MATHEMATICS OF COMPUTATION
Volume 65, Number 214
April 1996, Pages 785–800

CLASS NUMBER 5, 6 AND 7

CHRISTIAN WAGNER

Abstract. We outline the determination of all imaginary quadratic fields with class number 5, 6 or 7.

1. Introduction[1]

In this paper we outline the proof that the following table contains all discriminants $-d < 0$ of imaginary quadratic fields with class number $h = 5$, 6 or 7.

Table 1. Fundamental discriminants with class number 5, 6 or 7

h	d, $-d$ (fundamental) discriminant, $h(-d) = h$
5	47, 79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 619, 683, 691, 739, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683
6	87, 104, 116, 152, 212, 244, 247, 339, 411, 424, 436, 451, 472, 515, 628, 707, 771, 808, 835, 843, 856, 1048, 1059, 1099, 1108, 1147, 1192, 1203, 1219, 1267, 1315, 1347, 1363, 1432, 1563, 1588, 1603, 1843, 1915, 1963, 2227, 2283, 2443, 2515, 2563, 2787, 2923, 3235, 3427, 3523, 3763
7	71, 151, 223, 251, 463, 467, 487, 587, 811, 827, 859, 1163, 1171, 1483, 1523, 1627, 1787, 1987, 2011, 2083, 2179, 2251, 2467, 2707, 3019, 3067, 3187, 3907, 4603, 5107, 5923

Recently, S. Arno [1, 2] combined methods of H. Stark [16] for $h = 2$ and H. Montgomery and P. Weinberger [11] for $h = 2$ and 3 to solve the class number-4 problem[2].

Arno had to overcome the problem of d having up to three distinct prime divisors, but could profit from the small number of leading coefficients of the reduced binary quadatic forms. In our case, d is simplified to at most two prime divisors, but the larger number of coefficients results in certain estimates being much worse. Therefore, we have to combine the abovementioned methods in another way as Arno. For $h = 8$ both items take a negative turn: up to four prime divisors and eight coefficients, which makes this case seem impregnable.

Received by the editor December 16, 1993 and, in revised form, February 13, 1995.
1991 *Mathematics Subject Classification.* Primary 11R29.
Key words and phrases. Binary quadratic form, class number, discriminant, imaginary quadratic field.

[1]General information on Gauss' class number problem may be obtained from the articles by S. Böcherer [3], D. Goldfeld [6], J. Oesterlé [13] and J.-P. Serre [14].

[2]As the referee pointed out to me, S. Arno has now extended some of these results. Arno has treated the class number problem for all odd h from 5 to 23.

©1996 American Mathematical Society

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

2. Class number, discriminant and forms

The restriction of h to the values 5, 6 or 7 implies conditions that must be fulfilled by the discriminants and the leading coefficients in their reduced (binary quadratic) forms, which occur in certain formulas of the abovementioned methods. In this section we list some of these conditions. This will help us later in reducing the computer work for searching for discriminants with class number 5, 6 or 7.

We are interested in reduced forms $f(x,y) = ax^2 + bxy + cy^2$ with discriminant $-d$, i.e., forms with $-d = b^2 - 4ac$ and $-a < b \le a < c$ or $0 \le b \le a = c$ (which implies $a \le \sqrt{d/3}$). An example is the principal form

$$f_1(x,y) = \begin{cases} x^2 + \frac{d}{4}y^2 & \text{for } d \equiv 0 \pmod 4, \\ x^2 + xy + \frac{d+1}{4}y^2 & \text{for } d \equiv 3 \pmod 4, \end{cases}$$

which is the only reduced form with 1 as leading coefficient a.

Throughout this paper we let $f_i(x,y) = a_i x^2 + b_i xy + c_i y^2$, $i = 1, \dots, h$, be the reduced forms with discriminant $-d$, numbered in a way that their leading coefficients $1, a_2, a_3, \dots, a_h$ are in ascending order, i.e., $1 < a_2 \le a_3 \le \cdots \le a_h$.

Lemma 2.1. *Let $f(x,y) = ax^2 + bxy + cy^2$ be a reduced form with discriminant $-d$. Then $y \ne 0$ implies $f(x,y) \ge c$, and $y = 0, x \ne 0$ implies $f(x,y) \ge a$.*

Proof. See Stark [16, Lemma 3]. □

Lemma 2.2. *Let $h > 1$. Then a_2 is prime and* $a_2 = \min\left\{p \text{ prime} \,\middle|\, \left(\frac{-d}{p}\right) \ne -1\right\}$.

Proof. The integer a_2 is prime, otherwise choose p prime, $p \mid a_2$, $p < a_2$. The number of representations of p with forms with discriminant $-d$ is

$$R(p) = \sum_{t|p} \left(\frac{-d}{t}\right) = 1 + \left(\frac{-d}{p}\right) = 1 + \left(\frac{b_2^2 - 4a_2c_2}{p}\right) = 1 + \left(\frac{b_2^2}{p}\right) \ge 1. \tag{1}$$

From Lemma 2.1 only the principal form can represent p. This can happen only when $y \ne 0$. But then by Lemma 2.1

$$\frac{d}{4} \le p < a_2 \le \sqrt{\frac{d}{3}}, \tag{2}$$

consequently $d < \frac{16}{3}$ and $h = 1$. This is a contradiction.

Now a_2 is represented (by f_2), so $\left(\frac{-d}{a_2}\right) \ne -1$. Let p be prime, $p < a_2$, $\left(\frac{-d}{p}\right) \ne -1$. As in (1), $R(p) \ge 1$, but because of (2), p cannot be represented. Contradiction. □

Lemma 2.3 (Heilbronn)**.** *Let $-d < 0$ be a discriminant, $h = h(-d)$, p a rational prime number and $\left(\frac{-d}{p}\right) = 1$. Then $p \ge \left(\frac{d}{4}\right)^{\frac{1}{h}}$.*

Lemma 2.4 (Heilbronn)**.** *Let $-d < 0$ be a discriminant, $a > 0$ squarefree, $a \mid d$ and $a \le \left(\frac{d}{4}\right)^{\frac{1}{2}}$. Then there is exactly one reduced form with discriminant $-d$ and leading coefficient a.*

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

2.1. **Odd class number.** Suppose $h = 2n + 1$ with $n \in \mathbb{N}_0$. This is the simplest case. By the theory of genera ($2^{t-1} \mid h$, where t is the number of distinct prime divisors of d), $-d$ is a prime discriminant, thus has the form $-4, -8$ or $-p$ with $p \equiv 3 \pmod 4$ prime. It is easy to calculate $h(-4) = h(-8) = 1$; we confine our analysis to the discriminants with $d \equiv 3 \pmod 4$ prime. We will see that the coefficients of the reduced forms behave well.

Lemma 2.5. *Let $h > 1$ be odd, $f(x, y) = ax^2 + bxy + cy^2$ be a reduced form other than the principal form with discriminant $-d = b^2 - 4ac$. Then $0 < |b| < a < c$.*

Proof. We have $b \neq 0$, because d is odd. If $|b| = a$, then $-d = a(a - 4c)$. With $a > 1$ and d prime, this leads to $a = d$, a contradiction. If $a = c$, then $b \geq 0$ and $-d = (b - 2a)(b + 2a)$. This leads to $b + 2a = d$ and $b - 2a = -1$, so $b = 2a - 1 > a$, a contradiction. □

Remark 2.6. Let $h > 1$ be odd, $f(x, y) = ax^2 + bxy + cy^2$ with $0 < |b| < a < c$. Then obviously $ax^2 - bxy + cy^2$ is reduced too. So besides the principal form there are respectively pairs of reduced forms, which differ only by the sign of b (but are nonequivalent).

