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INTERPOLATION BETWEEN SOBOLEV SPACES IN LIPSCHITZ DOMAINS WITH AN APPLICATION TO MULTIGRID THEORY

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ABSTRACT. In this paper we describe an interpolation result for the Sobolev spaces $H_0^s(\Omega)$ where Ω is a bounded domain with a Lipschitz boundary. This result is applied to derive discrete norm estimates related to multilevel preconditioners and multigrid methods in the finite element method. The estimates are valid for operators of order $2m$ with Dirichlet boundary conditions.

1. INTRODUCTION

Let Ω be a bounded domain with a Lipschitz boundary. In this note we show that the Sobolev space $H_0^r(\Omega)$, for n an integer and $r = 1, \ldots, n-1$, is the same space as that obtained by interpolation between the spaces $H_0^n(\Omega)$ and $L_2(\Omega)$. The interpolation spaces are defined by means of the real method of interpolation of Lions and Peetre [9]. This is proved in [8] when the boundary of Ω is smooth. For plane domains with piecewise smooth boundaries it was proved by Zolesio in [11]. The resulting space $[H_0^n(\Omega), L_2(\Omega)]_{r/n}$ has a different Hilbert space structure from the usual space $H_0^r(\Omega)$. For $n = 2m$ and $r = m$ the regularity properties associated with the elliptic pseudodifferential operator defined by the corresponding inner product are generally better than those of the standard m th power of the Laplacian with Dirichlet boundary conditions. This will be used to prove some estimates connected with multilevel finite element methods.

In order to motivate the discussion, we shall consider the following. Employing the notation in [1], let M be a finite-dimensional space equipped with an inner product $(\cdot, \cdot)$ with corresponding norm $\|\cdot\|$. Let $A(\cdot, \cdot)$ be a symmetric bilinear form on M. Now set $M = M_J$ and suppose that we have subspaces M_k with

$$M_1 \subset M_2 \subset \cdots \subset M_J \equiv M.$$

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The linear operator $A_k : M_k \to M_k$ is defined by

$$(A_k\psi, \theta) = A(\psi, \theta) \quad \text{for all } \psi, \theta \in M_k.$$

Furthermore, the projectors $P_k : M_J \to M_k$ and $Q_k : M_J \to M_k$ are defined by

$$A(P_k u, v) = A(u, v)$$

and

$$(Q_k u, v) = (u, v),$$

for all $u \in M_J$ and all $v \in M_k$. In the problem considered in §4 of this paper, Q_k is the L_2-projection and P_k is the so-called elliptic projection. Let λ_k be the largest eigenvalue of A_k. The condition

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})\|^2 \le CA(v, v) \quad \text{for all } v \in M_J, \tag{1.1}$$

with the constant C independent of J, plays a central role in both the additive and muliplicative multilevel theory, cf. [1]. This was first pointed out in [3], where the additive multilevel preconditioner was introduced. It was proved there only in the case of "full elliptic regularity". This regularity property does not hold in the case of second-order elliptic operators on nonconvex polygonal plane domains or for higher-order operators even on convex nonsmooth domains. Inequality (1.1) was shown to hold much more generally in the second-order case by Oswald [10] using Besov space arguments. A different proof based on the knowledge of a modest amount of elliptic regularity was later provided in [2]. Here we show that for any Lipschitz domain we can define an operator in the equivalence class for which full elliptic regularity holds, thus proving (1.1) in general in the second-order case. This is done by using the above interpolation result.

Remark 1. In the above discussion and in what follows, the spaces $H_0^s(\Omega)$ may be replaced by $H^s(\Omega)$, thus allowing us to consider also the cases in which the boundary conditions are natural. In this case the development is either completely analogous or simpler. Thus, we will consider only the case of the Dirichlet problem.

