ON THE SHARPNESS OF $L^2$-ERROR ESTIMATES OF $H^1_0$-PROJECTIONS ONTO SUBSPACES OF PIECEWISE, HIGH-ORDER POLYNOMIALS

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Dedicated to Professor R. Bruce Kellogg on the occasion of his 60th birthday

Abstract. In a plane polygonal domain, consider a Poisson problem $-\Delta u = f$ with homogeneous Dirichlet boundary condition and the $p$-version finite element solutions of this. We give various upper and lower bounds for the error measured in $L^2$. In the case of a single element (i.e., a convex domain), we reduce the question of sharpness of these estimates to the behavior of a certain inf-sup constant, which is numerically determined, and a likely sharp estimate is then conjectured. This is confirmed during a series of numerical experiments also for the case of a reentrant corner. For a one-dimensional analogue problem (of rotational symmetry), sharp $L^2$-error estimates are proven directly and via an extension of the classical duality argument. Here, we give sharp $L^\infty$-error estimates in some weighted and unweighted norms also.

1. Introduction

The purpose of the paper is to study the influence of corner singularities on the accuracy of $p$-version finite element solutions, and the sharpness of $L^2$-norm error estimates. We note that in the context of the $h$-version finite element method for a corner problem, the sharpness of $L^2$-norm error estimates, and more generally of error estimates in negative-order Sobolev norms, has been fully studied in [17]. Some attempts have also been made regarding the $p$-version of the finite element method. In [13], error estimates in $L^2$ and $H^{-s}$ ($s > 0$) are derived, using the traditional duality technique. However, the question of the sharpness of $L^2$-norm error estimates is unanswered. This paper is a further attempt to answer the question. In §2, we give an $L^2$-norm error bound. In §3, we derive a lower bound for the $L^2$-norm error. By connecting to the optimality constant, the stability constant and inf-sup constants, we conjecture a likely optimal $L^2$-norm error estimate. The conjectured optimal $L^2$-norm error estimate is proved in §4 in a one-dimensional setting which mimics the two-dimensional corner singularities. The one-dimensional singular...
model problem we study is in some sense closer to two-dimensional corner problems than the one considered in [8]. In §5, we prove various sharp $L^\infty$-norm error estimates for the one-dimensional model problem. In the last section, we present numerical results on corner domains with various internal angles to confirm the likely optimal $L^2$-norm error estimate.

Let $\Omega$ be a bounded, simply connected polygonal domain in the plane. Consider the Dirichlet boundary value problem

\begin{equation}
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\end{equation}

with $f \in H^{s-1}(\Omega)$ for some $s \geq 0$. Let $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{M-1} \leq \omega_M$ denote the interior angles at the corners $A_j$ of $\Omega$, $1 \leq j \leq M$. Let

\begin{equation}
\alpha_j = \frac{\pi}{\omega_j} \in \left[ \frac{1}{2}, \infty \right), \quad 1 \leq j \leq M.
\end{equation}

The regularity of the solution $u$, in the presence of corners, is the subject of some classical treatments, cf. Dauge et al. [5], Grisvard [10], Kellogg [14] and Kondrat'ev [15]. Following Dauge et al., we introduce singular functions, written in terms of polar coordinates, centered at the vertex of a cone $\Gamma$. Within a neighborhood of a corner $A_j$, the domain $\Omega$ coincides with a cone $\Gamma_j = \{(r, \theta_j) : 0 < r, 0 < \theta_j < \omega_j \}$. Then, the singular functions are

\begin{equation}
S_{j, l} = \eta_j \sigma_{j, l} = \eta_j \sigma_{j, l} \log r_j + \theta_j \cos(\lambda_{j, l} \theta_j) \quad \text{if } \lambda_{j, l} \in \mathbb{Z}_+,
\end{equation}

where $\eta_j \in C^\infty$ is a cutoff function, $\eta_j = 1$ near $A_j$, and $\eta_j = 0$ outside some neighborhood of $A_j$,

\begin{equation}
\sigma_{j, l} = \begin{cases}
\lambda_{j, l} \sin(\lambda_{j, l} \theta_j) & \text{if } \lambda_{j, l} \not\in \mathbb{Z}_+, \\
\lambda_{j, l} (\sin(\lambda_{j, l} \theta_j) \log r_j + \theta_j \cos(\lambda_{j, l} \theta_j)) & \text{if } \lambda_{j, l} \in \mathbb{Z}_+
\end{cases}
\end{equation}

and

\begin{equation}
\lambda_{j, l} = l \alpha_j = \frac{l \pi}{\omega_j}.
\end{equation}

Then we can write the solution to the problem (1.1) as follows: supposing $f \in H^{s-1}(\Omega)$, $s \in R_+ \setminus Z$, we have

\begin{equation}
u = u_0 + \sum_{j=1}^{M} \sum_{\lambda_{j, l} \leq s} y_{j, l} S_{j, l}
\end{equation}

with

\begin{equation}
u_0 \in H^{s+1}(\Omega) \quad \text{and} \quad \|u_0\|_{s+1} + \sum_{j, l} |y_{j, l}| \leq c (\|f\|_{s-1} + \|u\|_1).
\end{equation}

The singular expansion coefficients $y_{j, l}$ are exhibited as linear functionals of $f$ in [5, §5]. The leading singularity in (1.3) is $S_{M, 1} \in H^{1+\alpha_M - \varepsilon}(\Omega)$, $\forall \varepsilon > 0$. We denote

\begin{equation}
\alpha \overset{\text{def}}{=} \alpha_M.
\end{equation}
2. Upper $L^2$-error bounds

Partition $\Omega$ into $E$ elements $\Omega = \bigcup_{i=1}^{E} \Omega_i$, where $\Omega_i$ are parallelograms or triangles with $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\Omega_i \cap \Omega_j$ is either empty, a vertex or a common side of both $\Omega_i$ and $\Omega_j$. Each vertex of $\Omega$ is assumed to be a vertex of some $\Omega_i$. We denote by $\mathcal{P}$ the set of parallelograms and by $\mathcal{T}$ the set of triangles in the Euclidean plane.

Let $R = (-1, 1)^2$ and $T = \{(x, y) : -1 < y < x, |x| < 1\}$ denote the standard square and triangle, respectively. Let $F_i$ be an affine, orientation preserving (i.e., $\det(DF_i) > 0$) mapping which maps $\Omega_i$ onto $R$ if $\Omega_i$ is a parallelogram, and onto $T$ if $\Omega_i$ is a triangle. Then we define the space of piecewise polynomials

$$(2.1) \quad \mathcal{S}_p = \left\{ v \in H_0^1(\Omega) : v|_{\Omega_i} \circ F_i^{-1} \in \begin{cases} Q_p(R) & \text{if } \Omega_i \in \mathcal{P} \\ P_p(T) & \text{if } \Omega_i \in \mathcal{T} \end{cases} \right\},$$

where $Q_p$ and $P_p$ denote the spaces of polynomials of separate degree, respectively total degree, less than or equal to $p$.