Corollary 2.7. *Let $h > 1$ be odd. Then $a_2 = a_3$ and $a_2, a_3 < a_4$, i.e., there is exactly one pair of forms with a_2 as leading coefficient.*

Proof. The equalities $a_3 = a_4 (= a_5)$ lead to $R(a_2) \geq 4$ (each form represents with $x = 1, y = 0$). But because of Lemma 2.2 and $(a_2, d) = 1$ we have $R(a_2) = 1 + \left(\frac{-d}{a_2}\right) = 2$. □

Lemma 2.8. *Let $h > 1$ be odd. Then $a_2 \geq \left(\frac{d}{4}\right)^{\frac{1}{h}}$.*

Proof. From Lemma 2.2 and Lemma 2.3, because $(a_2, d) = 1$. □

Lemma 2.9. *Let $h > 3$ be odd. Then a_4 is prime or $a_4 = a_2^n$ with $1 < n \leq \frac{h}{2} \frac{\log \frac{d}{3}}{\log \frac{d}{4}}$.*

The proofs of this lemma and of Lemma 2.11 are variations of the proof given for Lemma 2.13 and are therefore omitted. The upper bound for n can be obtained with the help of Lemma 2.8. We also omit the proof of Corollary 2.10, which works like the one for Corollary 2.7.

Corollary 2.10. *Let $h = 5$ or $h = 7$. Then $a_4 = a_5$ and in case of $h = 7$, $a_4, a_5 < a_6$, i.e., there is exactly one pair of forms with a_4 as leading coefficient.*

Lemma 2.11. *Let $h = 7$. Then a_6 is prime or $a_6 = a_2^n a_4^m$ with $n, m \in \mathbb{N}_0$, $1 < n + m < 4$.*

We can sum up the results of the preceding lemmas in a way which is useful for us later.

Lemma 2.12. 1) *In the case of $h = 5$ we have $1 < a_2 = a_3 < a_4 = a_5$, in the case of $h = 7$ we have $1 < a_2 = a_3 < a_4 = a_5 < a_6 = a_7$.*

2) *Suppose a_2 has a lower bound $\lambda \in \mathbb{R}$. Let $p_2 < p_4 < p_6$ be the three least prime numbers $\geq \lambda$. Then in the case of $h = 5$ or $h = 7$, $a_2 \geq p_2$ and $a_4 \geq p_4$; moreover, in the case of $h = 7$ and $\lambda \geq 4$, $a_6 \geq p_6$.*

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Proof. 1) From 2.6, 2.7 and 2.10.

2) We have $a_2 \geq p_2$ because a_2 is prime by Lemma 2.2. Now by Lemma 2.9 and 1), $a_4 \geq p_4$ or $a_4 \geq a_2^2$. Bertrand's postulate yields a prime number p, $a_2 < p < 2a_2 \leq a_2^2$, hence it is safe to take the first bound. Finally, in case $h = 7$, by Lemma 2.11 and 1), $a_6 \geq p_6$ or $a_6 \geq a_2^2$. Bertrand's postulate yields for $a_2 \geq 4$ two prime numbers p, q, $a_2 < p < q < 4a_2 \leq a_2^2$, so it is safe to take the first bound. □

The condition $\lambda \geq 4$ is satisfied except for very small discriminants (cf. Lemma 2.8).

2.2. **Even class number.** Suppose (restrictively) that $h = 2n$ with n odd (e.g., $h = 6$); then by the theory of genera the number t of distinct prime divisors of d equals 2. This is because on the one hand, $2^{t-1} \mid h$, so $t \leq 2$, and on the other hand, h is always odd for discriminants with $t = 1$ (prime discriminants). Further, take into consideration that $-d \equiv 1 \pmod 4$, or $-d \equiv 0 \pmod 4$ and $-\frac{d}{4} \equiv 2, 3 \pmod 4$; then d must have one of the forms

$$d = \begin{cases} 4p & \text{with } p \text{ prime, } p \equiv 1 \pmod 4, \\ 8p & \text{with } p \text{ prime, } p > 2, \\ pq & \text{with } p, q \text{ prime, } p \equiv 1 \pmod 4,\ q \equiv 3 \pmod 4. \end{cases}$$

When the class number is even, there are, unlike for odd class number, not necessarily pairs of reduced forms (see Lemma 2.4). For the moment, we collect all reduced forms with a_2 as leading coefficient under the term f_2.

Lemma 2.13. *Let $h \geq 4$ be even, $f(x, y) = ax^2 + bxy + cy^2$ be a reduced form with $-d = b^2 - 4ac$ and $a > a_2$. Suppose further that the leading coefficients of all reduced forms other than the principal form and f_2 with discriminant $-d$ are $\geq a$. Then a is prime or $a = a_2^n$ with $n > 1$.*

Proof. Let $d' = \frac{d}{a_2} (\in \mathbb{Q})$. Because $a_2 \leq \left(\frac{d}{3}\right)^{\frac{1}{2}} = a_2^{\frac{1}{2}} \left(\frac{d'}{3}\right)^{\frac{1}{2}}$ we have $a_2^{\frac{1}{2}} \leq \left(\frac{d'}{3}\right)^{\frac{1}{2}}$. Suppose a is not prime, say $p \mid a$, $p < a$, p prime. Like in (1), $R(p) \geq 1$. By Lemma 2.1 at most f_1 and f_2 can represent p. Like in (2) (with a instead of a_2), f_1 does not represent p, therefore f_2 must represent p. Here, with Lemma 2.1, $p = a_2$ or $p \geq c_2$. Thus, a has the form $a_2^n p_1 \cdots p_m$ with $n, m \in \mathbb{N}_0$, p_i prime, $p_i \geq c_2 \geq \frac{d}{4a_2} = \frac{d'}{4}$, $i = 1, \dots, m$, and so

$$\frac{d'}{3} \geq a_2^{\frac{1}{2}} \left(\frac{d'}{3}\right)^{\frac{1}{2}} = \left(\frac{d}{3}\right)^{\frac{1}{2}} \geq a \geq a_2^n \left(\frac{d'}{4}\right)^m.$$

If $m \geq 2$, then $\frac{d'}{3} \geq \left(\frac{d'}{4}\right)^2$, $d' \leq \frac{16}{3}$ and $a_2 = 1$. This is a contradiction. If $m = 1$, then $n \geq 1$, otherwise a is prime, which is contrary to the supposition. So $\frac{d'}{3} \geq a_2 \frac{d'}{4}$ and $a_2 = 1$. This is a contradiction. □

Lemma 2.14. *Let $h = 6$.*

1) *Suppose $a_2 \nmid d$. Then $a_2 \geq \left(\frac{d}{4}\right)^{\frac{1}{6}}$ and $a_3 = a_2$.*

2) *Suppose $a_2 \mid d$. Then a_3 is prime, $a_3 \geq \left(\frac{d}{4}\right)^{\frac{1}{6}}$. Moreover, if $a_2 < \left(\frac{3}{16}d\right)^{\frac{1}{2}}$, then $a_4 = a_3$.*

Proof. 1) By Lemma 2.2, $\left(\frac{-d}{a_2}\right) \neq -1$; since $a_2 \nmid d$, we have $\left(\frac{-d}{a_2}\right) \neq 0$; hence, $\left(\frac{-d}{a_2}\right) = 1$. Here, on the one hand, $a_2 \geq \left(\frac{d}{4}\right)^{\frac{1}{6}}$ by Lemma 2.3, and on the other

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

hand, the number of representations $R(a_2) = 1 + \left(\frac{-d}{a_2}\right) = 2$. *One* form with leading coefficient a_2 yields only one representation. The form f_1 cannot represent a_2 (cf. (2)). If $a_2 < a_3$, then the remaining forms cannot do so either (cf. Lemma 2.1).