2. Interpolation between the spaces $H_0^n(\Omega)$ and $L^2(\Omega)$

We recall here the real method of interpolation. Let B_0 and B_1 be two Banach spaces with B_1 continuously embedded and dense in B_0. An *intermediate space* B is any subspace of B_0 satisfying

$$B_1 \subset B \subset B_0.$$

Real Method of Interpolation. We shall define for $0 < s < 1$ a scale of spaces $[B_1, B_0]_s$ with

$$B_1 \subset [B_1, B_0]_s \subset B_0.$$

Define for each $t > 0$ and $u \in B_1$

$$K(t, u) = \inf_{u_0+u_1=u} (\|u_0\|_{B_0}^2 + t^2\|u_1\|_{B_1}^2)^{1/2}, \tag{2.1}$$

where $u_0 \in B_0$, $u_1 \in B_1$ and a standard notation is used to denote the norms.

For $0 < s < 1$ define the quantity

$$\|u\|_{[B_1, B_0]_s} = \left(\int_0^\infty t^{-2s} K^2(t, u) \frac{dt}{t} \right)^{1/2},$$

which is a norm and hence defines a Banach space $[B_1, B_0]_s$ which is intermediate to B_1 and B_0. It is elementary to see that if B_1 and B_0 are Hilbert spaces then this norm satisfies the parallelogram law and therefore induces a Hilbert space structure.

We now prove a result for Sobolev spaces on domains with Lipschitz boundaries. As previously noted, such a result may be found in [8] in the case of smooth boundaries and for plane domains with piecewise smooth boundaries in [11]. As usual, c and C will be used to denote generic constants, not necessarily the same in any two places.

Theorem 2.1. *Let Ω be a bounded domain with a Lipschitz boundary. Then, for n an integer and $r = 1, \dots, n-1$,*

$$[H_0^n(\Omega), L_2(\Omega)]_{r/n} = H_0^r(\Omega)$$

with equivalent norms.

Proof. For u defined in Ω, let $\tilde{E}$ denote extension by zero to R^d.

Define

$$\tilde{H}^r(\Omega) := \{u \in H^r(\Omega) | \quad \tilde{E}u \in H^r(R^d)\}$$

and set

$$\|u\|_{\tilde{H}^r(\Omega)} = \|\tilde{E}u\|_{H^r(R^d)}.$$

Recall that $H_0^r(\Omega)$ is the completion of $\mathscr{D}(\Omega)$ under the norm $\|\cdot\|_{H^r(\Omega)}$. By Theorem 1.4.2.2 in [7],

$$\tilde{H}^r(\Omega) = H_0^r(\Omega).$$

We will first show that

$$[H_0^n(\Omega), L_2(\Omega)]_{r/n} \subset H_0^r(\Omega).$$

To do this, let $u \in H_0^j(\Omega)$, for $j = 0$ or $j = 2m$. Since $\tilde{H}^j(\Omega) = H_0^j(\Omega)$, we have that

$$\|\tilde{E}u\|_{H^j(R^d)} = \|u\|_{\tilde{H}^j(\Omega)} \le C\|u\|_{H_0^j(\Omega)}.$$

Hence by interpolation of operators, since

$$[H^n(R^d), L_2(R^d)]_{r/n} = H^r(R^d),$$

we have that

$$c\|u\|_{H_0^r(\Omega)} \le \|u\|_{\tilde{H}^r(\Omega)} = \|\tilde{E}u\|_{H^r(R^d)} \le C\|u\|_{[H_0^n(\Omega), L_2(\Omega)]_{r/n}}.$$

This shows that $[H_0^n(\Omega), L_2(\Omega)]_{r/n} \subset H_0^r(\Omega)$.

