MATHEMATICS OF COMPUTATION
Volume 65, Number 216
October 1996, Pages 1749–1754

ON SEQUENCES WITHOUT GEOMETRIC PROGRESSIONS

BRIENNE E. BROWN AND DANIEL M. GORDON

ABSTRACT. Several papers have investigated sequences which have no k-term arithmetic progressions, finding bounds on their density and looking at sequences generated by greedy algorithms. Rankin in 1960 suggested looking at sequences without k-term geometric progressions, and constructed such sequences for each k with positive density. In this paper we improve on Rankin's results, derive upper bounds, and look at sequences generated by a greedy algorithm.

1. Introduction

Erdős and Turan [1] defined $r_k(n)$ to be the least r for which any sequence of r numbers less than n must contain a k-term arithmetic progression. Roth [7] showed that $r_3(n) = O(n/\log\log n)$, and Szemerédi [8] showed that $r_k(n) = o(n)$ for all k.

We will denote all sets of nonnegative integers without a k-term arithmetic progression by APF_k (for arithmetic progression-free). Erdős conjectured that the sum of reciprocals of the (nonzero) terms of any such sequence converge, and offered $3,000 for a proof or disproof.

One way to generate an arithmetic progression-free sequence is to use a greedy algorithm: start with 0, and add the smallest number which does not form a k-term arithmetic progression. Variations on the resulting sequences have been studied by several people [2, 3, 5]. For prime k, greedy sequences are just the integers whose base-k representation has no digits equal to $k-1$. For composite k their behavior is still mysterious.

In [4], the *span* of a set is defined to be the difference of its largest and smallest elements, and $\mathrm{sp}(k,n)$ to be the smallest span of a set in APF_k with n members, and a table of values for $\mathrm{sp}(k,n)$ for small k and n due to Usiskin is given. The value given for $\mathrm{sp}(3,10)$ in that table is wrong; Table 1 corrects it and gives $\mathrm{sp}(k,n)$ for a larger range of k and n.

The corresponding questions for sequences with no geometric progressions have received little attention. Rankin [6] used sequences in APF_k to form sequences with no k-term geometric progressions, and found their density. In §2 we review his methods, and show how sequences coming from a greedy method are superior to his for $k > 3$. In §3 we derive upper bounds for the density of such sequences.

Throughout this paper, A will denote an arbitrary sequence of nonnegative integers, A_k will be an arbitary sequence in APF_k, and A_k^* will be the greedy sequence described above.

Received by the editor May 1, 1995 and, in revised form, August 15, 1995.
1991 *Mathematics Subject Classification.* Primary 11B05; Secondary 11B83.

Table 1. Smallest span for APF_k

$k\backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	3	4	8	10	12	13	19	23	25	29	31	35	39	40	50
4		4	5	7	8	9	12	14	16	18	21	22	24	26	27
5			5	6	7	8	10	11	12	13	15	16	17	18	23
6				6	7	8	9	11	12	13	14	16	17	18	19

2. Geometric progression-free sequences

Let GPF_k denote all sets of positive integers with no k-term geometric progressions. The only previous consideration of geometric progression-free sequences we know of is by Rankin [6]. An obvious sequence in GPF_3 is the set of squarefree numbers, which have density $6/\pi^2 \approx 0.608$.

Rankin showed that sequences in APF_k can be used to form denser sequences in GPF_k:

For a nonnegative sequence of integers $A = \{a_1, a_2, \dots\}$, let $G(A)$ be the set of all integers

$$N = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, \tag{1}$$

where the p_i are distinct primes, r is any nonnegative integer, and $e_i \in A$ for $i = 1, \dots, r$.