2) We have $\left(\frac{-d}{a_2}\right) = 0$, so $R(a_2) = 1$, hence $a_3 > a_2$. By Lemma 2.13, a_3 is prime or $a_3 = a_2^n$ with $n > 1$. The latter yields $R(a_3) = \sum_{t|a_3} \left(\frac{-d}{t}\right) = \sum_{i=0}^n \left(\frac{-d}{a_2}\right)^i = 1$. Consequently, $a_4 > a_3$ and in $f_3(x, y) = a_3x^2 + b_3xy + c_3y^2$ it must be that $b_3 = 0$ or $b_3 = a_3$ (otherwise there are two forms with a_3, and $a_4 = a_3$, cf. the remark to Lemma 2.5). Both cases result in a contradiction: $b_3 = 0$ gives $-d = -4a_2^n c_3$, $b_3 = a_3$ gives $-d = a_2^n(a_2^n - 4c_3)$, and $-d$ is no discriminant. So a_3 must be prime. Because a_2 already divides the discriminant, a_3 cannot do the same, since otherwise, depending on the form of the discriminant (see the beginning of this section) and with the help of the inequalities $d \le \left(\frac{d}{3}\right)^{\frac{1}{2}} \left(\frac{d}{3}\right)^{\frac{1}{2}}$ resp. $d \le 8\left(\frac{d}{3}\right)^{\frac{1}{2}}$, we get a contradiction. Hence, $\left(\frac{-d}{a_3}\right) = \pm 1$. Furthermore, $\left(\frac{-d}{a_3}\right) \ne -1$, because a_3 is trivially represented. Now Lemma 2.3 gives the second part of the assertion.

Finally let $a_2 < \left(\frac{3}{16}d\right)^{\frac{1}{2}}$. We have $R(a_3) = 2$ like above. *One* form with leading coefficient a_3 yields only one representation. The form f_1 cannot represent a_3 (cf. (2)). If $a_3 < a_4$, at most a reduced form with a_2 as leading coefficient can represent a_3 (cf. Lemma 2.1). But then (cf. Lemma 2.1), $\left(\frac{d}{3}\right)^{\frac{1}{2}} \ge a_3 \ge c_2 \ge \frac{d}{4a_2}$ and $a_2 \ge \left(\frac{3}{16}d\right)^{\frac{1}{2}}$. This is a contradiction. □

Corollary 2.15. *Let $h = 6$. Then a_2 and a_3 are prime, and $a_3 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$.*

2.3. **Integers having prescribed quadratic character.** A result of D. H. Lehmer, E. Lehmer and D. Shanks [10] can be used to obtain effective lower bounds for discriminants with "small" class number and "big" leading coefficients of the associated binary quadratic forms. We will use this in §3 for class number 5.

Lemma 2.16. *Let p be a prime, $p > 2$. Define*

$$M_p = \min\left\{ n \in \mathbb{N} \,\middle|\, \left(\frac{-n}{q}\right) = -1 \text{ for all } q \text{ prime, } 2 \le q < p \right\}.$$

Then $a_2 \ge p$ implies $d \ge M_p$.

If d is a prime, then in the definition of M_p we restrict n to be prime, thus obtaining possibly greater values of M_p, i.e., better lower bounds for d.

Proof. Let $a_2 \ge p > 2$. Recall that by Lemma 2.2, a_2 is prime and $\left(\frac{-d}{q}\right) = -1$ for all prime q, $2 \le q < a_2$. Therefore, $d \ge M_{a_2}$. The inequality $M_{a_2} \ge M_p$ holds trivially. □

The prime values for M_p we will use are $M_{131} = 193310265163$, $M_{137,139} = 229565917267$, $M_{149} = 915809911867$ and $M_{191} = 30059924764123$. They are taken from D. H. Lehmer, E. Lehmer and D. Shanks [10]. It may be noted that our M_p equals Lehmer's N_q, if q is the greatest prime less than p. The value for N_{149} had to be corrected, see [16]. The number M_{191} is N_{181} of [15].

3. Encircling the range

We apply the methods of Stark and Montgomery-Weinberger to "midsized" discriminants. First, we restrict the size of d with the help of

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Theorem 3.1. *Let $-d < 0$ be a discriminant with class number h. Then*

$$h > \frac{1}{55} \prod_{p|d}{}^{*} \left(1 - \frac{\lfloor 2\sqrt{p} \rfloor}{p+1}\right) \log d,$$

*where the * indicates that the greatest prime divisor of d must be omitted.*

This theorem originates from the works of D. Goldfeld [4] in 1976, B. Gross and D. Zagier [7] in 1983, and J. Oesterlé [12, 13] in 1984.

Lemma 3.2. *Let $-d < 0$ be a discriminant. If $h = 5$, then $d < 10^{120}$; if $h = 6$, then $d < 10^{574}$; if $h = 7$, then $d < 10^{168}$.*

Proof. If $h = 5$ or $h = 7$, then d is prime (cf. §2.1). If $h = 6$, then d has exactly two distinct prime divisors (cf. §2.2). Now use Theorem 3.1. □

Now we turn to d with order of magnitude up to $10^{11} \dots 10^{14}$ (depending on the class number). They are searched with the use of a computer. It would be very time-consuming, for example, to examine all discriminants in the range $1 \le d \le 10^{11}$ for class number 5. However, Lemma 2.3 helps to reduce the work. Let p be a prime and $d > 4p^h$. Then by Lemma 2.3 we have $\left(\frac{-d}{q}\right) \neq 1$ for all prime q, $2 \le q \le p$. This means that in a first step we can discard a certain quantity of d's by looking at their quadratic character modulo small primes. In a second step we examine the remaining d for class number h by searching for reduced solutions ($-a < b \le a < c$ or $0 \le b \le a = c$) of $-d = b^2 - 4ac$.

Thus, in the range $1 \le d \le 1.33 \cdot 10^{11}$ we obtain the d-values for $h = 5$ as listed in Table 1. Using results of §2.3, we can slightly raise the bound $1.33 \cdot 10^{11}$: for $d > 1.33 \cdot 10^{11}$, by Lemma 2.8, we have $a_2 \ge 131$. By Lemma 2.16 this leads to $d \ge M_{131} \ge 1.9331 \cdot 10^{11}$. Repeating this argument, we get sucessively $a_2 \ge 139$, $d \ge M_{139} \ge 2.2956 \cdot 10^{11}$, $a_2 \ge 149$, $d \ge M_{149} \ge 9.1580 \cdot 10^{11}$, $a_2 \ge 191$, $d \ge M_{191} \ge 3.0059 \cdot 10^{13}$.

Lemma 3.3. *Suppose $h(-d) = 5$; then d is one of the numbers listed in Table 1 or $d > 3 \cdot 10^{13}$.*

In the case of $h = 6$ and $h = 7$, §2.3 cannot be used to reduce the range.

