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SIMPLE PERFECT SQUARED SQUARES AND 2×1 SQUARED RECTANGLES OF ORDER 26

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Abstract. In this note tables of all simple perfect squared squares and simple 2×1 perfect squared rectangles of order 26 are presented.

1. Introduction

For describing the problem of the dissection of squares and rectangles into unequal squares in a nontrivial way we use the terminology of Brooks, Smith, Stone and Tutte [6] and Bouwkamp [1, 3, 4, 2].

A dissection of a rectangle into a finite number $N > 1$ of nonoverlapping squares is called a *squared rectangle* or a *squaring* of *order N*. The N squares are called the *elements* of the dissection. The term "elements" is also used for the (length of the) sides of the elements. If all the elements are unequal, the squaring is called *perfect* and the rectangle is called a *perfect rectangle*; otherwise the squaring is *imperfect*. A squaring that contains a smaller rectangle or square dissected in squares is called *compound*. All other squared rectangles or squares are *simple*.

The lowest-order simple perfect squared square is of order 21 and was found in March 1978 [14]. The order-22 simple perfect 2×1 squared rectangle found in August 1978 appears to be of lowest order [13]. Recently, solutions of orders 22, 23 , 24 [15] and 25 were found by an exhaustive computer search [16]. A catalogue of solutions of simple perfect squared squares of orders 21 through 25, Bouwkamp codes and drawings, has been published by the university of Eindhoven, The Netherlands [7]. For squarings of order 26 we need c-nets of orders 27. Therefore, we first generated and identified c-nets of order 23, 24 , 25 and 26. These were stored on secondary storage. Those of order 27 were only generated and searched for the existence of perfect squared squares of order 26. I communicated my squared squares and 2×1 squared rectangles of orders 22, 23 , 24 and 25 to Bouwkamp. Bouwkamp then constructed 21 new solutions of order 26 from my results by means of transformation techniques [5]. Wilson found 24 particular solutions of order 26 [20] in his thesis. Skinner [9] found three solutions of order 26, Willcocks [9] found one solution of order 26 and Federico [9] found three solutions of order 26. For a historical overview of the squared-square and squared-rectangle problems, see Federico [18].

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2. Mathematical theory and computer procedures

Squared rectangles and squared squares can be obtained from so-called c-nets [6]. A c-net is a three-connected planar graph. The order of a c-net is its number of edges. The *dual* of a c-net is also a c-net. The c-nets are constructed using Tutte's theorem, known since 1947 and published in 1961 [19].

Let C_n be the set of c-nets of order n. If $s \in C_n$ is not a wheel, then at least one of the nets s and its dual s' can be constructed from $\sigma \in C_{n-1}$ by addition of an edge joining two vertices. A wheel is a c-net with an even number of edges E, with one edge of degree $E/2$ and $E/2$ vertices of degree 3. The *degree* of a vertex is the number of edges joining the vertex. Generation of c-nets of order $n+1$ out of order n gives rise to many duplicate c-nets. These can be removed using an identification method described in 1962 [10, 11] and improved in 1978 [12].

Squarings can be obtained from c-nets by considering them as electrical networks of unit resistances [6]. Basically, starting from c-nets of order n, those of order $n+1$ are generated and identified using electronic computers. Duplicates are removed, currents are calculated and simple perfect squared squares and 2×1 squared rectangles are filtered. For detailes see [10, 11].

During the period of January 7, 1993 to March 15, 1993 the order-26 squared-square solutions were calculated by means of four HP workstations connected to the university network. Their speed is 75 Mflops. The machines were only available to me during the nights and the weekends. From c-nets of order 26 those of order 27 were generated but kept in the machine and not stored on secondary storage. In all possible ways electromotive forces were placed in the branches and currents were calculated. Only squared squares and 2×1 squared rectangle solutions were stored on the disk. An album of solutions of simple perfect squared squares of orders 26, Bouwkamp codes and drawings, has been published by the university of Eindhoven, The Netherlands [8]. This paper is prepublished as Memorandum Informatica 15(1995) by the University Twente, The Netherlands [17].

3. Results

Results are presented in two tables: TableI and TableII, which are made available to you through internet in the directory "/pub/doc/dvs" of the anonymous ftp of the university Twente, The Netherlands. The internet address is: ftp.cs.utwente.nl[1].