Theorem 1. *If A is in APF_k, then $G(A)$ is in GPF_k.*

Proof. Let $\{a, as, as^2, \dots, as^{k-1}\}$ be any set of integers in a geometric progression. (Note that, while $a \in \mathbb{Z}$, s may be a rational noninteger, e.g. the progression 9,12,16). Any prime dividing the numerator or denominator of s occurs to powers $c, c+d, c+2d, \dots, c+(k-1)d$, for some $c \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$. These powers form a k-term arithmetic progression, which cannot be contained in A, and so the numbers in the geometric progression cannot all be in $G(A)$. $\square$

Let G_k^* be the set in GPF_k generated by the greedy algorithm; $g_1 = 1$, and g_i is the smallest integer which does not form a k-term geometric progression with $g_1, \dots, g_{i-1}$.

Theorem 2. *We have $G_k^* = G(A_k^*)$.*

Proof. Let m be the smallest number in G_k^* which is not in $G(A_k^*)$. We will show that m is in a geometric progression with $k-1$ numbers in $G(A_k^*)$. This contradicts the definition of G_k^*, since G_k^* is equal to $G(A_k^*)$ up to m, proving that no such m exists.

Let $m = \prod_j p_j^{e_j} \prod_l q_l^{f_l}$, where the e_j are in A_k^*, and the f_l are not. Then for each f_l, there is an arithmetic progression $\{f_{l,1}, f_{l,2}, \dots, f_{l,k} = f_l\}$ with $f_{l,1}, \dots, f_{l,k-1} \in$

A_k^*. Then

$$N_1 = \prod_j p_j^{e_j} \prod_l q_l^{f_{l,1}},$$
$$N_2 = \prod_j p_j^{e_j} \prod_l q_l^{f_{l,2}},$$
$$\vdots$$
$$N_{k-1} = \prod_j p_j^{e_j} \prod_l q_l^{f_{l,k-1}},$$

together with m would form a geometric progression. All of $N_1, \dots, N_{k-1}$ are less than m and in $G(A_k^*)$, and so are in G_k^*. They form an arithmetic progression with m, contradicting $m \in G_k^*$. □

Rankin also gave a method to compute the density of a sequence $G(A) \in \mathrm{GPF}_k$ of the form (1). The Dirichlet series

$$f_{G(A)}(s) = \sum_{n \in G} n^{-s}$$

has the Euler product

$$f_{G(A)}(s) = \prod_p F_A(p^{-s}),$$

where, for $|x| < 1$,

(2) $$F_A(x) = \sum_{q \in A} x^q.$$

When k is prime, $A = A_k^*$ consists of numbers with no digits equal to $k-1$ base k, and (2) becomes

$$F_{A_k^*}(x) = \prod_{v=0}^{\infty} \left(1 + x^{k^v} + x^{2k^v} + \cdots + x^{(k-2)k^v}\right)$$
$$= \prod_{v=0}^{\infty} \frac{1 - x^{(k-1)k^v}}{1 - x^{k^v}},$$

which implies

(3) $$f_{G_k^*}(s) = \prod_{v=0}^{\infty} \frac{\zeta(k^v s)}{\zeta((k-1)k^v s)}.$$

The asymptotic density of G equals the residue at $s = 1$ of $f_G(s)$. For $G = G_3^*$, this is 0.7197 (Rankin gave the same sequence). Even for composite k, where there is no known closed form for $f_{G_k^*}(s)$, we may still compute the residue to any desired precision. For example, for $k = 4$, $A_4^* = \{0, 1, 2, 4, 5, \dots\}$, and

$$f_{G_4^*}(s) = \prod_p \left(1 + p^{-s} + p^{-2s} + p^{-4s} + \cdots\right)$$
$$= \zeta(s) \prod_p \left(1 - p^{-3s} + p^{-4s} - p^{-6s} + \cdots\right),$$

which has residue ≈ 0.895.

This is better than the density 0.8626 GPF_4 sequence Rankin found. In fact, we can show that the greedy sequence is the best of the form (1):

Theorem 3. *If $G = G(A_k)$ for $k \geq 3$ and some* APF_k *sequence A_k, then its density is no greater than the greedy sequence.*

Proof. Any sequence $G = G(A)$ has a Dirichlet series of the form

$$f_G(s) = \prod_p \left(a_0 + a_1 p^{-s} + a_2 p^{-2s} + \cdots\right), \tag{4}$$

where $a_i = 1$ if $i \in A$, and $a_i = 0$ otherwise. As stated above, the residue at $s = 1$ of this function gives the density of the corresponding sequence.