For the opposite inclusion, we construct an operator $R : H^j(R^d) \mapsto H_0^j(\Omega)$ as follows. Let $\mathscr{B}$ be an open ball such that $\overline{\Omega} \subset \mathscr{B}$ and let E be a continuous extension operator from $H^j(\mathscr{B} \setminus \Omega)$ to $H^j(R^d)$ for $0 \le j \le n$. We can do this by virtue of the fact that Ω has a Lipschitz boundary. Now define R by

$$Ru = u - Eu$$

restricted to Ω. Let $\psi \in C_0^\infty(\mathscr{B})$ be a cutoff function which is equal to one on Ω. Then the function $\psi(u - Eu)$, defined on all of R^d, is in $H^j(R^d)$ and

$$\psi(u - Eu) = \tilde{E}Ru.$$

Hence, $Ru \in \tilde{H}^j(\Omega) = H_0^j(\Omega)$. Now

$$\begin{aligned}\|Ru\|_{H_0^j(\Omega)} &\le C(\|u\|_{H^j(\Omega)} + \|Eu\|_{H^j(\Omega)}) \\ &\le C\|u\|_{H^j(R^d)},\end{aligned}$$

for $j = 0$ or $j = n$. Hence,

$$\|Ru\|_{[H_0^n(\Omega),\, L_2(\Omega)]_{r/n}} \le C\|u\|_{H^r(R^d)}.$$

But if $u \in H_0^r(\Omega)$, then $\tilde{E}u \in H^r(R^d)$ and $R\tilde{E}u = u$. Hence,

$$\|u\|_{[H_0^n(\Omega),\, L_2(\Omega)]_{r/n}} = \|R\tilde{E}u\|_{[H_0^n(\Omega),\, L_2(\Omega)]_{r/n}} \le C\|\tilde{E}u\|_{H^r(R^d)} \le C\|u\|_{H_0^r(\Omega)}.$$

Hence it follows that $H_0^r(\Omega) \subset [H_0^n(\Omega),\, L_2(\Omega)]_{r/n}$. This proves the theorem.

Note that $H_0^r(\Omega)$ and $[H_0^n(\Omega),\, L_2(\Omega)]_{r/n}$ may be different Hilbert spaces.

3. Bilinear forms

It is well known that for $k < l$ the space $H_0^l(\Omega)$ is compactly imbedded in $H_0^k(\Omega)$ (cf. [7]). Denoting the usual inner product on $H_0^{2m}(\Omega)$ by $(\cdot, \cdot)_{2m}$, the operator $T : L_2(\Omega) \mapsto H_0^{2m}(\Omega)$ defined by

$$(Tu, \phi)_{2m} = (u, \phi) \quad \text{for all } \phi \in H_0^{2m}(\Omega)$$

is positive definite and compact, with positive eigenvalues $\{\Lambda_i^{-1}\}$. The corresponding eigenvectors $\{\phi_i\}$ are complete in $L_2(\Omega)$ and satisfy $\Lambda_i T\phi_i = \phi_i$. It is easy to see that for $u = \sum_{i=1}^\infty c_i\phi_i$, the norm on $H_0^{2m}(\Omega)$ is given by

$$\|u\|_{H_0^{2m}(\Omega)} = (\sum_{i=1}^\infty \Lambda_i c_i^2)^{1/2}.$$

The space $\hat{H}_0^m(\Omega) = [H_0^{2m}(\Omega),\, L_2(\Omega)]_{1/2}$ has the norm

$$\|u\|_{\hat{H}_0^m(\Omega)} = (\sum_{i=1}^\infty \Lambda_i^{1/2} c_i^2)^{1/2},$$

which induces the inner product, say $\mathscr{A}(\cdot, \cdot)$, with

$$\mathscr{A}(u, u) = \sum_{i=1}^\infty \Lambda_i^{1/2} c_i^2.$$

This norm is equivalent to the standard norm by Theorem 2.1. The spaces $\hat{H}_0^{m+s}(\Omega) = [H_0^{2m}(\Omega),\, L_2(\Omega)]_{1/2+s/2m}$ and $\hat{H}_0^{m-s}(\Omega) = [H_0^{2m}(\Omega),\, L_2(\Omega)]_{1/2-s/2m}$, for $0 \le s \le m$, have the norms

$$\|u\|_{\hat{H}_0^{m+s}(\Omega)} = (\sum_{i=1}^\infty \Lambda_i^{1/2+s/2m} c_i^2)^{1/2}$$

and

$$\|u\|_{\hat{H}_0^{m-s}(\Omega)} = (\sum_{i=1}^\infty \Lambda_i^{1/2-s/2m} c_i^2)^{1/2}.$$

The following is now obvious.

