-1}^1 w(x)\,|P_n^{(\alpha,\beta)}(x)|^2\,dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}.
$$

Also recall Rodrigues' formula

$$
P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\,\frac{1}{w(x)}\,\frac{d^n}{dx^n}\left[w(x)\,(1-x^2)^n\right].
$$

We will use in particular the Jacobi polynomials $\{P_n^{(0,1)}\}_{n=0}^\infty$, which are orthogonal in $L^2_{01}(-1, 1)$, cf. [6] or [16]. Let $\{j_n\}_{n=0}^\infty = \{P_n^{(0,1)} \circ t\}_{n=0}^\infty$ be the corresponding orthogonal sequence in $L^2_{01}(0, 1)$ obtained via the transformation (4.7). Then, we have

Theorem 4.2. *The following estimate holds for $u = r^\alpha$:*

$$
\|u' - u_N'\|_{L^2_{01}(0,1)} \leq c\,N^{-2\alpha}, \tag{4.8}
$$

and the estimate is sharp.

Proof. This follows from Lemma 4.1, the form of $u' = \alpha r^{\alpha-1}$ and standard approximation results for polynomials, see [11], [8] or later [4]: using Rodrigues' and Stirling's formulas, one gets

$$u' = \sum_{n=0}^{\infty} b_n j_n, \qquad u'_N = \sum_{n=0}^{N-1} b_n j_n, \tag{4.9}$$

where

$$\begin{aligned} b_n = \frac{(u', j_n)}{(j_n, j_n)} &= (-1)^{n+1} 2(n+1) \frac{\Gamma(\alpha+1)}{\Gamma(-\alpha)} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+2+\alpha)} \\ &= (-1)^{n+1} c_0(\alpha)\, n^{-2\alpha}(1 + (1+2\alpha)/n + O(n^{-2})) \end{aligned} \tag{4.10}$$

in which $c_0(\alpha) = \Gamma(\alpha+1)/\Gamma(-\alpha)$, and $\lim_{\alpha\to k} c_0(\alpha) = 0$ for $k > 0$ integer. We shall have occasion to use (4.10) again. □

The same argument yields:

Proposition 4.3. *The following N-distance holds for $u = r^\alpha$ in $L^2_{01}(0, 1)$ with respect to the subspace $\mathscr{P}_N$:*

$$\inf_{v\in\mathscr{P}_N} \|u - v\|_{L^2_{01}(0,1)} = c(\alpha)\, N^{-(2\alpha+2)}(1 + O(N^{-1})). \tag{4.11}$$

Remark 4.2. The results (4.8) and (4.11) give the "correct" rates also for the two-dimensional problems with dominant corner singularity $\tilde{u}$; this follows from the analysis in [3] and [7]. □

Let us calculate the $L^2_{01}(0, 1)$ error estimate directly and show to what extent this in turn is provable via the usual duality technique. Define

$$j_n^{(-1)}(r) = \int_1^r j_n(s)\, ds, \quad n \in Z_+ \cup \{0\}. \tag{4.12}$$

Then, using (4.9), we get

$$u - u_N = \sum_{n=N}^{\infty} b_n j_n^{(-1)}, \tag{4.13}$$

and it becomes beneficial to have a formula relating $j_n^{(-1)}$ to a linear combination of j_k.

Lemma 4.4. *Let $n \geq 2$. Then the following formulas hold:*

$$j_n^{(-1)} = c_{n,n+1} j_{n+1} + c_{n,n} j_n + c_{n,n-1} j_{n-1}, \tag{4.14}$$

where

$$c_{n,n+1} = \frac{n+2}{(2n+3)(2n+2)}, \quad c_{n,n} = \frac{-1}{(2n+3)(2n+1)}, \quad c_{n,n-1} = \frac{-n}{(2n+2)(2n+1)} \tag{4.15}$$

and

(4.16)

$$\int_1^x P_n^{(0,1)}(u)\, du = \bar{c}_{n,n+1} P_{n+1}^{(0,1)}(x) + \bar{c}_{n,n} P_n^{(0,1)}(x) + \bar{c}_{n,n-1} P_{n-1}^{(0,1)}(x),$$

where, $\bar{c}_{n,k} = 2c_{n,k}$, $k = n+1, n, n-1$.

