INTERPOLATION BETWEEN SOBOLEV SPACES
IN LIPSCHITZ DOMAINS WITH AN APPLICATION
TO MULTIGRID THEORY

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Abstract. In this paper we describe an interpolation result for the Sobolev
spaces $H^r_0(\Omega)$ where $\Omega$ is a bounded domain with a Lipschitz boundary. This
result is applied to derive discrete norm estimates related to multilevel precon-
ditioners and multigrid methods in the finite element method. The estimates
are valid for operators of order $2m$ with Dirichlet boundary conditions.

1. Introduction

Let $\Omega$ be a bounded domain with a Lipschitz boundary. In this note we
show that the Sobolev space $H^r_0(\Omega)$, for $n$ an integer and $r = 1, \ldots, n-1$, is
the same space as that obtained by interpolation between the spaces $H^0_0(\Omega)$ and
$L^2(\Omega)$. The interpolation spaces are defined by means of the real method of
interpolation of Lions and Peetre [9]. This is proved in [8] when the boundary
of $\Omega$ is smooth. For plane domains with piecewise smooth boundaries it was
proved by Zolesio in [11]. The resulting space $[H^0_0(\Omega), L^2(\Omega)]_{r/n}$ has a different
Hilbert space structure from the usual space $H^0_0(\Omega)$. For $n = 2m$ and $r = m$
the regularity properties associated with the elliptic pseudodifferential operator
defined by the corresponding inner product are generally better than those of the
standard $m$th power of the Laplacian with Dirichlet boundary conditions. This
will be used to prove some estimates connected with multilevel finite element
methods.

In order to motivate the discussion, we shall consider the following. Employ-
ing the notation in [1], let $M$ be a finite-dimensional space equipped with an
inner product $(\cdot, \cdot)$ with corresponding norm $\| \cdot \|$. Let $A(\cdot, \cdot)$ be a symmetric
bilinear form on $M$. Now set $M = M_J$ and suppose that we have subspaces
$M_k$ with

$$M_1 \subset M_2 \subset \ldots \subset M_J \equiv M.$$
The linear operator \( A_k : M_k \rightarrow M_k \) is defined by
\[
(A_k \psi, \theta) = A(\psi, \theta) \quad \text{for all } \psi, \theta \in M_k.
\]
Furthermore, the projectors \( P_k : M_J \rightarrow M_k \) and \( Q_k : M_J \rightarrow M_k \) are defined by
\[
A(P_k u, v) = A(u, v)
\]
and
\[
(Q_k u, v) = (u, v),
\]
for all \( u \in M_J \) and all \( v \in M_k \). In the problem considered in §4 of this paper, \( Q_k \) is the \( L_2 \)-projection and \( P_k \) is the so-called elliptic projection. Let \( \lambda_k \) be the largest eigenvalue of \( A_k \). The condition
\[
\sum_{k=1}^{J} \lambda_k \|Q_k - Q_{k-1}\|^2 \leq CA(v, v) \quad \text{for all } v \in M_J,
\]
with the constant \( C \) independent of \( J \), plays a central role in both the additive and multiplicative multilevel theory, cf. [1]. This was first pointed out in [3], where the additive multilevel preconditioner was introduced. It was proved there only in the case of "full elliptic regularity". This regularity property does not hold in the case of second-order elliptic operators on nonconvex polygonal plane domains or for higher-order operators even on convex nonsmooth domains. Inequality (1.1) was shown to hold much more generally in the second-order case by Oswald [10] using Besov space arguments. A different proof based on the knowledge of a modest amount of elliptic regularity was later provided in [2]. Here we show that for any Lipschitz domain we can define an operator in the equivalence class for which full elliptic regularity holds, thus proving (1.1) in general in the second-order case. This is done by using the above interpolation result.

Remark 1. In the above discussion and in what follows, the spaces \( H^2_0(\Omega) \) may be replaced by \( H^s(\Omega) \), thus allowing us to consider also the cases in which the boundary conditions are natural. In this case the development is either completely analogous or simpler. Thus, we will consider only the case of the Dirichlet problem.