The Galerkin solution $u_p \in \mathcal{S}_p$, characterized by varying $p$ (and fixed partition), is then defined by

$$(2.2) \quad \int_{\Omega} \nabla u_p \nabla v \, dx = \int_{\Omega} fv \, dx, \quad \forall v \in \mathcal{S}_p,$$

or we may write $u_p = \Pi_p u$, where $\Pi_p$ is the $H_0^1$-projection onto $\mathcal{S}_p$ (defined by the Dirichlet form in (2.2)).

It is well known, cf. [3], that, denoting by $e_p = u - u_p$, one has

$$(2.3) \quad \|e_p\|_1 \leq c \, p^{- \min(2\alpha, s)}$$

and that, in general, one cannot expect a better convergence rate than that in the estimate (2.3). In this paper, we are interested in the $L^2$-norm error estimate.

**Proposition 2.1** (as in [13, Theorem 2]). Let $f \in H^{s-1}(\Omega), s \in R_+ \setminus Z$. Assume $\alpha = \alpha_M \notin Z$. Let $u_p = \Pi_p u$ be defined as above. Then

$$(2.4) \quad \|e_p\|_0 \leq c \, p^{- \min(s+1, 2\alpha+1)}.$$

**Proof.** Define an auxiliary function $w_p \in H_0^1(\Omega)$ by

$$(2.5) \quad \begin{cases} -\Delta w_p = e_p & \text{in } \Omega, \\ w_p = 0 & \text{on } \partial \Omega. \end{cases}$$

In the case of a convex $\Omega$ (where the shift theorem $\|w_p\|_2 \leq c \|e_p\|_0$ holds),

$$\|e_p\|_0^2 = (e_p, -\Delta w_p) = (\nabla e_p, \nabla w_p) = (\nabla e_p, \nabla (w_p - z_p)) \leq \|e_p\|_1 \|\nabla (w_p - z_p)\|_0 \leq c \|e_p\|_1 p^{-1} \|w_p\|_2 \leq c \|e_p\|_1 p^{-1} \|e_p\|_0$$

holds for some $z_p \in \mathcal{S}_p$. So,

$$\|e_p\|_0 \leq c \, p^{-1} \|e_p\|_1$$

and (2.4) follows. (Cf. [12, Prop. 3.2] for a similar argument for higher-order problems.) In the case of a nonconvex $\Omega$ partitioned with multiple elements,
we split $w_p$—as done in [13]—according to (cf. (1.3))

$$w_p = \bar{w}_p + \sum_{j=1}^{M} \sum_{\lambda_j, l < 1} \gamma_{j, l} S_{j, l}$$

with $\bar{w}_p \in H^2(\Omega)$, and

$$\|\bar{w}_p\|_2 + \sum_{\lambda_j, l < 1} |\gamma_{j, l}| \leq c (\|e_p\|_0 + \|w_p\|_1) \leq c \|e_p\|_0.$$ 

Thus, for some $z_p \in S^p$,

$$\|\nabla (w_p - z_p)\|_0 \leq c \left( p^{-1} \|w_p\|_2 + \sum_{\lambda_j, l < 1} |\gamma_{j, l}| p^{-2l} \right) \leq c p^{-1} \|e_p\|_0.$$

So, again, $\|e_p\|_0 \leq c p^{-1} \|e_p\|_1$ and (2.4) follows. □

3. The $L^2$-distance of the $H^1_0$-projection from the exact solution bounded in terms of a stability constant

Let $u_p$ be the Galerkin solution (i.e., $H^1_0$-projection onto $S^p$). We shall derive some lower bounds on the error $u - u_p$ measured in $L^2$, which are of lesser convergence rate than those of the $L^2$-projection. We recall that, cf. [3], for a problem like (1.1) with leading singularity $r^\alpha \sin \alpha \theta$,

$$\inf_{v \in S^p} \|u - v\|_0 \leq c p^{-(2\alpha + 2)}.$$  \hfill (3.1)

Following an idea of Wahlbin [17], one may easily prove

**Proposition 3.1.** Let $u$ and $u_p$ be the exact and discrete solutions to (1.1) as defined before. Let $\Omega$ have reentrant corners. Then, if the coefficient of the leading term $r^\alpha \sin \alpha \theta$ in the singular expansion of $u$ is nonzero, there exists a $c > 0$ such that

$$\|u - u_p\|_0 \geq c p^{-4\alpha}. \hfill (3.2)$$

**Proof.** A mere transcription of the main idea in [17]. Like in [17], it is enough to prove the result for a particular solution $u^0 = a_0 r^\alpha \sin \alpha \theta + \cdots$, $a_0 \neq 0$. Let $K$ be an element in the triangulation of $\Omega$ with a vertex in common with a vertex of $\Omega$ (A$_M$, say) that supports the leading singularity. Let $x_0 \in K$ and let $B_0 \subset B_1 \subset K$ be disks centered at $x_0$ so that $\Omega_0 = B_1 \setminus B_0$ is an annulus. Let $\omega \in C^\infty(\Omega)$ be such that

$$\omega = \begin{cases} 1 & \text{outside } B_1, \\ 0 & \text{inside } B_0, \end{cases}$$

and $G(x; x_0)$ be the Green’s function for (1.1). Now let

$$u^0(x) = \omega(x) G(x; x_0).$$

Then, $u^0$ has a singularity at the vertex of the type $r^\alpha \sin \alpha \theta$, as was shown in [17]. Let the $H^1_0$-projection of $u^0$ onto $S^p$ be denoted by $u^0_p$, and let

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$e_p^0 = u^0 - u_p^0$. Then, noticing that supp $(\Delta u^0) \subset K$, we have

$$c p^{-4\alpha} \leq \|e_p^0\|^2 = \int_\Omega \nabla e_p^0 \nabla (u^0 - u_p^0) \, dx$$

$$= \int_\Omega \nabla e_p^0 \nabla u^0 \, dx = - \int_\Omega e_p^0 \Delta u^0 \, dx$$

$$\leq \|e_p^0\|_{0,K} \|\Delta u^0\|_{0,K},$$

yielding (3.2). □

**Remark 3.1.** The estimate (3.2) yields a stricter lower bound than (3.1) for $\alpha < 1$, i.e., the case of a reentrant corner. Based on the lower bound (3.2), we see that the estimate (2.4) is optimal in rate for the case of a crack domain. For other corner domains, the estimate (2.4) is generally not sharp, as will be seen from the consideration below and numerical results later. □

We next look at the convex domain case and assume for simplicity that $\Omega$ is a single parallelogram. We will then look at the situation where there is only one element, namely, $\Omega$ itself. Let $S_p = Q_p(F(\Omega)) \cap H^1_0(\Omega)$, where $F$ is affine, $\det (F) > 0$, $F(\Omega) = (-1,1)^2$. Then we may integrate (2.2) by parts to get

$$-\int_\Omega u_p \Delta v \, dx = \int_\Omega fv \, dx, \quad \forall v \in S_p.$$

We may alternatively formulate (3.3) as our finite element method with an asymmetric bilinear form. In this setting, an $L^2$-norm error estimate is the “natural” one for $e_p$, and this is intimately connected with the inf-sup or stability constant for the bilinear form in (3.3).