Proof. By the substitution $u = 2s - 1$,

$$j_n^{(-1)}(r) = \int_1^r j_n(s)\,ds = \int_1^r P_n^{(0,1)}(2s-1)\,ds = \frac{1}{2}\int_1^{2r-1} P_n^{(0,1)}(u)\,du,$$

and we need only verify (4.16). Clearly,

$$\int_1^x P_n^{(0,1)}(u)\,du = \sum_{m=0}^{n+1} \bar{c}_{n,m} P_m^{(0,1)}(x), \tag{4.17}$$

and we may integrate by parts to get with

$$(u, v) = \int_{-1}^1 u v\,(1+x)\,dx$$

that

$$\begin{aligned} \bar{c}_{n,m}(P_m^{(0,1)}, P_m^{(0,1)}) &= \sum_{k=0}^{n+1} \bar{c}_{n,k}(P_m^{(0,1)}, P_k^{(0,1)}) \\ &= \int_{-1}^1 \left(\int_1^x P_n^{(0,1)}(u)\,du\right) P_m^{(0,1)}(x)\,(1+x)\,dx \\ &= \left[\left(\int_1^x P_n^{(0,1)}(u)\,du\right)\left(\int_{-1}^x P_m^{(0,1)}(u)\,(1+u)\,du\right)\right]_{-1}^1 \\ &\quad - \int_{-1}^1 P_n^{(0,1)}(x)\left(\int_{-1}^x P_m^{(0,1)}(u)\,(1+u)\,du\right) dx \\ &= -\int_{-1}^1 P_n^{(0,1)}(x)\left(\int_{-1}^x P_m^{(0,1)}(u)\,(1+u)\,du\right) dx \end{aligned} \tag{4.18}$$

for $m \in Z_+ \cup \{0\}$, by orthogonality. Here, we may factor

$$\int_{-1}^x P_m^{(0,1)}(u)\,(1+u)\,du = (1+x)\,q_{m+1}(x)$$

for some $q_{m+1} \in \mathscr{P}^{m+1}$. Thus,

$$\bar{c}_{n,m} = 0 \quad \text{for } 0 \le m \le n-2. \tag{4.19}$$

Identity (4.16) in [16] specializes to

$$(2n+2)\,P_n^{(0,1)}(x) = (n+2)\,P_n^{(1,1)}(x) - (n+1)\,P_{n-1}^{(1,1)}(x)$$

and identity (4.14) in [16] gives

$$\frac{d}{dx}P_n^{(0,0)}(x) = \frac{1}{2}(n+1)\,P_{n-1}^{(1,1)}(x),$$

so that, with $l_n(x) = P_n^{(0,0)}(x)$, the Legendre polynomial of degree n,

$$P_n^{(0,1)}(x) = \frac{1}{n+1}(l'_{n+1}(x) - l'_n(x)). \tag{4.20}$$

Identity (4.16) in [16] also yields

$$(1+u)\,P_n^{(0,1)}(u) = P_n^{(0,0)}(u) + P_{n+1}^{(0,0)}(u)\,, \tag{4.21}$$

so that

$$\int_{-1}^{x} P_m^{(0,1)}(u)\,(1+u)\,du = \int_{-1}^{x} l_m(u)\,du + \int_{-1}^{x} l_{m+1}(u)\,du. \tag{4.22}$$

But, for $j \ge 1$,

$$\begin{aligned}
&\int_{-1}^{1} l_i'(x)\left(\int_{-1}^{x} l_j(u)\,du\right)dx \\
&= \left[l_i(x)\left(\int_{-1}^{x} l_j(u)\,du\right)\right]_{-1}^{1} - \int_{-1}^{1} l_i(x) l_j(x) dx \\
&= -\frac{2}{2i+1}\,\delta_{ij}.
\end{aligned} \tag{4.23}$$