2. Interpolation between the spaces \( H^0_0(\Omega) \) and \( L^2(\Omega) \)

We recall here the real method of interpolation. Let \( B_0 \) and \( B_1 \) be two Banach spaces with \( B_1 \) continuously embedded and dense in \( B_0 \). An intermediate space \( B \) is any subspace of \( B_0 \) satisfying
\[
B_1 \subset B \subset B_0.
\]

Real Method of Interpolation. We shall define for \( 0 < s < 1 \) a scale of spaces \([B_1, B_0]_s\) with
\[
B_1 \subset [B_1, B_0]_s \subset B_0.
\]
Define for each \( t > 0 \) and \( u \in B_1 \)
\[
K(t, u) = \inf_{u_0 + u_1 = u} (\|u_0\|_{B_0}^2 + t^2\|u_1\|_{B_1}^2)^{1/2},
\]
where \( u_0 \in B_0, u_1 \in B_1 \) and a standard notation is used to denote the norms.
For $0 < s < 1$ define the quantity
\[ ||u||_{[B_1, B_0]_{s}} = \left( \int_0^\infty t^{-2s} K^2(t, u) \frac{dt}{t} \right)^{1/2}, \]
which is a norm and hence defines a Banach space $[B_1, B_0]_{s}$, which is intermediate to $B_1$ and $B_0$. It is elementary to see that if $B_1$ and $B_0$ are Hilbert spaces then this norm satisfies the parallelogram law and therefore induces a Hilbert space structure.

We now prove a result for Sobolev spaces on domains with Lipschitz boundaries. As previously noted, such a result may be found in [8] in the case of smooth boundaries and for plane domains with piecewise smooth boundaries in [11]. As usual, $c$ and $C$ will be used to denote generic constants, not necessarily the same in any two places.

**Theorem 2.1.** Let $\Omega$ be a bounded domain with a Lipschitz boundary. Then, for $n$ an integer and $r = 1, \ldots, n - 1$,
\[ [H^n_0(\Omega), L^2(\Omega)]_{r/n} = H^r_0(\Omega) \]
with equivalent norms.

**Proof.** For $u$ defined in $\Omega$, let $\tilde{E}$ denote extension by zero to $R^d$.

Define
\[ H^r(D) := \{ u \in H^r(\Omega) \mid \tilde{E}u \in H^r(R^d) \} \]
and set
\[ \|u\|_{H^r(\Omega)} = \|\tilde{E}u\|_{H^r(R^d)}. \]

Recall that $H^r_0(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ under the norm $\| \cdot \|_{H^r(\Omega)}$. By Theorem 1.4.2.2 in [7],
\[ \tilde{H}^r(\Omega) = H^r_0(\Omega). \]

We will first show that
\[ [H^n_0(\Omega), L^2(\Omega)]_{r/n} \subset H^r_0(\Omega). \]

To do this, let $u \in H^r_0(\Omega)$, for $j = 0$ or $j = 2m$. Since $\tilde{H}^j(\Omega) = H^j_0(\Omega)$, we have that
\[ \|\tilde{E}u\|_{H^j(R^d)} = \|u\|_{H^j(\Omega)} \leq C\|u\|_{H^j_0(\Omega)}. \]

Hence by interpolation of operators, since
\[ [H^n(R^d), L^2(R^d)]_{r/n} = H^r(R^d), \]
we have that
\[ c\|u\|_{H^r_0(\Omega)} \leq \|u\|_{H^r(\Omega)} \leq \|\tilde{E}u\|_{H^r(R^d)} \leq C\|u\|_{H^r_0(\Omega), L^2(\Omega)}_{r/n}. \]

This shows that $[H^n_0(\Omega), L^2(\Omega)]_{r/n} \subset H^r_0(\Omega)$.