Denote, for convenience, the bilinear form by

$$B(u,v) = -\int_\Omega u \Delta v \, dx, \quad u \in L^2, \quad v \in H^2 \cap H^1_0.$$

We introduce two constants,

$$C(u) = \frac{\|u - u_p\|_0}{\inf_{v \in S_p} \|u - v\|_0}, \quad \text{the optimality constant at } u,$$

$$D(u) = \sup_{v \in S_p} \frac{\|(u + v)_p\|_0}{\|u + v\|_0}, \quad \text{the stability constant at } u,$$

where $u$ is given by (1.1), $u_p$ by (3.3) and $(u + v)_p \in S_p$ by

$$B((u + v)_p, w) = B(u + v, w), \quad \forall w \in S_p,$$

i.e., the projection of $u + v$ onto $S_p$ given by the inner product defined in (3.4). It was shown in [2] that $C$ and $D$ are interconnected,

$$D(u) - 1 \leq C(u) \leq D(u) + 1.$$
The constant $D$ turns out to be connected to the inf-sup constant. We introduce two new constants,

$$
\mu = \inf_{u \in L^2} \sup_{v \in H^1 \cap H^1_0} \frac{|B(u,v)|}{\|u\|_0 \|v\|_2},
$$

(3.7)

$$
\mu_p = \inf_{u \in S^p} \sup_{v \in S^p} \frac{|B(u,v)|}{\|u\|_0 \|v\|_2}.
$$

From the Buniakowsky-Cauchy-Schwarz inequality, we get

$$
|B(u,v)| \leq \sqrt{2} \|u\|_0 \|v\|_2,
$$

(3.8)

so that, via [2], the following relations hold,

$$
\frac{\mu}{\mu_p} \leq \sup_u D(u) \leq \frac{\sqrt{2}}{\mu_p}.
$$

(3.9)

We note that for any $u \in L^2(\Omega)$ there is a unique $v \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $\Delta v = u$, and since $\Omega$ is convex, we have the estimate $\|v\|_2 \leq c_\Omega \|u\|_0$. With (3.8), we easily get

$$
\frac{1}{c_\Omega} \leq \mu \leq \sqrt{2}.
$$

(3.10)

Remark 3.2. Since $\Omega$ is convex, $\|v\|_2$ is equivalent to $\|\Delta v\|_0$ on $H^2(\Omega) \cap H^1_0(\Omega)$. If we replace $\|v\|_2$ by $\|\Delta v\|_0$ in the definition of $\mu$, then we have $\mu = 1$. □

Clearly, the existence of quasi-optimal error estimates is linked with the inf-sup constant $\mu_p$. We will show that in some sense to be made more clear in the next lemma, it is enough to consider $\mu_p$ for a square.

Let $P_{\omega}$ be the parallelogram in Figure 3.1.

The affine mapping which maps $P_{\omega}$ onto $R = (-1,1)^2$ is

$$
F(x) = 2 \begin{pmatrix} 1 & -\cot \omega \\ 0 & \csc \omega \end{pmatrix} x - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

Let $S^p_{\omega} = \{ \hat{v} = v \circ F : v \in S^p(R) \}$ with

$$
S^p(R) = \bar{Q}_p(R) = (1-x_1^2)(1-x_2^2) Q_{p-2}(R).
$$
Define, as in (3.7),

\[
\mu_{p, \omega} = \inf_{u \in S_{p, \omega}} \sup_{\bar{v} \in S_{p, \omega}} \frac{|B(u, \bar{v})|}{\|u\|_0, P_{p, \omega} \|\bar{v}\|_2, P_{p, \omega}}.
\]

**Lemma 3.2.** Let \( \mu_p = \mu_{p, \pi/2} \) be as defined in (3.11) with respect to \( P_{\pi/2} = R \) and \( \mu_{p, \omega} \) as in (3.11). Then, given a \( \omega_0 \in (0, \pi/2) \), there exist \( c \) and \( C \), depending on \( \omega_0 \), so that

\[
\mu_p \leq \mu_{p, \omega} \leq C \mu_p, \quad \forall \omega \in [\omega_0, \pi - \omega_0].
\]

**Proof.** We prove the equivalence between the respective inner products in (3.11). Let \( x \approx y \) denote an equivalence in the sense that there exist positive constants \( c \) and \( C \) independent of \( x \) and \( y \), so that \( cx \leq y \leq Cx \). First we see that

\[
\int_{P_{p, \omega}} \bar{u} \bar{\bar{v}} \, dx = \int_R uv |J|^{-1} \, dx \approx \int_R uv \, dx,
\]

where \( J = DF \) is the Jacobian of \( F \) and \( |J| = 4 \csc \omega \), provided we require \( \omega_0 \leq \omega \leq \pi - \omega_0 \) for some \( \omega_0 \in (0, \pi/2) \). Secondly, one may prove that

\[
\int_{P_{p, \omega}} \nabla \bar{x} \nabla \bar{\bar{v}} \, dx = \int_R \nabla u J^T J \nabla v |J|^{-1} \, dx \approx \int_R \nabla u \nabla v \, dx
\]

by verifying that the eigenvalues of \( J^T J \) lie in a positive interval for \( \omega_0 \leq \omega \leq \pi - \omega_0 \). This gives the equivalence of the bilinear forms. Thirdly,

\[
|\bar{u}|^2_{H^2(P_{p, \omega})} = \int_R (D^2 u)^T H D^2 u |J|^{-1} \, dx \approx |u|^2_{H^2(R)},
\]

where \( H = \sum_{i,j=1}^2 f_{ij} f_{ij} \) with \( f_{ij} = (J_{1i} J_{1j}, J_{1i} J_{2j} + J_{2i} J_{1j}, J_{2i} J_{2j}) \) and \( D^2 u = (u_{11}, u_{12}, u_{22}) \). Through a lengthy calculation, it can be shown that the eigenvalues of \( H \) are in some positive interval for \( \omega_0 \leq \omega \leq \pi - \omega_0 \). We proved \( |\bar{u}|^2_{H^k(P_{p, \omega})} \approx |u|^2_{H^k(R)} \) for \( k = 0, 1 \) earlier. Thus, \( |\bar{u}|^2_{H^2(P_{p, \omega})} \approx |u|^2_{H^2(R)} \) and (3.12) follows from the definition (3.11).