Substituting (4.20) and (4.22) into (4.18), using (4.23) and the fact that $(P_m^{(0,1)}, P_m^{(0,1)}) = 2/(m+1)$, we have

$$\begin{aligned}
\bar{c}_{n,n+1} &= \frac{n+2}{(2n+3)(n+1)}, \\
\bar{c}_{n,n} &= -\frac{2}{(2n+3)(2n+1)}, \\
\bar{c}_{n,n-1} &= -\frac{n}{(2n+1)(n+1)}.
\end{aligned} \tag{4.24}$$

Combining (4.17), (4.19), and (4.24) yields (4.16). □

Inserting (4.14) into (4.13) gives

$$\begin{aligned}
u - u_N &= \sum_{n=N}^{\infty} b_n (c_{n,n+1} j_{n+1} + c_{n,n} j_n + c_{n,n-1} j_{n-1}) \\
&= \sum_{k=N+1}^{\infty} (c_{k-1,k} b_{k-1} + c_{k,k} b_k + c_{k+1,k} b_{k+1})\, j_k \\
&\quad + (c_{N,N} b_N + c_{N+1,N} b_{N+1}) j_N + c_{N,N-1} b_N j_{N-1}.
\end{aligned} \tag{4.25}$$

Since $|c_{k,k}| = 1/(4k^2) + O(k^{-3})$ and $-c_{k+1,k}$ and $c_{k-1,k}$ are both equal to $1/(4k) + O(k^{-2})$, we get

$$\begin{aligned}
\|u - u_N\|_{L^2_{01}}^2 &= c\left\{\sum_{k=N+1}^{\infty} k^{-4\alpha} k^{-3} k^{-1} + N^{-4\alpha} N^{-2} N^{-1}\right\}\left(1 + O(N^{-1})\right) \\
&= c\,N^{-4\alpha-3}\left(1 + O(N^{-1})\right),
\end{aligned}$$

so that

$$\|u - u_N\|_{L^2_{01}} = c\,N^{-2\alpha-3/2}\left(1 + O(N^{-1})\right). \tag{4.26}$$

This is half a power off the N-distance in (4.11). We will now show that the usual duality argument can be extended to recover (4.26) and not just one power better than the H^1 estimate (4.8).

Remark 4.3. One notes that the last two terms in (4.25) dominate asymptotically, cf. Lemma 5.2.

Theorem 4.5. *Let $u = r^\alpha$ and u_N be determined by* (4.3). *Then there exist positive constants independent of N so that*

$$c\,N^{-(2\alpha+3/2)} \le \|u - u_N\|_{L^2_{01}} \le C\,N^{-(2\alpha+3/2)}.$$

Proof. This is a restatement of (4.26). We will prove the second inequality once more, now via an extension of the usual duality argument, in the hope of providing a possible approach to deal with higher-dimensional problems. Let $e_N = u - u_N$. Then, defining w_N by

$$\begin{cases} -\dfrac{1}{r}(rw_N')' = e_N & \text{in } (0,1), \\ |w_N(0)| < \infty, & w_N(1) = 0, \end{cases}$$

we get for any $z_N \in \mathscr{P}_N^0$,

(4.27)

$$\begin{aligned}(e_N, e_N) &= \int_0^1 e_N\left(-\frac{1}{r}(rw_N')'\right) r\,dr = -[e_N(rw_N')]_0^1 + \int_0^1 e_N' w_N' r\,dr \\ &= (e_N', w_N' - z_N') = \left(e_N, -\frac{1}{r}\left(r(w_N' - z_N')\right)'\right) \\ &= \left(e_N, e_N + \frac{1}{r}(rz_N')'\right) \le \|e_N\|_{L^2_{01}}\left\|e_N + \frac{1}{r}(rz_N')'\right\|_{L^2_{01}}\end{aligned}$$

by repeated integration by parts and the H_0^1-projection nature of u_N. An immediate corollary is

$$\|e_N\|_{L^2_{01}} = \inf_{z_N \in \mathscr{P}_N^0}\left\|e_N + \frac{1}{r}(rz_N')'\right\|_{L^2_{01}}. \tag{4.28}$$