For the opposite inclusion, we construct an operator $R : H^j(R^d) \to H^j_0(\Omega)$ as follows. Let $\mathcal{B}$ be an open ball such that $\overline{\Omega} \subset \mathcal{B}$ and let $E$ be a continuous extension operator from $H^j(\mathcal{B} \setminus \Omega)$ to $H^j(R^d)$ for $0 \leq j \leq n$. We can do this by virtue of the fact that $\Omega$ has a Lipschitz boundary. Now define $R$ by
\[ Ru = u - E u \]
restricted to \( \Omega \). Let \( \psi \in C_0^\infty(\mathcal{B}) \) be a cutoff function which is equal to one on \( \Omega \). Then the function \( \psi(u - Eu) \), defined on all of \( R^d \), is in \( H^j(R^d) \) and 
\[
\psi(u - Eu) = \hat{E}Ru.
\]
Hence, \( Ru \in \hat{H}^j(\Omega) = H_0^j(\Omega) \). Now
\[
\|Ru\|_{H^j_0(\Omega)} \leq C(\|u\|_{H^j(\Omega)} + \|Eu\|_{H^j(\Omega)})
\]
for \( j = 0 \) or \( j = n \). Hence,
\[
\|Ru\|_{[H^j_0(\Omega), L_2(\Omega)]_{1/n}} \leq C\|u\|_{H^j(R^d)}.
\]
But if \( u \in H^j_0(\Omega) \), then \( \hat{E}u \in H^j(R^d) \) and \( R\hat{E}u = u \). Hence,
\[
\|u\|_{H^j_0(\Omega), L_2(\Omega)} = \|R\hat{E}u\|_{H^j_0(\Omega), L_2(\Omega)} \leq C\|\hat{E}u\|_{H^j(R^d)} \leq C\|u\|_{H^j_0(\Omega)}.
\]
Hence it follows that \( H^j_0(\Omega) \subset [H^j_0(\Omega), L_2(\Omega)]_{1/n} \). This proves the theorem.

Note that \( H^j_0(\Omega) \) and \([H^j_0(\Omega), L_2(\Omega)]_{1/n} \) may be different Hilbert spaces.

3. Bilinear forms

It is well known that for \( k < l \) the space \( H^k_0(\Omega) \) is compactly imbedded in \( H^l_0(\Omega) \) (cf. [7]). Denoting the usual inner product on \( H^2m_0(\Omega) \) by \( \langle \cdot, \cdot \rangle_{2m} \), the operator \( T: L_2(\Omega) \rightarrow H^{2m}_0(\Omega) \) defined by
\[
(Tu, \phi)_{2m} = \langle u, \phi \rangle_{2m}
\]
for all \( \phi \in H^{2m}_0(\Omega) \) is positive definite and compact, with positive eigenvalues \( \{\Lambda^{-1}_i\} \). The corresponding eigenvectors \( \{\phi_i\} \) are complete in \( L_2(\Omega) \) and satisfy \( \lambda_i T\phi_i = \phi_i \). It is easy to see that for \( u = \sum_{i=1}^\infty c_i\phi_i \), the norm on \( H^{2m}_0(\Omega) \) is given by
\[
\|u\|_{H^{2m}_0(\Omega)} = \left( \sum_{i=1}^\infty \lambda_i c_i^2 \right)^{1/2}.
\]
The space \( \hat{H}^{2m}_0(\Omega) = [H^{2m}_0(\Omega), L_2(\Omega)]_{1/2} \) has the norm
\[
\|u\|_{\hat{H}^{2m}_0(\Omega)} = \left( \sum_{i=1}^\infty \lambda_i^{1/2} c_i^2 \right)^{1/2},
\]
which induces the inner product, say \( \mathcal{A} \langle \cdot, \cdot \rangle \), with
\[
\mathcal{A}\langle u, u \rangle = \sum_{i=1}^\infty \lambda_i^{1/2} c_i^2.
\]
This norm is equivalent to the standard norm by Theorem 2.1. The spaces \( \hat{H}^{m+s}_0(\Omega) = [H^{m+2s}_0(\Omega), L_2(\Omega)]_{1/2+s/2m} \) and \( \hat{H}^{m-s}_0(\Omega) = [H^{2m}_0(\Omega), L_2(\Omega)]_{1/2-s/2m} \), for \( 0 \leq s \leq m \), have the norms
\[
\|u\|_{\hat{H}^{m+s}_0(\Omega)} = \left( \sum_{i=1}^\infty \lambda_i^{1/2+s/2m} c_i^2 \right)^{1/2}
\]
and
\[
\|u\|_{\hat{H}^{m-s}_0(\Omega)} = \left( \sum_{i=1}^\infty \lambda_i^{1/2-s/2m} c_i^2 \right)^{1/2}.
\]
The following is now obvious.
Theorem 3.1. Let $\Omega$ be a bounded domain with a Lipschitz boundary. Then for $u \in H_0^{m-1}(\Omega)$ and $\phi \in H_0^{m+1}(\Omega)$,
\[
\mathcal{A}(u, \phi) = \sum_{i=1}^{\infty} \Lambda_i^{1/2}(u, \phi_i)(\phi, \phi_i)
\]
is well defined and
\[
\|u\|_{H_0^{m-1}(\Omega)} = \sup_{0 \neq \phi \in H_0^{m+1}(\Omega)} \frac{\mathcal{A}(u, \phi)}{\|\phi\|_{H_0^{m+1}(\Omega)}}.
\]