**Remark 3.3.** We may equally well handle a reduction from an (oblique) triangle to a right-angled one.

**Remark 3.4.** If \( \Omega \in \mathcal{S} \), then \( \mu_p \geq c p^{-1} \) for some constant \( c \) independent of \( p \), by the equivalence in the above lemma and the lower bound in (3.9). For certain special cases involving \( C^1 \)-elements, one will not be able to obtain a better inequality, \( \mu_p \geq c p^{-\nu} \), for \( \nu \in (0, 1) \). Specifically, consider solving \(-\Delta u = f\) in the V-shaped domain of Figure 6.1, with smooth \( f \) and zero Dirichlet boundary conditions, using \( p \)-version \( C^1 \)-elements. Collecting several earlier results, we have

\[
c(\alpha) p^{-4\alpha} \leq \|u - u_p\|_0 \leq C(\alpha) \left( \frac{\sqrt{2}}{\mu_p} + 1 \right) p^{-2\alpha}.
\]

Now select \( \alpha = (1 + \varepsilon)/2 \) for small \( \varepsilon > 0 \); then isolating \( \mu_p \) and contracting the above inequalities, one finds

\[
\mu_p \leq c(\varepsilon) p^{-1+\varepsilon}.
\]
Remark 3.5. Lemma 3.2 is significant in the sense that in the guaranteed upper estimate $C(u) \leq \sqrt{2/\mu_p, \omega} + 1 \leq c/\mu_p, \pi/2 + 1$, one will observe the same degree of suboptimality for all angles in any compact subinterval of $(0, \pi)$. □

The inf-sup constant $\mu_p, \pi/2$, which by Lemma 3.2 characterizes the behavior of all convex, single-element cases, can be computed numerically for a given choice of finite element space, via a generalized eigenvalue problem. This was done for $Q_p$ in [13] with an emerging behavior of

$$c p^{-1/2} \leq \mu_p, \pi/2 \leq C p^{-1/2},$$

indicating the loss of a half power of $p$ relative to the $L^2$-distance (3.1). It is an open problem to prove (3.13) theoretically. If one were able to show (3.13), it would follow that $\|e_p\|_0 \leq c p^{-2a-3/2}$. We shall further justify this for a purely radial version of the problem in the next section and further demonstrate that the classical duality argument can be extended to obtain the optimal convergence estimate $\|e_p\|_0 \leq c p^{-2a-3/2}$, in contrast to (2.4).

4. A ONE-DIMENSIONAL ANALOGUE

It is well known that the singularities of (1.1) supported at the vertices of $\Omega$ are essentially radial and of the form $r^\alpha g(r, \theta)$ in local polar coordinates. We shall investigate a purely radial model problem with such a singularity and $g \equiv 1$ for simplicity.

The purely radial singular function $u = r^\alpha$, $\alpha > 0$, is the unique solution to

$$\frac{1}{r} (ru')' = \alpha^2 r^{\alpha-2} \quad \text{in } (0, 1),$$
$$|u(0)| < \infty, \quad u(1) = 1.$$

Let

$$R^0_N = \{v : v \text{ is a polynomial of degree } \leq N, \quad v(1) = 0\}.$$

Then, for $N \geq 1$, the Galerkin solution $u_N \in r + R^0_N$ satisfies

$$\int_0^1 (u_N' - u')v' r dr = 0, \quad \forall v \in R^0_N,$$

which is then our simple one-dimensional problem.

Remark 4.1. An indication of how (4.1), then (4.3), could arise naturally follows. Let $\Omega$ be the intersection of a cone with opening $\pi/\alpha$ ($\alpha \in \mathbb{Z}_+$), $\alpha \geq 1/2$, and the open unit ball centered at the apex of the cone whose right leg is assumed to coincide with the positive $x$-axis. Let $\hat{u} = r^\alpha \sin \alpha \theta$ be the exact solution to

$$\begin{cases}
-\Delta \hat{u} = 0 & \text{in } \Omega = \{(r, \theta) : r \in (0, 1), \quad \theta \in (0, \pi/\alpha)\}, \\
\hat{u} = 0 & \text{on } \partial \Omega \cap \{(r, \theta) : \theta = 0 \text{ or } \theta = \pi/\alpha\}, \\
\hat{u} = \sin \alpha \theta & \text{on } \partial \Omega \cap \{(r, \theta) : r = 1\}.
\end{cases}$$

The purely radial part of $\hat{u}$ is $u$. Suppose we used finite elements of the type $u_N(r) \sin \alpha \theta$, where $u_N(r)$ were a polynomial of degree at most $N$, equal to 1
at \( r = 1 \) (vanishing at 1 for the test functions). Then the Galerkin equation would read

\[
0 = \int_{\Omega} \nabla[(u_N - u) \sin \alpha \theta] \nabla[v \sin \alpha \theta] r \, dr \, d\theta
\]

(4.5)

\[
= \frac{\pi}{2\alpha} \int_{0}^{1} \left\{ (u_N' - u') v' + \frac{\alpha^2}{r^2} (u_N - u) v \right\} r \, dr.
\]

If we now drop the zeroth-order term, we get (4.3). □

Let us define the weighted inner product

(4.6) \( (u, v) = \int_{0}^{1} u v r \, dr \)

for \( u, v \in L^2_{01}(0, 1) \), subscripts corresponding to our weight \((1 - r)^0 r^1\) at the endpoints of \((0, 1)\). Then (4.3) implies:

**Lemma 4.1.** The derivative \( u_N' \) is the \( L^2_{01}(0, 1) \) projection of \( u' \) onto \( P_{N-1} \).

**Proof.** A constant is the derivative of a function in \( \Sigma_P^0 : 1 = d(r - 1)/dr \). The remaining monomials of \( P_{N-1} \) also belong to \( d(P_n^0)/dr \). □

Note that we may translate all statements to the interval \((-1, 1)\) by the affine transformation

(4.7) \( (0, 1) \ni r \mapsto x = t(r) = 2r - 1 \in (-1, 1) \).

We recall that on the standard interval \([-1, 1]\) the Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \), \( n \geq 0 \), are orthogonal with respect to the weight \( w(x) = (1 - x)^\alpha (1 + x)^\beta \), \( \alpha > -1 \), \( \beta > -1 \). The value of \( P_n^{(\alpha, \beta)}(x) \) at \( x = 1 \) is \( (n+\alpha) \), and

\[
\int_{-1}^{1} w(x) |P_n^{(\alpha, \beta)}(x)|^2 \, dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{n! \Gamma(n+\alpha+\beta+1)}.
\]

Also recall Rodrigues’ formula

\[
P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{1}{w(x)} \frac{d^n}{dx^n} \left[ w(x) (1 - x^2)^n \right].
\]

We will use in particular the Jacobi polynomials \( \{P_n^{(0,1)}\}_{n=0}^{\infty} \), which are orthogonal in \( L^2_{01}(-1, 1) \), cf. [6] or [16]. Let \( \{j_n\}_{n=0}^{\infty} = \{P_n^{(0,1)} \circ t\}_{n=0}^{\infty} \) be the corresponding orthogonal sequence in \( L^2_{01}(0, 1) \) obtained via the transformation (4.7). Then, we have

**Theorem 4.2.** The following estimate holds for \( u = r^\alpha \):

(4.8) \[
\|u' - u_N'\|_{L^2_{01}(0, 1)} \leq c N^{-2\alpha},
\]

and the estimate is sharp.