One gets a similar result for a single element in two dimensions. Now $\mathscr{P}_{N-2}$ is contained in the image of $\mathscr{P}_N^0$ under the map $z \mapsto (1/r)(rz')'$, and we get from (4.9) and (4.11), and by using Lemma 4.4, that

$$\begin{aligned}\left\|e_N + \frac{1}{r}(rz_N')'\right\|_{L^2_{01}} &\le \inf_{\xi_{N-2}\in\mathscr{P}_{N-2}} \|u - \xi_{N-2}\|_{L^2_{01}} + \inf_{\zeta_{N-2}\in\mathscr{P}_{N-2}} \|u_N - \zeta_{N-2}\|_{L^2_{01}} \\ &\le c\,N^{-2\alpha-2} + \inf_{\zeta_{N-2}\in\mathscr{P}_{N-2}}\left\|\sum_{n=0}^{N-1} b_n j_n^{(-1)} - \zeta_{N-2}\right\|_{L^2_{01}} \\ &\le c\,N^{-2\alpha-2} + \|b_{N-1}c_{N-1,N}j_N\|_{L^2_{01}} \\ &\quad + \|(b_{N-2}c_{N-2,N-1} + b_{N-1}c_{N-1,N-1})j_{N-1}\|_{L^2_{01}} \\ &\le c\,N^{-(2\alpha+3/2)}. \quad \square\end{aligned}$$

Remark 4.4. These results do not follow automatically from [2] or [8], since we are dealing with an entirely different inner product geared towards the two-dimensional setting. They could, on the other hand, be used to establish instability results like the ones in [2], now for the nonconstant coefficient operator $-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$.

Remark 4.5. It is not difficult to extend the results to a three-dimensional model scenario with the operator $-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$, or model d-dimensional problems with symmetry group S^{d-1}.

Remark 4.6. It is interesting that the Green's function no longer has the r^α singularity used to prove (3.2) in Proposition 3.1, but that we may still prove a Wahlbin-like lower bound along the same lines (with $u^0 = r^\alpha$) to obtain

$$\|e_N\|_{L^2_{01}} \geq c\,\frac{\sqrt{2\alpha-2}}{\alpha^2}N^{-4\alpha},$$

which is valid only for $\alpha \geq 1$ (i.e. corresponding to the convex case).

5. Sharp L^∞-error estimates for the one-dimensional model

To better understand how far one can go with error estimates for the p-version method, we derive various L^∞-norm error estimates for p-version finite element solutions for the one-dimensional model problem considered in the previous section. We need asymptotic expansions of the various coefficients that appeared.

Lemma 5.1. *For the coefficients defined in* (4.15), *we have*

$$\begin{aligned}
c_{n,n+1} &= \frac{1}{4n} - \frac{1}{8n^2} + O(n^{-3}),\\
c_{n,n} &= -\frac{1}{4n^2} + O(n^{-3}),\\
c_{n,n-1} &= -\frac{1}{4n} + \frac{3}{8n^2} + O(n^{-3}),\\
c_{n-1,n} &= \frac{1}{4n} + \frac{1}{8n^2} + O(n^{-3}),\\
c_{n+1,n} &= -\frac{1}{4n} + \frac{5}{8n^2} + O(n^{-3}).
\end{aligned}$$

The proof of the lemma is elementary, and hence is omitted.