Theorem 3.1. *Let Ω be a bounded domain with a Lipschitz boundary. Then for $u \in H_0^{m-s}(\Omega)$ and $\phi \in H_0^{m+s}(\Omega)$,*

$$\mathscr{A}(u, \phi) = \sum_{i=1}^{\infty} \Lambda_i^{1/2}(u, \phi_i)(\phi, \phi_i)$$

is well defined and

$$\|u\|_{\hat{H}_0^{m-s}(\Omega)} = \sup_{0 \neq \phi \in \hat{H}_0^{m+s}(\Omega)} \frac{\mathscr{A}(u, \phi)}{\|\phi\|_{\hat{H}_0^{m+s}(\Omega)}}.$$

4. Regularity for the form $\mathscr{A}$

We shall consider the solution $u \in H_0^m(\Omega)$ of the problem

$$\mathscr{A}(u, \phi) = (f, \phi) \quad \text{for all } \phi \in H_0^m(\Omega), \tag{4.1}$$

where $f \in L^2(\Omega)$. The following theorem is easy.

Theorem 4.1. *Let u be a solution of (4.1) with $f \in L^2(\Omega)$. Then $u \in H_0^{2m}(\Omega)$ and*

$$\|u\|_{H_0^{2m}(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

Proof. Choose $\phi = \sum_{i=1}^n \Lambda_i^{1/2}(u, \phi_i)\phi_i$ in (4.1) and, after applying the Cauchy–Schwarz inequality, let n tend to infinity.

Let $\{S_h\}$, $h > 0$, be a family of subspaces of $H_0^m(\Omega)$ with the following approximation property. There exists a constant $C > 0$ such that for any $v \in H_0^{m+s}(\Omega)$

$$\inf_{\chi \in S_h} \|v - \chi\|_{H_0^m(\Omega)} \leq Ch^s \|v\|_{H_0^{m+s}(\Omega)}, \tag{4.2}$$

for some s with $0 < s \leq m$. For many examples of such spaces see [5] or [6].

Now the discrete solution of (4.1) in S_h is the unique solution $U \in S_h$ such that

$$\mathscr{A}(U, \phi) = (f, \phi) \quad \text{for all } \phi \in S_h. \tag{4.3}$$

Define the orthogonal projection operator $P_h : H_0^m(\Omega) \mapsto S_h$ by means of

$$\mathscr{A}(P_h u, \phi) = \mathscr{A}(u, \phi) \quad \text{for all } \phi \in S_h.$$

Since $\mathscr{A}(\cdot, \cdot)$ defines a norm on $H_0^m(\Omega)$, we have, using (4.2),

$$\begin{aligned} \|u - P_h u\|_{H_0^m(\Omega)} &\leq C \inf_{\chi \in S_h} \|v - \chi\|_{H_0^m(\Omega)} \\ &\leq Ch^s \|u\|_{H_0^{m+s}(\Omega)} \quad \text{for all } u \in H_0^{m+s}(\Omega). \end{aligned} \tag{4.4}$$

For this projector we now have the following "regularity pickup".

Theorem 4.2. *Let $u \in H_0^{m+s}(\Omega)$. If (4.2) is satisfied, then*

$$\|u - P_h u\|_{H_0^{m-s}(\Omega)} \leq Ch^s \|u\|_{H_0^m(\Omega)}.$$

Proof. This follows from Theorem 3.1 and (4.4).