Lemma 3.4. *Suppose $h(-d) = 7$; then d is one of the numbers listed in Table 1 or $d > 8 \cdot 10^{12}$.*

Lemma 3.5. *Suppose $h(-d) = 6$; then d is one of the numbers listed in Table 1 or in case of*

1) $17923 \nmid d$, $a_2 = 2$; $d > 6.2 \cdot 10^{13}$,
2) $17923 \nmid d$, $a_2 = 3$; $d > 3 \cdot 10^{13}$,
3) $17923 \nmid d$, $a_2 \ge 5$; $d > 2 \cdot 10^{13}$,
4) $17923 \mid d$; $d > 1.1 \cdot 10^{14}$.

We omit the programming details here and refer to [17]. The programs ran about 5 minutes for $h = 5$, 55 hours for $h = 6$, and 11 hours for $h = 7$ on the IBM 3090 of the University of Freiburg i. Br. For control purposes the programs were also run on the CRAY 2 of the University of Stuttgart.

The rest of the paper deals with the remaining "midsized" d.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

4. Basics of Stark's method

Stark (for details, see [16]) starts from the following representation of the zeta function $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi_{-d})$ of $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$,

$$\begin{aligned} \zeta(s)L(s,\chi_{-d}) &= \zeta(2s)\sum_f a^{-s} + \zeta(2-2s)\frac{\Gamma(1-s)}{\Gamma(s)}\left(\frac{d}{4\pi^2}\right)^{\frac{1}{2}-s}\sum_f a^{s-1} \\ &\quad + \sum_f h(s,f). \end{aligned} \tag{3}$$

The summation is over a complete system of nonequivalent binary quadratic forms $f = f(x,y) = ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = -d$. The error terms $h(s,f)$ are

$$h(s,f) = a^{-s}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\left(x - [x] - \frac{1}{2}\right)\frac{d}{dx}\left\{\left(\left(x + \frac{bj}{2a}\right)^2 + \frac{dj^2}{4a^2}\right)^{-s}\right\}dx. \tag{4}$$

For these error terms, Stark [16] gave estimates. We adopt them with slight modifications.

Lemma 4.1. *Let $s = \sigma + i\tau$, $\sigma \geq \frac{1}{2}$ and $k \in \mathbb{N}$, $k \geq 3$. Then*

$$|h(s,f)| < 2\left(\frac{4a}{d}\right)^{\sigma-\frac{1}{2}}\left(\frac{2\pi}{ak}\right)^{\frac{1}{2}}\left(\frac{k-1}{k-2}\right)\left(\frac{2|s| + \frac{k-1}{2}}{\pi\left(\frac{d}{a^2}\right)^{\frac{1}{2}}}\right)^k \exp\left(\frac{1}{4k} + \frac{\sqrt{2}}{3\pi^3k^2}\right).$$

Proof. See Stark [16, Lemma 1]. Simply change the estimate for $\zeta(m)$ to $\zeta(m) < \sqrt{(m-1)/(m-2)}$ $(m \geq 3)$. ◻

Lemma 4.2. *Let $a \leq \left(\frac{d}{3}\right)^{\frac{1}{2}}$ and $s = \frac{1}{2} + i\tau$. Then for all $J \in \mathbb{N}$*

$$|h(s,f)| < \frac{4|s|}{(3d)^{\frac{1}{4}}}\left(\sum_{j=1}^{J-1}\frac{1}{j} + \frac{\frac{2}{3}\tau + 0.77}{3\sqrt{3}}\sum_{j=J}^{\infty}\frac{1}{j^2}\right).$$

Proof. See Stark [16, Lemma 2]. We do not use the estimate for

$$\left|(2s+1) - \frac{2s+2}{u^2+1}\right|; \tag{5}$$

instead, we let $s = \frac{1}{2} + i\tau$ in (5) and obtain

$$\begin{aligned} &\left|\int_{-\infty}^{\infty}\left(x - [x] - \frac{1}{2}\right)\frac{d}{dx}\left\{\left(\left(x + \frac{bj}{2a}\right)^2 + \frac{dj^2}{4a^2}\right)^{-s}\right\}dx\right| \\ &\leq \frac{4a^2|s|}{3dj^2}\int_0^{\frac{\pi}{2}}\sqrt{\left((2 - 3\cos^2 x)\cos x\right)^2 + \left(2\tau\sin^2 x\cos x\right)^2}\,dx \\ &\leq \frac{4a^2|s|}{3dj^2}\left(\frac{4}{3}\sin\left(\arccos\sqrt{\frac{2}{3}}\right) + \frac{2}{3}\tau\right). \end{aligned} \tag{6}$$

◻

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

5. Basics of Montgomery-Weinberger

Let $-k < 0$ be a discriminant with $(d, k) = 1$. Further, let θ be a complex number with $|\theta| \leq 1$, the value of which depends on the context in which it occurs, e.g., θ in Lemma 5.1 and θ in Lemma 5.3 need not be the same. Montgomery and Weinberger [11] proceed from

Lemma 5.1. *Let* $(d, k) = 1$, $t \geq 0$. *Then*

(7)
$$it\, L(\tfrac{1}{2} + it, \chi_k)\, L(\tfrac{1}{2} + it, \chi_{kd})\, \Gamma(\tfrac{1}{2} + it) \left(\frac{kd^{\frac{1}{2}}}{2\pi} \right)^{it} = iM(t) \sin \varphi(t) + \theta t E(t),$$

in which

$$M(t) = \left| 2t\, \zeta(1 + 2it)\, \Gamma(\tfrac{1}{2} + it)\, P_k(\tfrac{1}{2} + it)\, A(\tfrac{1}{2} + it) \right|,$$

$$\varphi(t) = \arg \left(i\, \zeta(1 + 2it)\, \Gamma(\tfrac{1}{2} + it)\, P_k(\tfrac{1}{2} + it)\, A(\tfrac{1}{2} + it) \left(\frac{kd^{\frac{1}{2}}}{2\pi} \right)^{it} \right),$$

with

$$P_k(s) = \prod_{p|k} (1 - p^{-2s}), \qquad A(s) = \sum_f \chi_k(a) a^{-s},$$

and

$$E(t) = \frac{4\pi^{\frac{1}{2}}}{k} \sum_f a^{-\frac{1}{2}} \sum_{n=1}^{\infty} K_0 \left(\frac{\pi n d^{\frac{1}{2}}}{ak} \right) \sum_{y|n} \left| \sum_{j=1}^{k} \chi_k(f(j, y)) \exp\left(\frac{2\pi i j n}{ky} \right) \right|,$$

where K_0 *is the modified Bessel function of the second kind (note that* $E(t) \geq 0$*).*

If in (7) we let t be the imaginary part of a zero $\frac{1}{2} + it$ of $L(s, \chi_k)$, then

$$|\sin \varphi(t)| \leq \frac{t\, E(t)}{M(t)}. \tag{8}$$

This relation, with sufficiently large d and appropriate choice of k and t, may be used to get a contradiction to the assumption of a small class number. The following lemmas provide estimates for $M(t), \varphi(t)$ and $E(t)$. For proofs see [11].

Lemma 5.2. *Let* $0 < t \leq \frac{1}{20}$. *Then* $M(t) \geq \frac{7}{4} \prod_{p|k} (1 - p^{-1}) \left| A(\frac{1}{2} + it) \right|$.

