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THE NUMBER OF POLYHEDRAL (3-CONNECTED PLANAR) GRAPHS

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Abstract. Data is presented on the number of 3-connected planar graphs, isomorphic to the graphs of convex polyhedra, with up to 26 edges. Results have been checked with the the number of rooted c-nets of R.C. Mullin and P.J. Schellenberg and Liu Yanpei.

1. Introduction

The set of 3-connected planar graphs is isomorphic with the set of polyhedral graphs. Tables of low-order polyhedral graphs were first given by Steiner in 1828, for references see [8]. Polyhedral graphs play an important role in the calculation of solutions of the squared rectangle and squared square problem as shown by Brooks, Smith, Stone, and Tutte [1]. They use the term c-nets for polyhedral graphs. In 1981 Duijvestijn and Federico published tables of polyhedral graphs up to order 22 and parts of order 23 and 24 [8].

A c-net is a three-connected planar graph. The order of a c-net is its number of edges. The *dual* of a c-net is also a c-net. The c-nets are constructed using Tutte's theorem, known since 1947 and published in 1961 [12].

Let C_n be the set of c-nets of order n. If $s \in C_n$ is not a wheel, then at least one of the nets s and its dual s' can be constructed from $\sigma \in C_{n-1}$ by addition of an edge joining two vertices. A wheel is a c-net with an even number of edges E, with one edge of degree $E/2$ and $E/2$ vertices of degree 3. The *degree* of a vertex is the number of edges joining the vertex. Generation of c-nets of order $n+1$ out of order n gives rise to many duplicate c-nets. These can be removed using an identification method described in 1962 [3, 4] and improved in 1978 [5]. The results presented here go as far as order 26. The number tabulated in [8] contains a printing error in the order 22 and is corrected in the new tables. Independently, Dillencourt calculated 3-connected planar graphs up to order 26. He reminded me that a discrepancy occurred in the printed data in the tables of order 22. At that time we could compare results up to order 25. Both our results were the same. Recently, we could also compare results on order 26. The number of c-nets of order 26 found by me was the same as the number found by Dillencourt[1] [10].

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2. Rooted c-nets

The concept of edge rooted graph was introduced by Tutte [13]. One edge is specified as the root and is directed by an arrow, and the two sides distinguished by labels l(left) and r(right). Since the arrow can be directed in two ways and in each case the sides can be labelled in two ways, four rooted graphs are produced from each edge, and the total number from a given graph is four times the number of edges. If the graph is symmetric, i.e., has a nontrivial automorphism, some of these will be isomorphic; the total number of distinct rooted graphs is $4E/h$ where E is the number of edges and h is the order of the automorphism group of the graph. The number of rooted c-nets can be calculated (without constructing them) by Tutte's formula, and column 3 of Tutte's table gives these figures (R) up to 25 edges as given by Tutte [13]. Table 1 in Liu Yanpei [9] gives rooted c-nets up to order 36. Mullin and Schellenberg [11] have derived an explicit formula for calculating the rooted c-nets in each Euler class. That paper includes a table giving these numbers for all classes up to order 16. Additional data have been calculated using a formula for $q^*_{m,n}$ on page 216 with the help of the language "maple". The rooted c-nets can be calculated as soon as the order of the automorphism group is known. Table 1 is given in §4.

3. Computer aspects

The motivation for the calculation of c-nets of such high orders was the calculation of squared squares and 2×1 squared rectangles [6, 7, 2]. The generation of c-nets of orders 22, 23 and 24 was done on Sun Sparc workstations in the period September 5, 1990 to December 1990. The generation of c-nets of order 25 was carried out during the Christmas vacation week 1991, using four Sun Sparc workstations of the Faculty of Computing Sciences of the University Twente connected to the university network. The generation and identification of c-nets of order 26 was completed April 1992 on four HP workstations of the Faculty of Computing Sciences of the University Twente. The speed of the machines is 75 Mflops. It took more than 600MB of disk space in compressed form to store the c-nets.