**Proof.** This follows from Lemma 4.1, the form of \( u' = \alpha r^{\alpha-1} \) and standard approximation results for polynomials, see [11], [8] or later [4]: using Rodrigues’
and Stirling's formulas, one gets

\begin{equation}
\label{eq:4.9}
\begin{aligned}
\quad \quad u' = \sum_{n=0}^{\infty} b_n j_n, \\
\quad \quad u'_N = \sum_{n=0}^{N-1} b_n j_n,
\end{aligned}
\end{equation}

where

\begin{equation}
\label{eq:4.10}
\begin{aligned}
\quad b_n &= \frac{(u', j_n)}{(j_n, j_n)} = (-1)^{n+1} 2^{(n+1)} (n+1) \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+2+\alpha)} \\
&= (-1)^{n+1} c_0(\alpha) n^{-2\alpha} (1 + (1 + 2\alpha)/n + O(n^{-2}))
\end{aligned}
\end{equation}

in which \( c_0(\alpha) = \Gamma(\alpha+1)/\Gamma(-\alpha) \), and \( \lim_{\alpha \to k} c_0(\alpha) = 0 \) for \( k > 0 \) integer. We shall have occasion to use (4.10) again. \( \Box \)

The same argument yields:

**Proposition 4.3.** The following \( N \)-distance holds for \( u = r^\alpha \) in \( L^2_0(0, 1) \) with respect to the subspace \( \mathcal{P}_N \) :

\begin{equation}
\label{eq:4.11}
\inf_{v \in \mathcal{P}_N} \|u - v\|_{L^2_0(0, 1)} = c(\alpha) N^{-2\alpha+2} (1 + O(N^{-1})).
\end{equation}

**Remark 4.2.** The results (4.8) and (4.11) give the "correct" rates also for the two-dimensional problems with dominant corner singularity \( \hat{u} \); this follows from the analysis in [3] and [7]. \( \Box \)

Let us calculate the \( L^2_0(0, 1) \) error estimate directly and show to what extent this in turn is provable via the usual duality technique. Define

\begin{equation}
\label{eq:4.12}
\begin{aligned}
\quad \quad j_n^{(-1)}(r) &= \int_1^r j_n(s) \, ds, \quad n \in Z_+ \cup \{0\}.
\end{aligned}
\end{equation}

Then, using (4.9), we get

\begin{equation}
\label{eq:4.13}
\begin{aligned}
\quad \quad u - u_N &= \sum_{n=N}^{\infty} b_n j_n^{(-1)},
\end{aligned}
\end{equation}

and it becomes beneficial to have a formula relating \( j_n^{(-1)} \) to a linear combination of \( j_k \).

**Lemma 4.4.** Let \( n \geq 2 \). Then the following formulas hold:

\begin{equation}
\label{eq:4.14}
\begin{aligned}
\quad \quad j_n^{(-1)} &= c_{n,n+1} j_{n+1} + c_{n,n} j_n + c_{n,n-1} j_{n-1},
\end{aligned}
\end{equation}

where

\begin{equation}
\label{eq:4.15}
\begin{aligned}
\quad \quad c_{n,n+1} &= \frac{n+2}{(2n+3)(2n+2)}, \\
\quad \quad c_{n,n} &= \frac{-1}{(2n+3)(2n+1)}, \\
\quad \quad c_{n,n-1} &= \frac{-n}{(2n+2)(2n+1)}
\end{aligned}
\end{equation}

and

\begin{equation}
\label{eq:4.16}
\begin{aligned}
\quad \quad \int_1^x P_n^{(0,1)}(u) \, du &= c_{n,n+1} P_{n+1}^{(0,1)}(x) + c_{n,n} P_n^{(0,1)}(x) + c_{n,n-1} P_{n-1}^{(0,1)}(x),
\end{aligned}
\end{equation}

where \( c_{n,k} = 2c_{n,k} \), \( k = n+1, n, n-1 \).
Proof. By the substitution $u = 2s - 1$,
\[
  j_n^{(-1)}(r) = \int_0^r j_n(s) \, ds = \int_1^r P_n^{(0,1)}(2s - 1) \, ds
  = \frac{1}{2} \int_1^{2r-1} P_n^{(0,1)}(u) \, du,
\]
and we need only verify (4.16). Clearly,
\[
  (4.17) \quad \int_1^x P_n^{(0,1)}(u) \, du = \sum_{m=0}^{n+1} c_{n,m} P_m^{(0,1)}(x),
\]
and we may integrate by parts to get with
\[
  (u, v) = \int_{-1}^1 u v (1 + x) \, dx
\]
that
\[
  c_{n,m}(P_m^{(0,1)}, P_m^{(0,1)}) = \sum_{k=0}^{n+1} c_{n,k}(P_m^{(0,1)}, P_k^{(0,1)})
  = \int_{-1}^1 \left( \int_{-1}^x P_n^{(0,1)}(u) \, du \right) P_m^{(0,1)}(x) (1 + x) \, dx
  = \left[ \left( \int_{-1}^x P_n^{(0,1)}(u) \, du \right) \left( \int_{-1}^x P_m^{(0,1)}(u) (1 + u) \, du \right) \right]_{-1}^1
  - \int_{-1}^1 P_n^{(0,1)}(x) \left( \int_{-1}^x P_m^{(0,1)}(u) (1 + u) \, du \right) \, dx
  = \int_{-1}^1 P_n^{(0,1)}(x) \left( \int_{-1}^x P_m^{(0,1)}(u) (1 + u) \, du \right) \, dx
\]
for $m \in \mathbb{Z}_+ \cup \{0\}$, by orthogonality. Here, we may factor
\[
  \int_{-1}^x P_m^{(0,1)}(u) (1 + u) \, du = (1 + x) q_{m+1}(x)
\]
for some $q_{m+1} \in \mathcal{B}^{m+1}$. Thus,
\[
  (4.19) \quad c_{n,m} = 0 \quad \text{for } 0 \leq m \leq n - 2.
\]
Identity (4.16) in [16] specializes to
\[
  (2n + 2) P_n^{(0,1)}(x) = (n + 2) P_n^{(1,1)}(x) - (n + 1) P_n^{(1,1)}(x)
\]
and identity (4.14) in [16] gives
\[
  \frac{d}{dx} P_n^{(0,0)}(x) = \frac{1}{2} (n + 1) P_n^{(1,1)}(x),
\]
so that, with $l_n(x) = P_n^{(0,0)}(x)$, the Legendre polynomial of degree $n$,
\[
  (4.20) \quad P_n^{(0,1)}(x) = \frac{1}{n+1} (l_{n+1}(x) - l'_n(x)).
\]
Identity (4.16) in [16] also yields

\[(4.21) \quad (1 + u) p_n^{(0, 1)}(u) = p_n^{(0, 0)} (u) + p_{n+1}^{(0, 0)} (u) , \]

so that

\[(4.22) \quad \int_{-1}^{x} p_m^{(0, 1)} (u) (1 + u) \, du = \int_{-1}^{x} l_m(u) \, du + \int_{-1}^{x} l_{m+1}(u) \, du . \]