Lemma 5.3. *Let* $0 \leq t \leq 6$. *Then*

$$\varphi(t) = t \left(C + \log \left(\frac{kd^{\frac{1}{2}}}{8\pi} \right) \right) + 3\theta t^3 + \theta t \left(c(h) + 2 \sum_{p|k} \frac{\log p}{p - 1} \right),$$

in which $C = 0.577215664\ldots$ *is Euler's constant and* $c(h) \leq \left| \frac{A'}{A}(\frac{1}{2} + it) \right|$.

Lemma 5.4. *Let* $k \geq 3060$, $0 < t \leq \frac{1}{4}$. *Define*

$$\delta(a) = \left(\frac{a}{d} \right)^{\frac{1}{2}} \exp \left(-\frac{\pi d^{\frac{1}{2}}}{2ak} \right).$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Then, if all forms f are reduced,

$$E(t) \le 8\left(\frac{k}{\pi}\right)^{\frac{1}{2}} \log k \prod_{p|k}(2+3p^{-\frac{3}{2}}) \sum_{f} \delta(a).$$

Proof. While proving Lemma 9 of [11], Montgomery and Weinberger get

$$E(t) \le 4\left(\frac{\pi}{k}\right)^{\frac{1}{2}} \prod_{p|k}(2+3p^{-\frac{3}{2}}) \sum_{f} a^{-\frac{1}{2}} g\left(\frac{\pi d^{\frac{1}{2}}}{2ak}\right),$$

with $g(x) = \frac{e^{-x}}{x}\left(1+\log\left(1+\frac{1}{x}\right)\right)$. So the summands of the sum over f have the form

$$\frac{2a^{\frac{1}{2}}k}{\pi d^{\frac{1}{2}}} \exp\left(-\frac{\pi d^{\frac{1}{2}}}{2ak}\right)\left(1+\log\left(1+\frac{2ak}{\pi d^{\frac{1}{2}}}\right)\right).$$

The third factor, by means of the inequality $a \le \sqrt{\frac{d}{3}}$, is bounded uniformly for all a by $\log k$ (for $k \ge 3060$). □

If a is not effectively known, we will use the last lemma either under the (nontrivial) assumption $a \le \left(\frac{d}{4}\right)^{\frac{1}{3}}$ or under the (trivial) assumption $a \le \left(\frac{d}{3}\right)^{\frac{1}{2}}$ in $\delta(a)$.

6. Class number 5 and 7

In case of an odd class number both Stark's and Montgomery-Weinberger's method can be applied. In [17] we used the first method for class number 5 and the second method for class number 7. However, the adaption of Stark's method was somewhat more tedious, so for brevity here we proceed like in [11]. This is straightforward; therefore, we can confine ourselves to a short survey. Details on working with Montgomery-Weinberger's method appear in §7.

For $h = 5$ we examine the range $2 \cdot 10^{12} \le d \le 10^{52}$ and distinguish three cases for the leading coefficients a_2 $(= a_3)$ and a_4 $(= a_5)$ of the reduced forms by comparing them with $\left(\frac{d}{4}\right)^{\frac{1}{3}}$. If the a_i are "small", then Lemma 5.4 will give us a "good" bound for $E(t)$. If the a_i are "big", then the bounds for $\left|A(\frac{1}{2}+it)\right|$ and $c(5)$ in Lemmas 5.2 and 5.3 will be "good". In all cases we use $k = 17923$ and $t = 0.030986$ (see Table 2). Finally, we examine d in the range $10^{52} \ldots 10^{120}$, using $k = 115147$ and $t = 0.003158$. The result is

Theorem 6.1. *Suppose* $2 \cdot 10^{12} \le d \le 10^{120}$. *Then* $h(-d) \ne 5$.

Table 2. Zeros $\frac{1}{2}+it$ of $L(s,\chi_k)$ for various k; from [11]

k			$t + 10^{-6}\theta$
17923			0.030986
28963	=	$11 \cdot 2633$	0.033774
30895	=	$5 \cdot 37 \cdot 167$	0.018494
37427	=	$13 \cdot 2879$	0.019505
115147	=	$113 \cdot 1019$	0.003158
123204	=	$4 \cdot 3 \cdot 10267$	0.010650
139011	=	$3 \cdot 46337$	0.012930

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

For $h = 7$ we examine the range $8 \cdot 10^{12} \le d \le 10^{168}$ and distinguish four cases for the leading coefficients a_2 $(= a_3)$, a_4 $(= a_5)$, a_6 $(= a_7)$ of the reduced forms, completely analogous to $h = 5$. We get

Theorem 6.2. *Suppose* $8 \cdot 10^{12} \le d \le 10^{168}$. *Then* $h(-d) \neq 7$.

7. Class number 6

7.1. **Results with Montgomery-Weinberger.** We first turn to $d \le 10^{52}$ and distinguish certain cases for the coefficients $a_2, a_3, \ldots, a_6$ (like we did for $h = 5, 7$). The problem is that a_2 no longer needs to be relatively prime to d, and therefore the lower bound of Lemma 2.8 will fail. We distinguish the cases $17923 \nmid d$ and $17923 \mid d$. In the first case we further distinguish $a_2 = 2$, $a_2 = 3$, $5 \le a_2 < \left(\frac{d}{4}\right)^{\frac{1}{6}}$ and $a_2 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$, and in the latter case $a_2 = 17923$ and $a_2 < 17923$.

1. $17923 \nmid d$

1(a) $a_2 = 2$

Here we restrict our argument to d being even, because if d is odd, then by Lemmas 2.2 and 2.3, $2 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$. Further, by Lemma 2.14 we have a_3 prime, $a_3 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$ and $a_4 = a_3$.

Lemma 7.1. *Let* $17923 \nmid d$, $a_2 = 2$, *and suppose* $2.3 \cdot 10^{14} \le d \le 10^{52}$, $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \ldots, 6$. *Then* $h(-d) \neq 6$.

Proof. A sample proof is given in the next lemma. We have $a_i \ge 38598$, $i = 3, \ldots, 6$. Take $k = 17923$ and $t = 0.030986$. □

Lemma 7.2. *Let* $17923 \nmid d$, $a_2 = 2$, *and suppose* $6.2 \cdot 10^{13} \le d \le 10^{52}$, $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, 4$, $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 5, 6$. *Then* $h(-d) \neq 6$.

Proof. We have $a_3, a_4 \ge 163$, $a_5, a_6 \ge 24934$. Take $k = 30895 = 5 \cdot 37 \cdot 167$ ($(d, k) = 1$, else $d = (4p$ or $8p) \le 8 \cdot 167$) and $t = 0.018494$. Then

$$\left|A\left(\frac{1}{2} + it\right)\right| \ge \underbrace{\left|1 + \underbrace{\left(\frac{-30895}{2}\right)}_{=1} 2^{-\frac{1}{2} - it}\right|}_{\ge 1.707} - \frac{2}{\sqrt{163}} - \frac{2}{\sqrt{24934}},$$

and thus $M(t) \ge 2.0820$ by Lemma 5.2,

$$c(6) \le \frac{\frac{\log 2}{\sqrt{2}} + 2\frac{\log 163}{\sqrt{163}} + 2\frac{\log 24934}{\sqrt{24934}}}{1.707 - \frac{2}{\sqrt{163}} - \frac{2}{\sqrt{24934}}} \le 0.9211$$

and $E(t) \le 43.5542$ by Lemma 5.4, $|\sin \varphi(t)| \le 0.3869$ by (8). But $0.3991 \le \varphi(t) \le 1.2863$ by Lemma 5.3. □

Lemma 7.3. *Let* $17923 \nmid d$, $a_2 = 2$, *and suppose* $6.2 \cdot 10^{13} \le d \le 10^{52}$ *and*

1) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, 4, 5$, $a_6 \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$ *or*

2) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \ldots, 6$.