4. Results

Results are tabulated by number of edges and the number of vertices. The number of c-nets as well as the number of rooted c-nets are given in Table 1, where NS means not selfdual and S means selfdual, V stands for number of vertices. Auto means order of the automorphism group. Since the dual of a c-net is a c-net, all c-nets can be obtained from Table 1. The result has been verified by means of the formula on the number of rooted c-nets by Mullin and Schellenberg and Liu Yanpei.

TABLE 1. Number of c-nets and rooted c-nets of orders 22 to 26

Order 22			C-nets	Rooted C-nets
V=10	NS	auto=1	4052	356576
V=10	NS	auto=2	365	16060
V=10	NS	auto=4	24	528
V=10	NS	auto=8	1	11
			4442	373175
V=11	NS	auto=1	102524	9022112
V=11	NS	auto=2	1663	73172
V=11	NS	auto=4	26	572
			104213	9095856
V=12	NS	auto=1	131718	11591184
V=12	NS	auto=2	1486	65384
V=12	NS	auto=4	36	792
V=12	NS	auto=8	3	33
			133243	11657393
V=12	S	auto=1	1817	150896
V=12	S	auto=2	80	3520
V=12	S	auto=4	10	220
V=12	S	auto=22	1	4
			1908	163640

Order 23		C-nets	Rooted C-nets
V=10	auto=1	1235	113620
V=10	auto=2	156	7176
V=10	auto=4	13	299
		1404	121095
V=11	auto=1	110015	10121380
V=11	auto=2	2023	93058
V=11	auto=4	44	1012
		112082	10215450
V=12	auto=1	704267	64792564
V=12	auto=2	4977	228942
V=12	auto=4	58	1334
		709302	65022840

Order 24			C-nets	Rooted C-nets
V=10		auto=1	137	13152
		auto=2	69	3312
		auto=3	1	32
		auto=4	13	312
		auto=6	6	96
		auto=8	4	48
		auto=16	1	6
		auto=24	1	4
		auto=32	1	3
			233	16965
V=11		auto=1	78169	7504224
		auto=2	1559	74832
		auto=3	5	160
		auto=4	25	600
		auto=6	13	208
		auto=12	2	16
			79773	7580040
V=12		auto=1	1255238	120502848
		auto=2	7631	366288
		auto=3	14	488
		auto=4	123	2952
		auto=6	12	192
		auto=8	5	60
		auto=12	7	56
		auto=24	1	4
		auto=48	1	2
			1263032	120872850
V=13	NS	auto=1	1460152	140174592
	NS	auto=2	5199	249552
	NS	auto=3	15	480
	NS	auto=4	27	648
	NS	auto=6	18	288
	NS	auto=8	3	36
			1465414	140425596
V=13	S	auto=1	6490	623040
	S	auto=2	144	6912
	S	auto=3	12	384
	S	auto=4	7	168
	S	auto=6	8	128
	S	auto=8	3	36
	S	auto=12	2	16
	S	auto=24	1	4
			6667	630688

TABLE 1. (continued)

Order 25		C-nets	Rooted C-nets
V=11	auto=1	35199	3519900
	auto=2	1287	64350
	auto=4	40	1000
	auto=10	2	20
		36528	3585270
V=12	auto=1	1548735	154873500
	auto=2	8141	407050
	auto=4	73	1825
	auto=10	2	20
	auto=20	1	5
		1556952	155282400
V=13	auto=1	8065553	806555300
	auto=2	20041	1002050
	auto=4	131	3275
		8085725	807560625

Order 26			C-nets	Rooted C-nets
V=11	NS	auto=1	9176	954304
	NS	auto=2	516	26832
	NS	auto=4	22	572
			9714	981708
V=12	NS	auto=1	1329899	138309496
	NS	auto=2	8774	456248
	NS	auto=4	171	4446
	NS	auto=8	9	117
			1338853	138770307
V=13	NS	auto=1	15507471	1612776984
	NS	auto=2	27975	1454700
	NS	auto=4	126	3276
			15535572	1614234960
V=14	NS	auto=1	16620453	1728527112
	NS	auto=2	22889	1190228
	NS	auto=4	152	3952
	NS	auto=8	3	39
			16643497	1729721331
V=14	S	auto=1	23199	2412696
	S	auto=2	343	17836
	S	auto=4	13	338
	S	auto=26	1	4
			23556	2430874

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