Now let us define

$$d_{N,k} = c_{k-1,k}b_{k-1} + c_{k,k}b_k + c_{k+1,k}b_{k+1}, \quad k \geq N+1, \tag{5.1}$$

$$d_{N,N} = c_{N,N}b_N + c_{N+1,N}b_{N+1}, \tag{5.2}$$

$$d_{N,N-1} = c_{N,N-1}b_N. \tag{5.3}$$

Then, the error expression (4.25) can be rewritten as

$$u - u_N = \sum_{k=N+1}^{\infty} d_{N,k}j_k + d_{N,N}j_N + d_{N,N-1}j_{N-1}. \tag{5.4}$$

For the asymptotic behavior of $\{d_{N,k}\}$, we have

Lemma 5.2. *There holds*

$$d_{N,k} = (-1)^k(1+\alpha)\,c_0(\alpha)k^{-2\alpha-2}(1+O(k^{-1})), \quad k \geq N+1,$$
$$d_{N,N} = (-1)^{N+1}\frac{c_0(\alpha)}{4}\,N^{-2\alpha-1}(1+O(N^{-1})),$$
$$d_{N,N-1} = (-1)^{N}\frac{c_0(\alpha)}{4}\,N^{-2\alpha-1}(1+O(N^{-1})).$$

The proof of Lemma 5.2 can be made by combining the results from Lemma 5.1 and (4.10).

Now, we are ready to give L^∞-norm error estimates on $u-u_N$.

Theorem 5.3. *Let* $\alpha \notin Z_+$. *Then there exists a positive* c *so that*

$$\|u-u_N\|_{L^\infty(0,1)} \leq c\,N^{-2\alpha}, \tag{5.5}$$

and the estimate is sharp.

Proof. From [1, Formula 22.14.1], we have

$$|P_n^{(0,1)}(x)| \leq n+1, \quad x \in [-1,1].$$

Thus,

$$|j_n(r)| \leq n+1, \quad r \in [0,1].$$

Then, from (5.4) and Lemma 5.2, we get

$$\begin{aligned}|u-u_N| &\leq \sum_{k=N+1}^{\infty} c_0(\alpha)(1+\alpha)\,k^{-2\alpha-1}(1+O(k^{-1}))\\ &\quad + \frac{c_0(\alpha)}{4}\,N^{-2\alpha}(1+O(N^{-1})) + \frac{c_0(\alpha)}{4}\,N^{-2\alpha}(1+O(N^{-1}))\\ &\leq c(\alpha)\,N^{-2\alpha}.\end{aligned}$$

To show that the estimate is sharp, we compute the error at $r=0$. We have

$$j_n(0) = P_n^{(0,1)}(-1) = (-1)^n\,(n+1).$$

Hence,

$$\begin{aligned}u(0)-u_N(0) &= \sum_{k=N+1}^{\infty} (-1)^k(1+\alpha)\,c_0(\alpha)k^{-2\alpha-2}(1+O(k^{-1}))\,(-1)^k(k+1)\\ &\quad + (-1)^{N+1}\frac{c_0(\alpha)}{4}\,N^{-2\alpha-1}(1+O(N^{-1}))\,(-1)^N(N+1)\\ &\quad + (-1)^{N}\frac{c_0(\alpha)}{4}\,N^{-2\alpha-1}(1+O(N^{-1}))\,(-1)^{N-1}N\\ &= (1+\alpha)\,c_0(\alpha)\sum_{k=N+1}^{\infty} k^{-2\alpha-1}(1+O(k^{-1}))\\ &\quad - \frac{c_0(\alpha)}{2}\,N^{-2\alpha}(1+O(N^{-1}))\\ &= \frac{1}{2\alpha}\,c_0(\alpha)\,N^{-2\alpha} + O(N^{-2\alpha-1}),\end{aligned}$$

where we have used the fact that

$$\sum_{k=N+1}^{\infty} k^{-2\alpha-1} = \frac{1}{2\alpha} N^{-2\alpha}(1+O(N^{-1})),$$

which is evident from the following inequalities:

$$\int_{N+1}^{\infty} t^{-2\alpha-1}\,dt \le \sum_{k=N+1}^{\infty} k^{-2\alpha-1} \le \int_{N}^{\infty} t^{-2\alpha-1}\,dt.$$