Then $h(-d) \neq 6$.

Proof. In both cases take $k = 30895$. □

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

In §7.2 we will use Stark's method to close the gap $6.2 \cdot 10^{13} \le d \le 2.3 \cdot 10^{14}$, which was left by Lemma 7.1; so for all d with $a_2 = 2$ we will have the uniform bound $6.2 \cdot 10^{13}$.

1(b) $a_2 = 3$

Here we have (for $d > 2916$) $d \equiv 0 \pmod 3$ by Lemmas 2.2 and 2.3. Further, by Lemma 2.14 we have a_3 prime, $a_3 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$ and $a_4 = a_3$.

Lemma 7.4. *Let* $17923 \nmid d$, $a_2 = 3$, *and suppose* $3 \cdot 10^{13} \le d \le 10^{52}$ *and*

1) $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \dots, 6$, *or*
2) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, 4$, $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 5, 6$, *or*
3) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, 4, 5$, $a_6 \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, *or*
4) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \dots, 6$.

Then $h(-d) \ne 6$.

Proof. In all cases, let $k = 37427 = 13 \cdot 2879$ ($(d, k) = 1$, else $d = 3p \le 3 \cdot 2879$), $t = 0.019505$ and use $\left(\frac{-37427}{3}\right) = 1$ to estimate $\left|A(\frac{1}{2} + it)\right|$. □

1(c) $5 \le a_2 < \left(\frac{d}{4}\right)^{\frac{1}{6}}$

Here, $d \equiv 0 \pmod{a_2}$ by Lemmas 2.2 and 2.3. Further, by Lemma 2.14, a_3 is prime, $a_3 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$ and $a_4 = a_3$.

Lemma 7.5. *Let* $17923 \nmid d$, $5 \le a_2 < \left(\frac{d}{4}\right)^{\frac{1}{6}}$; *suppose* $2 \cdot 10^{13} \le d \le 10^{52}$ *and*

1) $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \dots, 6$, *or*
2) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, 4$, $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 5, 6$, *or*
3) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, 4, 5$, $a_6 \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, *or*
4) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \dots, 6$.

Then $h(-d) \ne 6$.

Proof. In all cases, let $k = 17923$. □

1(d) $a_2 \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$

Lemma 7.6. *Let* $17923 \nmid d$, $2 \cdot 10^{13} \le d \le 10^{52}$ *and*

1) $a_2 \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, *or*
2) $\left(\frac{d}{4}\right)^{\frac{1}{6}} \le a_2 < \left(\frac{d}{4}\right)^{\frac{1}{3}}$.

Then $h(-d) \ne 6$.

Proof. In both cases, take $k = 17923$. □

2. $17923 \mid d$

Here, d has exactly two distinct prime divisors. Without loss of generality we may assume 17923 to be the smaller one, since otherwise $d < 4 \cdot 10^8$. Further, assume $17923 \le \left(\frac{d}{4}\right)^{\frac{1}{2}}$, since otherwise $d < 2 \cdot 10^9$. Lemma 2.4 yields the existence

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

of a reduced form with discriminant $-d$ and 17923 as leading coefficient. Therefore, $a_2 \le 17923$.

2(a) $a_2 = 17923$

Lemma 7.7. *Let* $17923 \mid d$, $a_2 = 17923$, *and suppose* $1.1 \cdot 10^{14} \le d \le 10^{52}$. *Then* $h(-d) \ne 6$.

Proof. Take $k = 28963 = 11 \cdot 2633$. □

2(b) $a_2 < 17923$

Here, $d \not\equiv 0 \pmod{a_2}$, else 17923 is not the smallest prime divisor of d. By Lemma 2.14 it follows that $a_3 = a_2$. Both coefficients are prime and $\ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$.

Lemma 7.8. *Let* $17923 \mid d$, $a_2 < 17923$; *suppose* $1.1 \cdot 10^{14} \le d \le 10^{52}$ *and*

1) $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 4, 5, 6$, *or*
2) $a_4 < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 5, 6$, *or*
3) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 4, 5$, $a_6 \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, *or*
4) $a_i < \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 4, 5, 6$.

Then $h(-d) \ne 6$.

Proof. In all cases, take $k = 28963$. □

Finally, we consider d from 10^{52} to 10^{574}. It is necessary to examine just two cases.

Lemma 7.9. *Suppose* $10^{52} \le d \le 10^{574}$ *and* $(d, 115147) = 1$. *Then* $h(-d) \ne 6$.

Proof. Clearly, $a_2 \ge 2$. By Corollary 2.15 we have $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{6}}$, $i = 3, \dots, 6$. Take $k = 115147 = 113 \cdot 1019$. □

Lemma 7.10. *Suppose* $10^{52} \le d \le 10^{574}$ *and* $(d, 115147) > 1$. *Then* $h(-d) \ne 6$.

Proof. We have $113 \mid d$ or $1019 \mid d$. But then $(d, 123204 = 4 \cdot 3 \cdot 10267) = 1$, otherwise d is too small. Take $k = 123204$, $t = 0.003158$. We have $a_i \ge 113$, $i = 2, \dots, 6$ ($a_2 < 113 \Rightarrow a_2 < \left(\frac{d}{4}\right)^{\frac{1}{6}} \Rightarrow a_2 \mid d \Rightarrow d$ too small; but $a_2 = 113$ is possible, cf. Lemma 2.4). We get $M(t) \ge 0.5282$, $c(6) \le 0.4956$, $E(t) \le 2 \cdot 10^{-7}$ and $|\sin\varphi(t)| \le 5 \cdot 10^{-9}$. But $0.7024 \le \varphi(t) \le 7.1664$. Therefore, $\varphi = \pi + \theta 10^{-8}$ or $\varphi = 2\pi + \theta 10^{-8}$. By Lemma 5.3 we see that $10^{122} < d < 10^{126}$ or $10^{250} < d < 10^{254}$ must hold.

However, we can also work with $k = 139011 = 3 \cdot 46337$, $t = 0.003158$. Then we get $M(t) \ge 1.0566$, $c(6) \le 0.4956$, $E(t) \le 4 \cdot 10^{-8}$ and $|\sin\varphi(t)| \le 5 \cdot 10^{-10}$. But also $0.8723 \le \varphi(t) \le 8.6843$. Therefore, $\varphi = \pi + \theta 10^{-8}$ or $\varphi = 2\pi + \theta 10^{-8}$. By Lemma 5.3, $10^{100} < d < 10^{103}$ or $10^{206} < d < 10^{208}$ must hold. The contradiction of the results for the two k's proves the lemma. □

7.2. **Results with Stark's method.** We will only examine the case $a_2 = 2$, $a_i \ge \left(\frac{d}{4}\right)^{\frac{1}{3}}$, $i = 3, \dots, 6$, $6.2 \cdot 10^{13} \le d \le 2.3 \cdot 10^{14}$ (remember Lemma 7.1 and

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

preceding remarks), which could not be treated with the method of Montgomery-Weinberger. (The condition $17923 \nmid d$ is insignificant here.)