4. Regularity for the form $\mathcal{A}$

We shall consider the solution $u \in H_0^m(\Omega)$ of the problem
\[
(4.1) \quad \mathcal{A}(u, \phi) = (f, \phi) \quad \text{for all } \phi \in H_0^m(\Omega),
\]
where $f \in L^2(\Omega)$. The following theorem is easy.

Theorem 4.1. Let $u$ be a solution of (4.1) with $f \in L^2(\Omega)$. Then $u \in H_0^{2m}(\Omega)$ and
\[
\|u\|_{H_0^{2m}(\Omega)} \leq \|f\|_{L^2(\Omega)}.
\]

Proof. Choose $\phi = \sum_{i=1}^{n} \Lambda_i^{1/2}(u, \phi_i)\phi_i$ in (4.1) and, after applying the Cauchy-Schwarz inequality, let $n$ tend to infinity.

Let $\{S_h\}$, $h > 0$, be a family of subspaces of $H_0^m(\Omega)$ with the following approximation property. There exists a constant $C > 0$ such that for any $v \in H_0^{m+s}(\Omega)$
\[
(4.2) \quad \inf_{\chi \in S_h} \|v - \chi\|_{H^s(\Omega)} \leq C h^s \|v\|_{H_0^{m+s}(\Omega)},
\]
for some $s$ with $0 < s \leq m$. For many examples of such spaces see [5] or [6].

Now the discrete solution of (4.1) in $S_h$ is the unique solution $U \in S_h$ such that
\[
(4.3) \quad \mathcal{A}(U, \phi) = (f, \phi) \quad \text{for all } \phi \in S_h.
\]

Define the orthogonal projection operator $P_h : H_0^m(\Omega) \mapsto S_h$ by means of
\[
\mathcal{A}(P_h u, \phi) = \mathcal{A}(u, \phi) \quad \text{for all } \phi \in S_h.
\]

Since $\mathcal{A}(\cdot, \cdot)$ defines a norm on $H_0^m(\Omega)$, we have, using (4.2),
\[
\|u - P_h u\|_{H_0^m(\Omega)} \leq C \inf_{\chi \in S_h} \|v - \chi\|_{H_0^m(\Omega)}
\]
\[
\quad \leq C h^s \|u\|_{H_0^{m+s}(\Omega)} \quad \text{for all } u \in H_0^{m+s}(\Omega).
\]

For this projector we now have the following "regularity pickup".