Thus, as long as $\alpha \notin Z_+$ (which excludes the trivial cases of smooth monomial solutions), the estimate (5.5) is sharp. □

It is a well-known phenomenon in error estimations of finite element solutions that usually the order of an L^∞-norm error estimate is almost the same as that of an L^2-norm error estimate. Comparing the above result with Theorem 4.5, we expect we can get higher-order error estimates in an L^∞-like norm which is compatible with the L^2_{01} norm. Let us introduce a weighted L^∞ space,

$$L^\infty_{01}(0,1) = \{u \text{ measurable} : \|u\|_{L^\infty_{01}(0,1)} = \operatorname{ess\,sup}_{r\in(0,1)} |r\,u(r)| < \infty\}.$$

Theorem 5.4. *Let $\alpha \notin Z_+$. Then there exists a positive constant c such that*

$$\|u-u_N\|_{L^\infty_{01}(0,1)} \le c\,N^{-2\alpha-3/2}, \tag{5.6}$$

and the estimate is sharp. As a consequence, we have a superconvergence result

$$\max_{r\in I} |u(r)-u_N(r)| \le c(I)\,N^{-2\alpha-3/2}$$

for any closed interval $I \subset (0,1]$.

Proof. From (4.21), we get

$$r\,j_n(r) = (l_n(2r-1) + l_{n+1}(2r-1))/2.$$

Thus, by using (5.4), we find

$$r\,(u(r)-u_N(r)) = \sum_{k=N+1}^{\infty} d_{N,k} r\,j_k(r) + d_{N,N} r\,j_N(r) + d_{N,N-1} r\,j_{N-1}(r) = E_1 + E_2,$$

where

$$E_1 = \frac{1}{2}\left\{ \sum_{k=N+2}^{\infty} (d_{N,k}+d_{N,k-1})\,l_k(2r-1) \right.$$
$$\left. + d_{N,N+1} l_{N+1}(2r-1) + (d_{N,N}+d_{N,N-1})\,l_N(2r-1) \right\},$$

$$E_2 = \frac{1}{2}\left\{ d_{N,N} l_{N+1}(2r-1) + d_{N,N-1} l_{N-1}(2r-1) \right\}.$$

From Lemma 5.2, we find

$$\begin{aligned} &d_{N,k} + d_{N,k-1} = O(k^{-2\alpha-3}), \quad k \ge N+2, \\ &d_{N,N+1} = O(N^{-2\alpha-2}), \\ &d_{N,N} + d_{N,N-1} = O(N^{-2\alpha-2}). \end{aligned}$$

Thus,

$$|E_1| \le c\, N^{-2\alpha-2}.$$

To deal with the term E_2, we use the following two identities (cf. [1]):

$$l_n(x) - l_{n+1}(x) = (1-x)\, P_n^{(1,0)}(x)\,, \tag{5.7}$$

$$P_n^{(1,0)}(x) + P_{n+1}^{(1,0)}(x) = \frac{n+3/2}{n+1}\,(1+x)\, P_n^{(1,1)}(x). \tag{5.8}$$

From (5.7) and (5.8), we find that

$$l_n(x) - l_{n+2}(x) = \frac{n+3/2}{n+1}\,(1-x)\,(1+x)\, P_n^{(1,1)}(x). \tag{5.9}$$

From Formula 8.965 in [9, page 1037], we have

$$(1-x)^{3/4}(1+x)^{3/4} P_n^{(1,1)}(x) = g(x)\, n^{-1/2} + O(n^{-3/2})$$

for some bounded, nonzero function $g(x)$. Using (5.9), we then have

$$l_n(x) - l_{n+2}(x) = \frac{n+3/2}{n+1}\,(1-x)^{1/4}(1+x)^{1/4} g(x)\, n^{-1/2} + O(n^{-3/2}).$$