Let $\phi_m = \frac{1}{2} + i\tau_m$ be a zero of the Riemann zeta function $\zeta(s)$. Putting ϕ_m in (3) gives

$$\left(\frac{d}{4\pi^2}\right)^{i\tau_m}\left(1+\frac{\sum_f h(\frac{1}{2}+i\tau_m,f)}{\zeta(1+2i\tau_m)\left(1+2^{-\frac{1}{2}-i\tau_m}+2a_3^{-\frac{1}{2}-i\tau_m}+a_5^{-\frac{1}{2}-i\tau_m}+a_6^{-\frac{1}{2}-i\tau_m}\right)}\right)$$
$$=-\frac{\zeta(1-2i\tau_m)}{\zeta(1+2i\tau_m)}\frac{\Gamma(\frac{1}{2}-i\tau_m)}{\Gamma(\frac{1}{2}+i\tau_m)}\frac{\left(1+2^{-\frac{1}{2}+i\tau_m}+2a_3^{-\frac{1}{2}+i\tau_m}+a_5^{-\frac{1}{2}+i\tau_m}+a_6^{-\frac{1}{2}+i\tau_m}\right)}{\left(1+2^{-\frac{1}{2}-i\tau_m}+2a_3^{-\frac{1}{2}-i\tau_m}+a_5^{-\frac{1}{2}-i\tau_m}+a_6^{-\frac{1}{2}-i\tau_m}\right)}.$$

Define

$$\alpha_m \equiv \pi - 2\arg\zeta(2\phi_m) - 2\arg\Gamma(\phi_m) \pmod{2\pi}, \qquad 0 \le \alpha_m < 2\pi,$$

$$A_n = \frac{1}{2\pi}\left(\frac{\tau_n}{\tau_1}\alpha_1 - \alpha_n\right).$$

Let $|\theta| \le 1$ as in §5. For $a_3 \ge 191$ let

$$\delta_m(a_3,a_5,a_6)=\frac{|h(\phi_m,f_1)|+|h(\phi_m,f_2)|+2|h(\phi_m,f_3)|+|h(\phi_m,f_5)|+|h(\phi_m,f_6)|}{|\zeta(2\phi_m)|\left(\left|1+2^{-\frac{1}{2}-i\tau_m}\right|-2a_3^{-\frac{1}{2}}-a_5^{-\frac{1}{2}}-a_6^{-\frac{1}{2}}\right)}.$$

Lemma 7.11. *Suppose $a_2 = 2$, $a_3 \ge 191$, $\delta_m(a_3,a_5,a_6) < \frac{1}{2}$. Then there is an integer x_m with*

$$\begin{aligned}\tau_m \log\left(\frac{d}{4\pi^2}\right) &= \alpha_m + 2\pi x_m \\ &\quad + 2\arg\left(1+2^{-\frac{1}{2}+i\tau_m}+2a_3^{-\frac{1}{2}+i\tau_m}+a_5^{-\frac{1}{2}+i\tau_m}+a_6^{-\frac{1}{2}+i\tau_m}\right) \\ &\quad + \frac{\pi}{3}\delta_m(a_3,a_5,a_6)\theta.\end{aligned}$$

Proof. Cf. Stark [16, Lemma 6]. □

Lemma 7.12. *Suppose $a_2 = 2$, $a_3 \ge 751$, $\delta_m(a_3,a_5,a_6) < \frac{1}{2}$ for $m = 1$ and $m = n$. Then*

$$\begin{aligned}x_n &= \frac{\tau_n}{\tau_1}x_1 + A_n + \frac{1}{\pi}\left(\frac{\tau_n}{\tau_1}\arg\left(1+2^{-\frac{1}{2}+i\tau_1}\right)-\arg\left(1+2^{-\frac{1}{2}+i\tau_n}\right)\right) \\ &\quad + \frac{\theta}{3}\left(\frac{\tau_n}{\tau_1}\frac{2a_3^{-\frac{1}{2}}+a_5^{-\frac{1}{2}}+a_6^{-\frac{1}{2}}}{\left|1+2^{-\frac{1}{2}+i\tau_1}\right|}+\frac{2a_3^{-\frac{1}{2}}+a_5^{-\frac{1}{2}}+a_6^{-\frac{1}{2}}}{\left|1+2^{-\frac{1}{2}+i\tau_n}\right|}\right) \\ &\quad + \frac{\theta}{6}\left(\frac{\tau_n}{\tau_1}\delta_1(a_3,a_5,a_6)+\delta_n(a_3,a_5,a_6)\right).\end{aligned}$$

Proof. Cf. Stark [16, Lemma 7]. However, here we use the relation

$$\arg(1+z+z') = \arg(1+z) + \frac{\pi}{3}\theta\frac{|z'|}{|1+z|} \qquad \text{for } \frac{|z'|}{|1+z|} \le \frac{1}{2}. \tag{9}$$ □

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

TABLE 3. τ_m and related quantities; from Stark [16]

m	$\tau_m + 5\cdot 10^{-10}\theta$	$\frac{\alpha_m}{2\pi} + 10^{-7}\theta$	$\lvert\zeta(2\phi_m)\rvert + 10^{-4}\theta$	$\frac{\tau_m}{\tau_1} + 5\cdot 10^{-10}\theta$	$A_m + 5\cdot 10^{-10}\theta$
1	14.134725142	0.189940085	1.9488	—	—
2	21.022039639	0.744277023	0.8310	1.487262004	-0.461786352

Lemma 7.13. *Suppose $a_2 = 2$, $a_3 \geq 191$ and $d \geq 20000$. Then*

$$\delta_m(a_3,a_5,a_6) < \begin{cases} 10^{-27} + 200.0\, d^{-\frac14}\left(0.427 - 2a_3^{-\frac12} - a_5^{-\frac12} - a_6^{-\frac12}\right)^{-1} & \text{for } m=1,\\ 10^{-27} + 807.4\, d^{-\frac14}\left(0.951 - 2a_3^{-\frac12} - a_5^{-\frac12} - a_6^{-\frac12}\right)^{-1} & \text{for } m=2.\end{cases}$$

Proof. For $a_1 = 1$ and $a_2 = 2$ we could use Lemma 4.1 (with $\frac{d}{a^2} \geq 5000$) to get $|h(\phi_m, f)| < 10^{-30}$ for $m \leq 2$ (with $k = 137$). For $a_3, \dots, a_6$ we used Lemma 4.2 to get $|h(\phi_m, f)| < C\, d^{-\frac14}$ with $C = 97.4$ for $m = 1$ (with $J = 2$) and $C = 167.7$ for $m = 2$ (with $J = 3$). By direct computation we obtained $\left|1 + 2^{-\frac12 - i\tau_1}\right| \geq 0.427$, $\left|1 + 2^{-\frac12 - i\tau_2}\right| \geq 0.951$. See also Table 3. □

Lemma 7.14. *Suppose $d > 6.2 \cdot 10^{13}$, $a_2 = 2$, $a_3 \geq \left(\frac{d}{4}\right)^{\frac13}$. Then*

$$x_1 > 2.249 \log d - 8.543 - 0.398\left(2^{-\frac12} + 2a_3^{-\frac12} + a_5^{-\frac12} + a_6^{-\frac12}\right),$$

$$d > 41.39 \exp\left(0.444\, x_1 - 0.177\left(2^{-\frac12} + 2a_3^{-\frac12} + a_5^{-\frac12} + a_6^{-\frac12}\right)\right).$$

Under the assumptions made, we always have $x_1 \geq 63$, and if $x_1 \geq 67$, then $d > 2.3 \cdot 10^{14}$.