But, for \( j \geq 1 \),

\[(4.23) \quad \int_{-1}^{1} l'(x) \left( \int_{-1}^{x} l_j(u) \, du \right) \, dx
\]

\[= \left[ l(x) \left( \int_{-1}^{x} l_j(u) \, du \right) \right]_{-1}^{1} - \int_{-1}^{1} l_i(x) l_j(x) \, dx
\]

\[-= \frac{2}{2i + 1} \delta_{ij} . \]

Substituting (4.20) and (4.22) into (4.18), using (4.23) and the fact that

\[(P_m^{(0, 1)}, P_m^{(0, 1)}) = 2/(m + 1) , \]

we have

\[c_{n,n+1} = \frac{n + 2}{(2n + 3)(n + 1)} , \]

\[(4.24) \quad c_{n,n} = -\frac{2}{(2n + 3)(2n + 1)} , \]

\[c_{n,n-1} = -\frac{n}{(2n + 1)(n + 1)} . \]

Combining (4.17), (4.19), and (4.24) yields (4.16). \( \square \)

Inserting (4.14) into (4.13) gives

\[(4.25) \quad u - u_N = \sum_{n=N}^{\infty} b_n (c_{n,n+1} j_{n+1} + c_{n,n} j_n + c_{n,n-1} j_{n-1})
\]

\[= \sum_{k=N+1}^{\infty} (c_{k-1,k} b_{k-1} + c_{k,k} b_k + c_{k+1,k} b_{k+1}) j_k
\]

\[+ (c_{N,N} b_N + c_{N+1,N} b_{N+1}) j_N + c_{N,N-1} b_N j_{N-1} . \]

Since \( |c_{k,k}| = 1/(4k^2) + O(k^{-3}) \) and \( -c_{k+1,k} \) and \( c_{k-1,k} \) are both equal to \( 1/(4k) + O(k^{-2}) \), we get

\[\|u - u_N\|_{L^2_{\delta_1}} = c \left\{ \sum_{k=N+1}^{\infty} k^{-4\alpha} k^{-3} k^{-1} + N^{-4\alpha} N^{-2} N^{-1} \right\} \left( 1 + O(N^{-1}) \right)
\]

\[= c N^{-4\alpha - 3} \left( 1 + O(N^{-1}) \right) , \]

so that

\[(4.26) \quad \|u - u_N\|_{L^2_{\delta_1}} = c N^{-2\alpha - 3/2} \left( 1 + O(N^{-1}) \right) . \]
This is half a power off the $N$-distance in (4.11). We will now show that the usual duality argument can be extended to recover (4.26) and not just one power better than the $H^1$ estimate (4.8).

**Remark 4.3.** One notes that the last two terms in (4.25) dominate asymptotically, cf. Lemma 5.2.

**Theorem 4.5.** Let $u = r^\alpha$ and $u_N$ be determined by (4.3). Then there exist positive constants independent of $N$ so that

$$c N^{-(2\alpha+3/2)} \leq \|u - u_N\|_{L^2_{01}} \leq C N^{-(2\alpha+3/2)}.$$

**Proof.** This is a restatement of (4.26). We will prove the second inequality once more, now via an extension of the usual duality argument, in the hope of providing a possible approach to deal with higher-dimensional problems. Let $e_N = u - u_N$. Then, defining $w_N$ by

$$\begin{cases}
-\frac{1}{r} (rw_N')' = e_N & \text{in } (0, 1), \\
|w_N(0)| < \infty, \quad w_N(1) = 0,
\end{cases}$$

we get for any $z_N \in \mathcal{P}_N^0$,

\begin{equation}
(e_N, e_N) = \int_0^1 e_N \left( -\frac{1}{r} (rw_N')' \right) r \, dr = -[e_N(rw_N')]_0^1 + \int_0^1 e_N' w_N' r \, dr
\end{equation}

by repeated integration by parts and the $H^1$-projection nature of $u_N$. An immediate corollary is

\begin{equation}
\|e_N\|_{L^2_{01}} = \inf_{z_N \in \mathcal{P}_N^0} \left\|e_N + \frac{1}{r} (r z_N')' \right\|_{L^2_{01}}.
\end{equation}

One gets a similar result for a single element in two dimensions. Now $\mathcal{P}_{N-2}$ is contained in the image of $\mathcal{P}^0_N$ under the map $z \mapsto (1/r) (rz')'$, and we get from (4.9) and (4.11), and by using Lemma 4.4, that

$$\left\|e_N + \frac{1}{r} (r z_N')' \right\|_{L^2_{01}} \leq \inf_{\zeta_N \in \mathcal{P}_{N-2}} \|u - \zeta_N\|_{L^2_{01}} + \inf_{\zeta_N \in \mathcal{P}_{N-2}} \|u_N - \zeta_N\|_{L^2_{01}}$$

$$\leq c N^{-2\alpha-2} + \sum_{n=0}^{N-1} b_n j_n^{(-1)} \| \zeta_N - \zeta_N' \|_{L^2_{01}}$$

$$\leq c N^{-2\alpha-2} + \| b_{N-1} c_{N-1} N j_N \|_{L^2_{01}}$$

$$+ \| (b_{N-2} c_{N-2}, N-1 + b_{N-1} c_{N-1}, N-1) j_N \|_{L^2_{01}}$$

$$\leq c N^{-(2\alpha+3/2)}.$$

\[ \Box \]
Remark 4.4. These results do not follow automatically from [2] or [8], since we are dealing with an entirely different inner product geared towards the two-dimensional setting. They could, on the other hand, be used to establish instability results like the ones in [2], now for the nonconstant coefficient operator \(-\frac{1}{r} \frac{d}{dr} (r \frac{d}{dr})\).

Remark 4.5. It is not difficult to extend the results to a three-dimensional model scenario with the operator \(-\frac{1}{r} \frac{d}{dr} (r^2 \frac{d}{dr})\), or model \(d\)-dimensional problems with symmetry group \(S^{d-1}\).