TableI shows the Bouwkamp codes of 441 simple perfect squared squares of order 26. We found 62 doublets with side 270, 271, 274, 275, 280, 282, 286, 312, 316, 345, 357, 360, 362, 363, 364, 368, 382, 388, 392, 416, 432, 440, 444, 456, 471, 492, 505, 509, 519, 528, 537, 538, 546, 548, 562, 570, 572, 652, 657, 668, 675, 680, 684, 685, 687, 688, 689, 692, 695, 699, 707, 727, 737, 742, 757, 762,768, 769, 784, 793, 806, and 812. We found 27 triplets with side 363, 372, 396, 452, 484, 497, 502, 512, 525, 527, 540, 544, 568, 655, 700, 704, 705, 708, 713, 718, 732, 752, 755, 756, 759, 796, and 797. Furthermore, we found 10 quadruplets with side 508, 524, 532, 536, 556, 660, 728, 748, 776, and 792 and 3 quintuplets with side 696, 697, and 716 and 1 sextet with side 780 and 2 septets with side 516 and 760.

[1]Apply the command: ftp ftp.cs.utwente.nl. After the prompt >ftp you type ftp. Then enter your own internet address. After the response of our computer you give the command: cd /pub/doc/dvs. You can inspect the directory by the command ls. By means of get TableI or get TableII one can copy the tables to your own area.

length = 456 width = 456
(231,225)(111,114)(126,105)(35,58,120,3)(117)(99,27)(12,23)(1,11)(18,10)(8,32,62)(26)(2,30)(28)

length = 456 width = 456
(231,225)(111,114)(126,105)(35,58,120,3)(117)(99,27)(12,23)(1,11)(28)(30,62)(18,10)(8,2)(32)(26)

FIGURE 1. Pair of simple perfect squared squares of order 26 with the same reduced side and elements differently arranged

length = 536 width = 536

(203,154,179)(41,74,39)(14,165)(53)(8,33)(97,114)(89,18)(71)(52,45)(28,219,27)(192)(7,66)(59)(125)

length = 536 width = 536

(286,250)(36,65,149)(106,68,98,50)(21,44)(48,23)(38,30)(25,42)(152,49)(144)(12,137)(32,10)(22)(103)

length = 728 width = 728

(264,245,219)(26,193)(19,85,167)(217,66)(151)(185,175)(247,121)(11,23,141)(5,131,48,1)(12)(126)(35)(83)

length = 728 width = 728

(309,192,112,115)(80,32)(29,86)(61)(168,91,13)(4,82)(78)(77,174)(182,127)(148,97)(55,72)(271)(237)(220)

length = 752 width = 752

(323,242,187)(55,132)(216,81)(23,33,76)(188,135)(62,19)(32,10)(43)(11,21)(73)(140)(53,294,77)(241)(217)

length = 752 width = 752

(348,228,176)(52,124)(208,72)(196)(199,149)(61,87,60)(256)(50,94,40,26)(14,99)(54)(205,44)(161,31)(130)

length = 759 width = 759

(426,333)(123,210)(188,172,66)(36,87)(52,50)(2,48)(81,216)(54)(14,34)(59,161,6)(20)(145,43)(135)(102)

length = 759 width = 759

(440,319)(136,183)(177,140,62,46,15)(31,120)(16,61)(78)(73,110)(33,28)(184,37)(72,179)(147)(142,35)(107)

length = 760 width = 760

(280,183,120,177)(63,57)(83,151)(169,77)(92,68)(148,132)(219)(60,201)(16,176)(164)(6,213)(207)(4,172)(168)

length = 760 width = 760

(345,217,198)(19,42,137)(143,59,34)(11,31)(25,20)(51)(84)(38,13)(150)(190,155)(140,125)(275)(35,260)(225)

length = 797 width = 797

(307,216,274)(128,88)(30,244)(118)(190,80,37)(43,44,78)(110,12,1)(11,34)(23)(157,96)(300)(61,279)(218)