Therefore, we have, again using Lemma 5.2,

$$\begin{aligned} E_2 &= (-1)^N \frac{c_0}{8}\, N^{-2\alpha-1}\,(l_{N-1}(2r-1) - l_{N+1}(2r-1)) + O(N^{-2\alpha-2}) \\ &= \tilde g(r)\, N^{-2\alpha-3/2} + O(N^{-2\alpha-2})\,, \end{aligned}$$

where $\tilde g$ is a bounded, nonzero function. Thus, the theorem is proved. □

Similarly, from the expansion for the error $u' - u_N'$, we have the following results. We omit the proofs of these results.

Theorem 5.5. *If* $\alpha > 1$, *we have the error estimate*

$$\|u' - u_N'\|_{L^\infty(0,1)} \le c\, N^{-2\alpha+2}\,,$$

and the estimate is sharp. When $\alpha < 1$, *there is no convergence of* $u_N' \to u'$ *in* $L^\infty(0,1)$.

Remark 5.1. It follows that the $W^{1,\infty}$-stability holds: $\|u_N\|_{W^{1,\infty}(0,1)} \le c\,\|u\|_{W^{1,\infty}(0,1)}$, if $\alpha > 1$.

Theorem 5.6. *There holds*

$$\|u' - u_N'\|_{L^\infty_{01}(0,1)} \le c\, N^{-2\alpha-1/2}\,,$$

and the estimate is sharp. As a consequence, for any closed interval $I \subset (0,1]$, *we have*

$$\max_{r\in I} |u'(r) - u_N'(r)| \le c(I)\, N^{-2\alpha-1/2}.$$

6. Numerical experiments

We report the numerical results of a two-dimensional problem related to (1.1), with the same sort of majorizing singularity, but with Neumann boundary conditions imposed weakly in order to guarantee a simple numerical implementation of the boundary conditions for an exact solution

$$u = r^{\alpha} \sin(\alpha\theta).$$

Consider the V-shaped domain, Ω, depicted in Figure 6.1.

In our figure, $\alpha = 2/3$, but we will vary α in the interval $(\frac{1}{2}, 1)$. We then use the p-version of the Galerkin method with merely one element $(= \frac{1}{2}\Omega)$ to compute numerical solutions to the problem indicated in Figure 6.2.

If we let $\partial\Omega^{+}$ denote the union of those three line segments bounding $\frac{1}{2}\Omega$ on which we do not have $u = 0$ imposed, then we may define the numerical solution, $u_N \in V_N$, to be

$$\int_{\frac{1}{2}\Omega} \nabla u_N \cdot \nabla v \, dx = \int_{\partial\Omega+} gv \, d\gamma, \quad \forall v \in V_N,$$

where $V_N = W_N \circ F$. Here, F is the affine mapping defined in Figure 3.1 and

$$W_N = \operatorname*{span}_{\substack{i=0,\dots,p \\ j=1,\dots,p}} \{L_i(s)L_j(t)\}.$$

We use here integrals of the Legendre polynomials, $L_i(\xi) = \int_{-1}^{\xi} P_{i-1}^{(0,0)}(\eta)\, d\eta$, for $i \geq 1$ and $L_0(\xi) = 1$, in order to have convenient analytical expressions for the Dirichlet forms and L^2 inner products acting on pairs of basis functions (leading to sparse stiffness and mass matrices). The load (the boundary integrals involving g_i) and the errors are computed using Simpson quadrature with 600 nodes.