Remark 4.6. It is interesting that the Green's function no longer has the \(r^\alpha\) singularity used to prove (3.2) in Proposition 3.1, but that we may still prove a Wahlbin-like lower bound along the same lines (with \(u^0 = r^\alpha\)) to obtain
\[
\|e_N\|_{L_0^\alpha} \geq c \frac{\sqrt{2\alpha - 2}}{\alpha^2} N^{-4\alpha},
\]
which is valid only for \(\alpha \geq 1\) (i.e. corresponding to the convex case).

5. Sharp \(L^\infty\)-error estimates for the one-dimensional model

To better understand how far one can go with error estimates for the \(p\)-version method, we derive various \(L^\infty\)-norm error estimates for \(p\)-version finite element solutions for the one-dimensional model problem considered in the previous section. We need asymptotic expansions of the various coefficients that appeared.

Lemma 5.1. For the coefficients defined in (4.15), we have
\[
c_{n,n+1} = \frac{1}{4n} - \frac{1}{8n^2} + O(n^{-3}),
\]
\[
c_{n,n} = -\frac{1}{4n^2} + O(n^{-3}),
\]
\[
c_{n,n-1} = -\frac{1}{4n} + \frac{3}{8n^2} + O(n^{-3}),
\]
\[
c_{n-1,n} = \frac{1}{4n} + \frac{1}{8n^2} + O(n^{-3}),
\]
\[
c_{n+1,n} = -\frac{1}{4n} + \frac{5}{8n^2} + O(n^{-3}).
\]

The proof of the lemma is elementary, and hence is omitted.

Now let us define
\[
d_{N-k} = c_{k-1,k} b_{k-1} + c_{k,k} b_k + c_{k+1,k} b_{k+1}, \quad k \geq N + 1,
\]
\[
d_{N} = c_{N,N} b_{N} + c_{N+1,N} b_{N+1},
\]
\[
d_{N-1} = c_{N,N-1} b_{N}.
\]

Then, the error expression (4.25) can be rewritten as
\[
u - u_N = \sum_{k=N+1}^{\infty} d_{N-k} j_k + d_{N,N} j_N + d_{N-1,N-1} j_{N-1}.
\]
For the asymptotic behavior of \( \{d_{N,k}\} \), we have

**Lemma 5.2.** There holds

\[
\begin{align*}
d_{N,k} &= (-1)^k (1 + \alpha) c_0(\alpha) k^{-2\alpha - 2} (1 + O(k^{-1})), \quad k \geq N + 1, \\
d_{N,N} &= (-1)^{N+1} \frac{c_0(\alpha)}{4} N^{-2\alpha-1} (1 + O(N^{-1})), \\
d_{N,N-1} &= (-1)^N \frac{c_0(\alpha)}{4} N^{-2\alpha-1} (1 + O(N^{-1})).
\end{align*}
\]

The proof of Lemma 5.2 can be made by combining the results from Lemma 5.1 and (4.10).

Now, we are ready to give \( L^\infty \)-norm error estimates on \( u - u_N \).

**Theorem 5.3.** Let \( \alpha \notin \mathbb{Z}_+ \). Then there exists a positive \( c \) so that

\[
\|u - u_N\|_{L^\infty(0,1)} \leq c N^{-2\alpha},
\]

and the estimate is sharp.

**Proof.** From [1, Formula 22.14.1], we have

\[
|P_n(0,1)(x)| \leq n + 1, \quad x \in [-1, 1].
\]

Thus,

\[
|j_n(r)| \leq n + 1, \quad r \in [0, 1].
\]

Then, from (5.4) and Lemma 5.2, we get

\[
|u - u_N| \leq \sum_{k=N+1}^{\infty} c_0(\alpha) (1 + \alpha) k^{-2\alpha - 1} (1 + O(k^{-1}))
\]

\[
+ \frac{c_0(\alpha)}{4} N^{-2\alpha} (1 + O(N^{-1})) + \frac{c_0(\alpha)}{4} N^{-2\alpha} (1 + O(N^{-1}))
\]

\[
\leq c(\alpha) N^{-2\alpha}.
\]

To show that the estimate is sharp, we compute the error at \( r = 0 \). We have

\[
j_n(0) = P_n(0,1)(-1) = (-1)^n (n + 1).
\]

Hence,

\[
u(0) - u_N(0) = \sum_{k=N+1}^{\infty} (-1)^k (1 + \alpha) c_0(\alpha) k^{-2\alpha - 2} (1 + O(k^{-1})) (-1)^k (k + 1)
\]

\[
+ (-1)^{N+1} \frac{c_0(\alpha)}{4} N^{-2\alpha-1} (1 + O(N^{-1})) (-1)^N (N + 1)
\]

\[
+ (-1)^N \frac{c_0(\alpha)}{4} N^{-2\alpha-1} (1 + O(N^{-1})) (-1)^{N-1} N
\]

\[
= (1 + \alpha) c_0(\alpha) \sum_{k=N+1}^{\infty} k^{-2\alpha - 1} (1 + O(k^{-1}))
\]

\[
- \frac{c_0(\alpha)}{2} N^{-2\alpha} (1 + O(N^{-1}))
\]

\[
= \frac{1}{2\alpha} c_0(\alpha) N^{-2\alpha} + O(N^{-2\alpha-1}),
\]
where we have used the fact that
\[
\sum_{k=N+1}^{\infty} k^{-2a-1} = \frac{1}{2\alpha} N^{-2\alpha} (1 + O(N^{-1})),
\]
which is evident from the following inequalities:
\[
\int_{N+1}^{\infty} t^{-2a-1} \, dt \leq \sum_{k=N+1}^{\infty} k^{-2a-1} \leq \int_{N}^{\infty} t^{-2a-1} \, dt.
\]
Thus, as long as \( \alpha \notin \mathbb{Z}^+ \) (which excludes the trivial cases of smooth monomial solutions), the estimate (5.5) is sharp. \( \square \)

It is a well-known phenomenon in error estimations of finite element solutions that usually the order of an \( L^\infty \)-norm error estimate is almost the same as that of an \( L^2 \)-norm error estimate. Comparing the above result with Theorem 4.5, we expect we can get higher-order error estimates in an \( L^\infty \)-like norm which is compatible with the \( L^2 \) norm. Let us introduce a weighted \( L^\infty \) space,

\[
L^\infty_0(0, 1) = \{ u \text{ measurable} : \| u \|_{L^\infty_0(0, 1)} = \text{ess sup}_{r \in (0, 1)} | u(r) | < \infty \}.
\]

**Theorem 5.4.** Let \( \alpha \notin \mathbb{Z}^+ \). Then there exists a positive constant \( c \) such that
\[
\| u - u_N \|_{L^\infty_0(0, 1)} \leq c N^{-2a-3/2},
\]
and the estimate is sharp. As a consequence, we have a superconvergence result
\[
\max_{r \in I} | u(r) - u_N(r) | \leq c(I) N^{-2a-3/2}
\]
for any closed interval \( I \subset (0, 1] \).