length = 797 width = 797

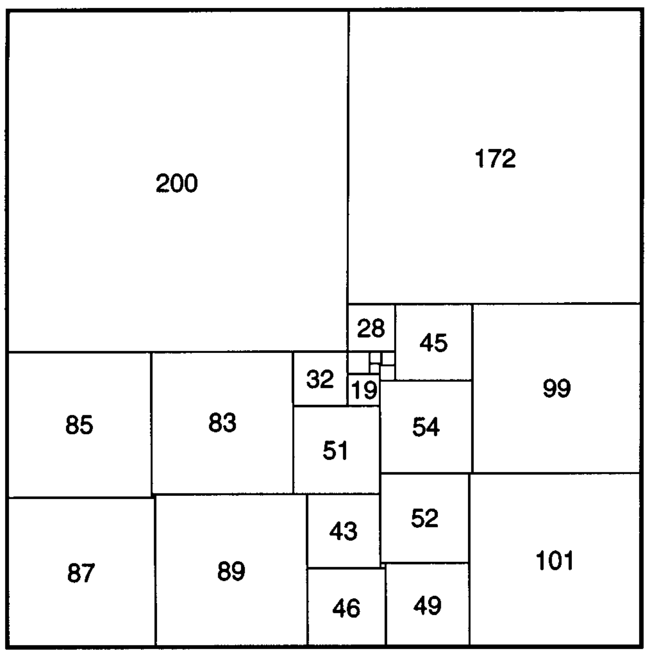
(324,212,261)(163,49)(162,148)(200,124)(47,68,48)(14,134)(26,21)(104,120)(5,84)(76,79)(273,3)(270)(254)

FIGURE 2. Pairs of simple perfect squared squares of order 26 with the same reduced side and no common elements

The smallest reduced side we found is 212, the largest is 825. TableII shows the Bouwkamp codes of 107 simple 2 ×1 perfect squared rectangles of order 26. Among the 2 ×1 squared rectangles we found 11 doublets with sides 312 × 156, 452 × 226, 464 × 232, 500 × 250, 654 × 327, 668 × 334, 704 × 352, 708 × 354, 810 × 405, 822 × 411, 864 × 432, and 2 quadruplets with sides 1122 × 561 and 792 × 396.

Figure 1 shows a pair of simple perfect squared squares with the same reduced sides and the same elements but differently arranged. There is only one example of order 26 with this property. It is not of lowest order since there exist three examples of order 25 [16]. Bouwkamp codes are also given in Figure 1.

There are 6 pairs of simple perfect squared squares with pairwise no common elements. Since no such solutions exist for orders lower than 26, they are of lowest order. Reduced sides and Bouwkamp codes are given in Figure 2.

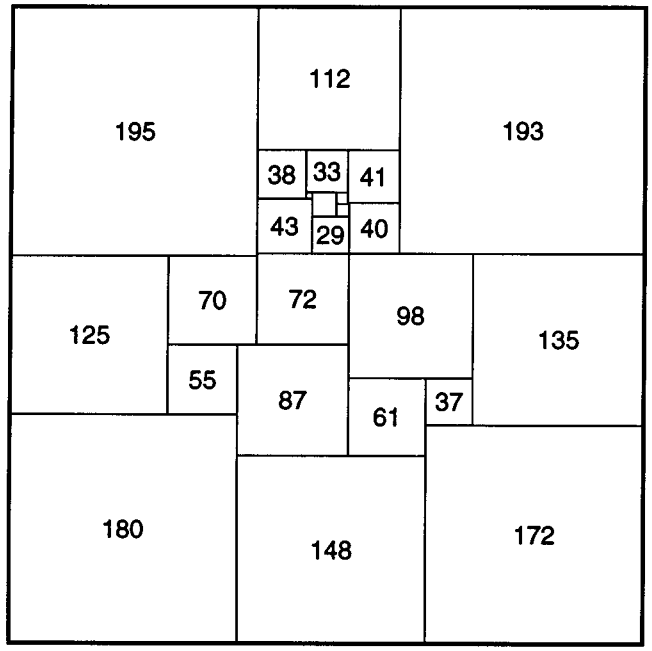


26:372

FIGURE 3. Simple perfect squared square of order 26 with a cross

Figure 3 shows a simple perfect squared square of order 26 with side 372 with a cross. Only one such example exists for order 26. Since no such solutions exist for orders lower than 26, it is of lowest order.

Figure 4 shows a compound perfect squared square of order 26 with side 500 and a cross on the perimeter of the compound rectangle. Only one such example exists for order 26. Since no such solutions exist for orders lower than 26, it is of lowest order.



26:500

FIGURE 4. Compound perfect squared square of order 26 with a cross on the periphermeter of the compound rectangle

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