Proof. Analogous to [16, Lemma 8]. We have $\delta_1(a_3, a_5, a_6) < \frac12$ by Lemma 7.13, because $a_3, \dots, a_6 \geq 24934$. Now use Lemma 7.11 and the inequality $|\arg(1+z)| < \frac54 |z|$ for $|z| \leq \frac{9}{10}$. □

Lemma 7.15. *Suppose $d \geq 6.2 \cdot 10^{13}$, $a_2 = 2$, $a_i \geq \left(\frac{d}{4}\right)^{\frac13}$, $i = 3, \dots, 6$. Then*

$$\left|x_2 - \frac{\tau_2}{\tau_1}x_1 - A_2 - \frac{1}{\pi}\left(\frac{\tau_2}{\tau_1}\arg\left(1 + 2^{-\frac12 + i\tau_1}\right) - \arg\left(1 + 2^{-\frac12 + i\tau_2}\right)\right)\right|$$
$$< 10^{-27} + \frac{2a_3^{-\frac12} + a_5^{-\frac12} + a_6^{-\frac12}}{3}\left(\frac{\tau_2}{\tau_1}\frac{1}{\left|1 + 2^{-\frac12 + i\tau_1}\right|} + \frac{1}{\left|1 + 2^{-\frac12 + i\tau_2}\right|}\right)$$
$$+ \frac16 d^{-\frac14}\left(\frac{\tau_2}{\tau_1}\frac{200.0}{0.427 - 2a_3^{-\frac12} - a_5^{-\frac12} - a_6^{-\frac12}} + \frac{807.4}{0.951 - 2a_3^{-\frac12} - a_5^{-\frac12} - a_6^{-\frac12}}\right).$$

Proof. Directly from Lemma 7.12, because $\delta_1(a_3, a_5, a_6) < \frac12$ and $\delta_2(a_3, a_5, a_6) < \frac12$ by Lemma 7.13. □

Lemma 7.16. *Suppose $6.2 \cdot 10^{13} \leq d \leq 2.3 \cdot 10^{14}$, $a_2 = 2$, $a_i \geq \left(\frac{d}{4}\right)^{\frac13}$, $i = 3, \dots, 6$. Then $h(-d) \neq 6$.*

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Proof. We have $a_i \geq 24934$, $i = 3, \dots, 6$, and $x_1 \geq 63$ by Lemma 7.14. We want to show that $x_1 \geq 67$, for then $d > 2.3 \cdot 10^{14}$ by Lemma 7.14. By Lemma 7.15 we have

$$\left| x_2 - \frac{\tau_2}{\tau_1} x_1 - A_2 - \frac{1}{\pi} \left(\frac{\tau_2}{\tau_1} \arg\left(1 + 2^{-\frac{1}{2}+i\tau_1}\right) - \arg\left(1 + 2^{-\frac{1}{2}+i\tau_2}\right)\right)\right| < 0.14.$$

But from

$$\frac{1}{\pi} \left(\frac{\tau_2}{\tau_1} \arg\left(1 + 2^{-\frac{1}{2}+i\tau_1}\right) - \arg\left(1 + 2^{-\frac{1}{2}+i\tau_2}\right)\right) = -0.5414 + 10^{-4}\theta$$

and $\frac{\tau_2}{\tau_1} = 1.487262004 + 5 \cdot 10^{-10}\theta$, $A_2 = -0.461786352 + 5 \cdot 10^{-10}\theta$ (cf. Table 3) it follows that

$$\left| x_2 - \frac{\tau_2}{\tau_1} x_1 - A_2 - \frac{1}{\pi} \left(\frac{\tau_2}{\tau_1} \arg\left(1 + 2^{-\frac{1}{2}+i\tau_1}\right) - \arg\left(1 + 2^{-\frac{1}{2}+i\tau_2}\right)\right)\right| > 0.15$$

for $x_1 = 63, \dots, 66$, because x_2 is an integer. □

In a last theorem we are summing up the results of this section.

Theorem 7.17. *Suppose*

1) $17923 \nmid d$, $a_2 = 2$, $6.2 \cdot 10^{13} \leq d < 10^{574}$, *or*
2) $17923 \nmid d$, $a_2 = 3$, $3 \cdot 10^{13} \leq d < 10^{574}$, *or*
3) $17923 \nmid d$, $a_2 \geq 5$, $2 \cdot 10^{13} \leq d < 10^{574}$, *or*
4) $17923 \mid d$, $1.1 \cdot 10^{14} \leq d < 10^{574}$.

Then $h(-d) \neq 6$.

Bibliography

1. S. Arno, *Class number 4,* Dissertation, Stanford, 1986.
2. S. Arno, *The imaginary quadratic fields of class number 4,* Acta Arith. **60** (1992), 321-334. MR **93b:**11144
3. S. Böcherer, *Das Gauß'sche Klassenzahlproblem,* Mitt. Math. Ges. Hamburg **11** (1988), 565-589. MR **89b:**11082
4. D. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer,* Ann. Scuola Norm. Sup. Pisa (IV) **3** (1976), 623-663. MR **56:**8529
5. D. Goldfeld, *The conjectures of Birch and Swinnerton-Dyer and the class numbers of quadratic fields,* Journées Arithmétiques de Caen **41-42** (1977), 219-227. MR **56:**5491
6. D. Goldfeld, *Gauss' class number problems for imaginary quadratic fields,* Bull. Amer. Math. Soc. **13** (1985), 23-37. MR **86k:**11065
7. B. Gross and D. Zagier, *Points de Heegner et dérivées de fonctions L,* C. R. Acad. Sci. Paris **297** (1983), 85-87. MR **85d:**11062
8. B. Gross and D. Zagier, *Heegner points and derivatives of L-series,* Invent. Math. **84** (1986), 225-230. MR **87j:**11057
9. H. Heilbronn, *On the class number in imaginary quadratic fields,* Quart. J. Math. Oxford Ser. **5** (1934), 150-160.
10. D. H. Lehmer, E. Lehmer and D. Shanks, *Integer sequences having prescribed quadratic character,* Math. Comp. **24** (1970), 433-451. MR **42:**5889
11. H. Montgomery and P. Weinberger, *Notes on small class numbers,* Acta Arith. **24** (1974), 529-542. MR **50:**9841
12. J. Oesterlé, *Nombres de classes des corps quadratiques imaginaires,* Séminaire Bourbaki **631** (1984), 309-323. MR **86k:**11064
13. J. Oesterlé, *Le problème de Gauss sur le nombre de classes,* Enseign. Math. **34** (1988), 43-67. MR **89j:**11108
14. J.-P. Serre, $\triangle = b^2 - 4ac$, Math. Medley **13** (1985), 1-10. MR **87g:**11148
15. D. Shanks, *Systematic examination of Littlewood's bounds on L(1,χ),* Proc. Sympos. Pure Math., Analytic Number Theory **24** (1972), 267-269. MR **49:**2596

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

16. H. Stark, *On complex quadratic fields with class-number two,* Math. Comp. **29** (1975), 289-302. MR **51:**5548
17. Ch. Wagner, *Klassenzahl 5, 6 und 7,* Dissertation, Freiburg i. Br., 1992.
18. D. Zagier, *L-series of elliptic curves, the Birch–Swinnerton-Dyer conjecture, and the class number problems of Gauss,* Notices Amer. Math. Soc. **31** (1984), 739-743. MR **86f:**11047

Mathematisches Institut, Albertstrasse 23b, 79104 Freiburg i. Br., Germany
Current address: Am Krayenrain 8, 4056 Basel, Switzerland

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use