5. Application to estimates for multilevel methods

We will follow the notation in [1] and give a proof that (1.1) is satisfied in the case that Ω is a polygonal domain whose boundary is Lipschitz (i.e., all angles are less than 2π), the spaces M_k are nested and defined by conforming piecewise polynomials. That is, $M_{k-1} \subset M_k$ and $M_J \subset H_0^m(\Omega)$. We assume that the initial triangulation is fixed independently of J and gives rise to the others by means of a halving strategy. The form $A(\cdot, \cdot)$ is the standard generalized Dirichlet form of order m. Finally, we suppose that (4.2) is satisfied with $S_{h_k} = M_k$ and $h_k = h_0 2^{-k}$, the mesh size associated with M_k and, for simplicity of notation, write P_k for P_{h_k}.

We will prove the following theorem.

Theorem 5.1. *There exists $C > 0$ such that*

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \le C\|v\|^2_{H_0^m(\Omega)} \quad \textit{for all } v \in M_J,$$

where Q_k is the $L^2(\Omega)$ orthogonal projection onto M_k and $Q_0 = 0$.

Proof. The proof of this theorem is similar to that given in [2]. We note that $h_k^{2m} \le C\lambda_k^{-1}$. Using standard properties of Q_k and the fact that $Q_k = Q_{k-1}Q_k = Q_kQ_{k-1}$, we see that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \le C \sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1})v\|^2_{H_0^{m-1}(\Omega)}.$$

By techniques similar to those of [4] it follows that $\|Q_k u\|_{H_0^{m-1}(\Omega)} \le C\|u\|_{H_0^{m-1}(\Omega)}$. Hence, using this and the fact that $(Q_k - Q_{k-1})v = 0$ for $v \in M_l$ with $l < k$,

$$\begin{aligned}
\sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1})v\|^2_{H_0^{m-1}(\Omega)} &= \sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1}) \sum_{l=k}^{J} (P_k - P_{k-1})v\|^2_{H_0^{m-1}(\Omega)} \\
&\le C \sum_{k=1}^{J} \Big(\sum_{l=k}^{J} (h_l/h_k) h_l^{-1} \|(P_l - P_{l-1})v\|_{H_0^{m-1}(\Omega)} \Big)^2 \\
&= C \sum_{k=1}^{J} \Big(\sum_{l=k}^{J} (h_l/h_k) h_l^{-1} \|(I - P_{l-1})(P_l - P_{l-1})v\|_{H_0^{m-1}(\Omega)} \Big)^2 \\
&\le C \sum_{k=1}^{J} \Big(\sum_{l=k}^{J} (h_l/h_k) \|(P_l - P_{l-1})v\|_{H_0^{m}(\Omega)} \Big)^2.
\end{aligned}$$

In the last inequality we used the regularity pickup error estimate of Theorem 4.2. Defining E to be the symmetric $J \times J$ matrix with entries $E_{kl} = h_l/h_k = 2^{k-l}$ for $l \ge k$, and $\vec{\alpha}$ to be the vector with components $\|(P_l - P_{l-1})v\|_{H_0^m(\Omega)}$, we see that

$$\sum_{k=1}^{J} \Big(\sum_{l=k}^{J} (h_l/h_k) \|(P_l - P_{l-1})v\|_{H_0^m(\Omega)} \Big)^2 \le |E\vec{\alpha}|^2.$$

Since the row sums are uniformly bounded in J, the eigenvalues of E are

bounded and hence

$$|E\vec{\alpha}|^2 \le C|\vec{\alpha}|^2 \le C\sum_{k=1}^{J} \mathscr{A}((P_l - P_{l-1})v\,,\,(P_l - P_{l-1})v)$$
$$= C\sum_{k=1}^{J} \mathscr{A}((P_l - P_{l-1})v\,,\,v) = C\mathscr{A}(v\,,\,v) \le C\|v\|^2_{H_0^m(\Omega)}.$$

This proves the theorem.

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