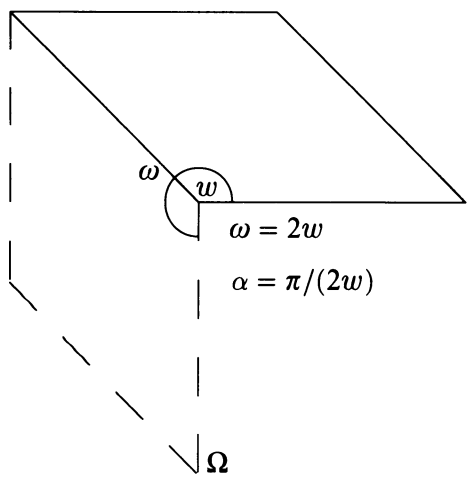

FIGURE 6.1. V-shaped domain Ω with reentrant corner of angle $2w$

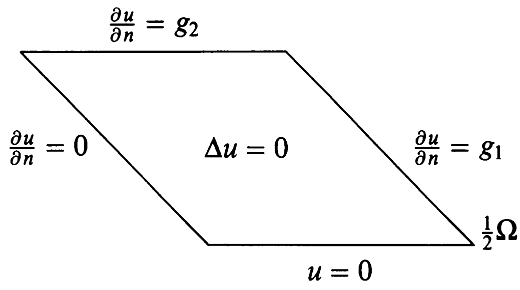

FIGURE 6.2. Elliptic boundary value problem posed over $\frac{1}{2}\Omega$

We have computed up to $p = 20$; the observed rates of convergence in the L^2 norm are depicted below:

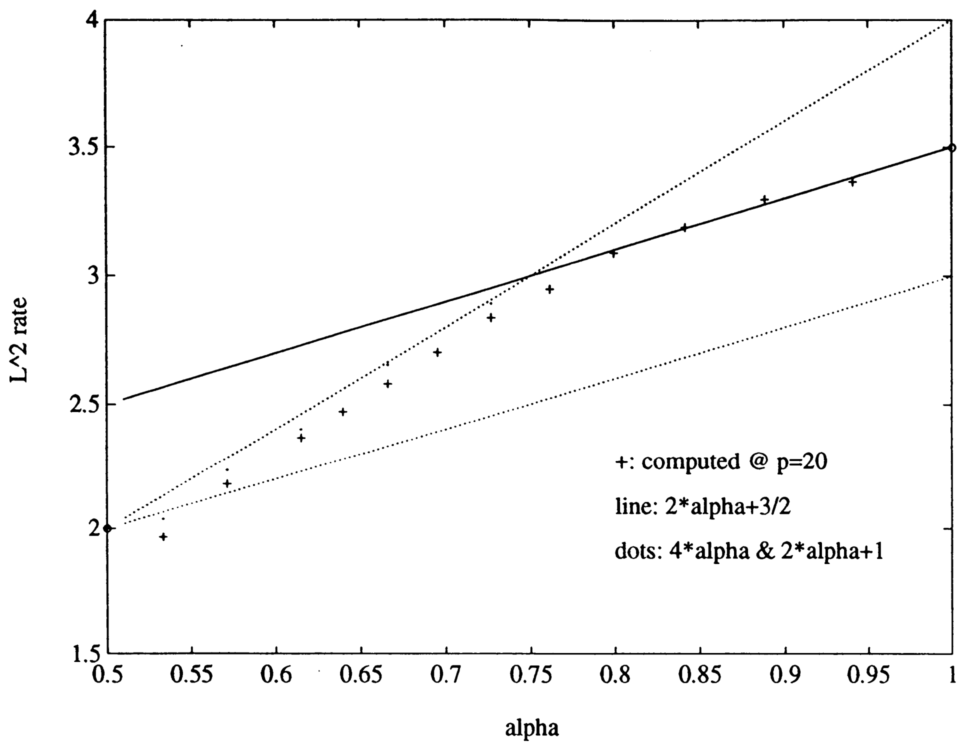

FIGURE 6.3. Numerically observed L^2 rate of convergence as dependent on α

The rates of convergence in H^1 all confirm the known rate of -2α. Although the numerical results for L^2 are somewhat inconclusive, there is some indication that a conjectured rate of $-\min\{4\alpha, 2\alpha + 3/2\}$, when the corner singularity is predominant, would be worth a proof.

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