**Proof.** From (4.21), we get
\[
r j_n(r) = (l_n(2r - 1) + l_{n+1}(2r - 1))/2.
\]
Thus, by using (5.4), we find
\[
r (u(r) - u_N(r)) = \sum_{k=N+1}^{\infty} d_{N,k} j_k(r) + d_{N,N} j_N(r) + d_{N,N-1} j_{N-1}(r) = E_1 + E_2,
\]
where
\[
E_1 = \frac{1}{2} \left\{ \sum_{k=N+2}^{\infty} (d_{N,k} + d_{N,k-1}) l_k(2r - 1) + d_{N,N+1} l_{N+1}(2r - 1) + (d_{N,N} + d_{N,N-1}) l_N(2r - 1) \right\},
\]
\[
E_2 = \frac{1}{2} \left\{ d_{N,N} l_{N+1}(2r - 1) + d_{N,N-1} l_{N-1}(2r - 1) \right\}.
\]
From Lemma 5.2, we find
\[
d_{N,k} + d_{N,k-1} = O(k^{-2a-3}), \quad k \geq N + 2,
\]
\[
d_{N,N+1} = O(N^{-2a-2}),
\]
\[
d_{N,N} + d_{N,N-1} = O(N^{-2a-2}).
\]
Thus,

$$|E_1| \leq c N^{-2a-2}.$$ 

To deal with the term $E_2$, we use the following two identities (cf. [1]):

$$l_n(x) - l_{n+1}(x) = (1 - x) P_n^{(1,0)}(x), \quad (5.7)$$

$$P_n^{(1,0)}(x) + P_{n+1}^{(1,0)}(x) = \frac{n + 3/2}{n + 1} (1 + x) P_n^{(1,1)}(x). \quad (5.8)$$

From (5.7) and (5.8), we find that

$$l_n(x) - l_{n+2}(x) = \frac{n + 3/2}{n + 1} (1 - x) (1 + x) P_n^{(1,1)}(x). \quad (5.9)$$

From Formula 8.965 in [9, page 1037], we have

$$(1 - x)^{3/4}(1 + x)^{3/4} P_n^{(1,1)}(x) = g(x) n^{-1/2} + O(n^{-3/2})$$

for some bounded, nonzero function $g(x)$. Using (5.9), we then have

$$l_n(x) - l_{n+2}(x) = \frac{n + 3/2}{n + 1} (1 - x)^{1/4}(1 + x)^{1/4} g(x) n^{-1/2} + O(n^{-3/2}).$$

Therefore, we have, again using Lemma 5.2,

$$E_2 = (-1)^N \frac{C_0}{8} N^{-2a-1} (l_{N-1}(2r - 1) - l_{N+1}(2r - 1)) + O(N^{-2a-2})$$

$$= \tilde{g}(r) N^{-2a-3/2} + O(N^{-2a-2}),$$

where $\tilde{g}$ is a bounded, nonzero function. Thus, the theorem is proved. $\square$

Similarly, from the expansion for the error $u' - u'_N$, we have the following results. We omit the proofs of these results.

**Theorem 5.5.** If $\alpha > 1$, we have the error estimate

$$\|u' - u'_N\|_{L^\infty(0, 1)} \leq c N^{-2a+2},$$

and the estimate is sharp. When $\alpha < 1$, there is no convergence of $u'_N \rightarrow u'$ in $L^\infty(0, 1)$.

**Remark 5.1.** It follows that the $W^{1, \infty}$-stability holds: $\|u_N\|_{W^{1, \infty}(0, 1)} \leq c \|u\|_{W^{1, \infty}(0, 1)}$, if $\alpha > 1$.

**Theorem 5.6.** There holds

$$\|u' - u'_N\|_{L^\infty(0, 1)} \leq c N^{-2a-1/2},$$

and the estimate is sharp. As a consequence, for any closed interval $I \subset (0, 1)$, we have

$$\max_{r \in I} |u'(r) - u'_N(r)| \leq c(I) N^{-2a-1/2}.$$
6. Numerical experiments

We report the numerical results of a two-dimensional problem related to (1.1), with the same sort of majorizing singularity, but with Neumann boundary conditions imposed weakly in order to guarantee a simple numerical implementation of the boundary conditions for an exact solution

\[ u = r^a \sin(\alpha \theta). \]

Consider the V-shaped domain, \( \Omega \), depicted in Figure 6.1.

In our figure, \( \alpha = 2/3 \), but we will vary \( \alpha \) in the interval \((\frac{1}{2}, 1)\). We then use the \( p \)-version of the Galerkin method with merely one element (= \( \frac{1}{2} \Omega \)) to compute numerical solutions to the problem indicated in Figure 6.2.

If we let \( \partial \Omega^+ \) denote the union of those three line segments bounding \( \frac{1}{2} \Omega \) on which we do not have \( u = 0 \) imposed, then we may define the numerical solution, \( u_N \in V_N \), to be

\[
\int_{\Omega} \nabla u_N \cdot \nabla v \, dx = \int_{\partial \Omega^+} gv \, d\gamma, \quad \forall v \in V_N,
\]

where \( V_N = W_N \circ F \). Here, \( F \) is the affine mapping defined in Figure 3.1 and

\[
W_N = \text{span} \{ L_i(s)L_j(t) \},
\]

\( i = 0, \ldots, p \)

\( j = 1, \ldots, p \)

We use here integrals of the Legendre polynomials, \( L_i(\xi) = \int_{-1}^{\xi} P_i^{(0,0)}(\eta) \, d\eta \), for \( i \geq 1 \) and \( L_0(\xi) = 1 \), in order to have convenient analytical expressions for the Dirichlet forms and \( L^2 \) inner products acting on pairs of basis functions (leading to sparse stiffness and mass matrices). The load (the boundary integrals involving \( g_i \)) and the errors are computed using Simpson quadrature with 600 nodes.

![Figure 6.1. V-shaped domain \( \Omega \) with reentrant corner of angle \( 2w \)](image-url)
We have computed up to $p = 20$; the observed rates of convergence in the $L^2$ norm are depicted below:

![Graph showing convergence rates](image)

**Figure 6.3.** Numerically observed $L^2$ rate of convergence as dependent on $\alpha$

The rates of convergence in $H^1$ all confirm the known rate of $-2\alpha$. Although the numerical results for $L^2$ are somewhat inconclusive, there is some indication that a conjectured rate of $-\min\{4\alpha, 2\alpha + 3/2\}$, when the corner singularity is predominant, would be worth a proof.
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Bibliography