Theorem 4.2. Let $u \in H_0^{m+s}(\Omega)$. If (4.2) is satisfied, then
\[
\|u - P_h u\|_{H_0^{m-1}(\Omega)} \leq C h^s \|u\|_{H_0^m(\Omega)}.
\]

Proof. This follows from Theorem 3.1 and (4.4).
5. APPLICATION TO ESTIMATES FOR MULTILEVEL METHODS

We will follow the notation in [1] and give a proof that (1.1) is satisfied in the case that $\Omega$ is a polygonal domain whose boundary is Lipschitz (i.e., all angles are less than $2\pi$), the spaces $M_k$ are nested and defined by conforming piecewise polynomials. That is, $M_{k-1} \subset M_k$ and $M_J \subset H^m_0(\Omega)$. We assume that the initial triangulation is fixed independently of $J$ and gives rise to the others by means of a halving strategy. The form $A(\cdot, \cdot)$ is the standard generalized Dirichlet form of order $m$. Finally, we suppose that (4.2) is satisfied with $S_{h_k} = M_k$ and $h_k = h_0 2^{-k}$, the mesh size associated with $M_k$ and, for simplicity of notation, write $P_k$ for $P_{h_k}$.

We will prove the following theorem.

**Theorem 5.1.** There exists $C > 0$ such that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \leq C \|v\|^2_{H^m_0(\Omega)} \text{ for all } v \in M_J,$$

where $Q_k$ is the $L^2(\Omega)$ orthogonal projection onto $M_k$ and $Q_0 = 0$.

**Proof.** The proof of this theorem is similar to that given in [2]. We note that $h_k^{-m} \leq C \lambda_k^{-1}$. Using standard properties of $Q_k$ and the fact that $Q_k = Q_{k-1}Q_k = Q_kQ_{k-1}$, we see that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \leq C \sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1})v\|^2_{H^m_0(\Omega)}.$$

By techniques similar to those of [4] it follows that $\|Q_ku\|_{H^m_0(\Omega)} \leq C \|u\|_{H^m_0(\Omega)}$. Hence, using this and the fact that $(Q_k - Q_{k-1})v = 0$ for $v \in M_l$ with $l < k$,

$$\sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1})v\|^2_{H^m_0(\Omega)} \leq \sum_{k=1}^{J} \sum_{l=k}^{J} h_k^{-2} \|(Q_k - Q_{k-1})\sum_{l=k}^{J} (P_k - P_{l-1})v\|^2_{H^m_0(\Omega)}$$

$$\leq C \sum_{k=1}^{J} \left( \sum_{l=k}^{J} \frac{h_l}{h_k} h_l^{-1} \|(P_l - P_{l-1})v\|^2_{H^m_0(\Omega)} \right)^2$$

$$= C \sum_{k=1}^{J} \left( \sum_{l=k}^{J} \frac{h_l}{h_k} h_l^{-1} \|(I - P_{l-1})(P_l - P_{l-1})v\|^2_{H^m_0(\Omega)} \right)^2$$

$$\leq C \sum_{k=1}^{J} \left( \sum_{l=k}^{J} \frac{h_l}{h_k} \|(P_l - P_{l-1})v\|^2_{H^m_0(\Omega)} \right)^2.$$

In the last inequality we used the regularity pickup error estimate of Theorem 4.2. Defining $E$ to be the symmetric $J \times J$ matrix with entries $E_{kl} = h_l/h_k = 2^{k-l}$ for $l \geq k$, and $\alpha$ to be the vector with components $\|(P_l - P_{l-1})v\|_{H^m_0(\Omega)}$, we see that

$$\sum_{k=1}^{J} \left( \sum_{l=k}^{J} \frac{h_l}{h_k} \|(P_l - P_{l-1})v\|_{H^m_0(\Omega)} \right)^2 \leq |E\alpha|^2.$$

Since the row sums are uniformly bounded in $J$, the eigenvalues of $E$ are
bounded and hence
\[ |E\bar{a}|^2 \leq C|\bar{a}|^2 \leq C \sum_{k=1}^{J} \mathcal{A}((P_k - P_{k-1})v, (P_k - P_{k-1})v) \]
\[ = C \sum_{k=1}^{J} \mathcal{A}((P_k - P_{k-1})v, v) = C \mathcal{A}(v, v) \leq C \|v\|_{H^m_0(\Omega)}^2. \]

This proves the theorem.

**BIBLIOGRAPHY**


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