Suppose there is another sequence A' for which $G' = G(A')$ has density greater than the greedy sequence $G(A)$. Let a'_i be the coefficients for the Dirichlet series $f_{G'}(s)$. The density of G' is greater than G if and only if the residue of $f_{G'}(s)$ at $s = 1$ is greater than the residue of $f_G(s)$.

At some point A' diverges from the greedy sequence, and we have $a_i = 1$ and $a'_i = 0$ for some i. Let H be the greedy sequence truncated at i, and H' be the same sequence with i removed and containing all $j > i$. Then H has density less than G and H' has density greater than G', so it suffices to show that

$$f_H(s) = \prod_p \left(a_0 + a_1 p^{-s} + \cdots + a_{i-1} p^{-(i-1)s} + p^{-is}\right) \tag{5}$$

has a larger residue at $s = 1$ than

$$\begin{aligned} f_{H'}(s) &= \prod_p \left(a_0 + \cdots + a_{i-1} p^{-(i-1)s} + p^{i+1)s} + p^{-(i+2)s} + \cdots\right) \\ &= \prod_p \left(a_0 + \cdots + a_{i-1} p^{-(i-1)s} + \frac{p^{-(i+1)s}}{1 - p^{-s}}\right). \end{aligned} \tag{6}$$

This is equivalent to showing that

$$\lim_{s \to 1} \frac{f_H(s)}{f_{H'}(s)} > 1.$$

But this is obvious, since for $p = 2$ the terms in (5) and (6) are equal at $s = 1$, and for all $p > 2$ and $s \geq 1$ the term in (5) is larger. $\square$

This leaves open the question of whether geometric progression-free sequences not of the form (1) have better density than greedy sequences. They can certainly do better over finite ranges; the greedy GPF_3 sequence:

1	2	3	5	6	7	8	10	11	13
14	15	16	17	19	21	22	23	24	26
27	29	30	31	33	34	35	37	38	39
40	41	42	43	46					

may be improved by removing 5 and adding 25 and 45.

3. Upper bounds

It is easy to show that the density of a GPF_k sequence is strictly less than one:

Theorem 4. *For any $k \geq 3$, the density of a sequence in GPF_k is at most $1-2^{-k}$.*

Proof. For any N, let a be an odd number less than $N/2^{k-1}$. Then the k numbers $a, 2a, 4a, \ldots, 2^{k-1}a$ cannot all appear in a GPF_k sequence. There are $N/2^k$ different a's, so this excludes $N/2^k$ numbers less than N from the sequence. □

Theorem 4 can be improved slightly:

Theorem 5. *For any $k \geq 3$, the density of a sequence in GPF_k is at most*

$$1 - 2^{-k} - \frac{5^{-(k-1)} - 6^{-(k-1)}}{2}.$$

Proof. Let b be an odd number, $N/6^{k-1} < b < N/5^{k-1}$. Then the numbers $3^{k-1}b, 3^{k-2}5b, \ldots, 5^{k-1}b$ cannot all appear in the sequence. There are $N/(2\cdot 5^{k-1}) - N/(2\cdot 6^{k-1})$ such b's, and none of them are the numbers $a, 2a, \ldots, 2^{k-1}a$ from Theorem 4, since they are all odd, and $3^{k-1}b > a$ for a and b in the ranges chosen. Moreover, since $6^{k-1}/5^{k-1} < 5/3$, the numbers $3^{k-1}b, 3^{k-2}5b, \ldots, 5^{k-1}b$ are distinct for different b in the range. □

Table 2. Densities for geometric progression-free sequences

k	greedy density	upper bound
3	0.71974	0.868889
4	0.89537	0.935815
5	0.95805	0.968336
6	0.98085	0.984279
7	0.99116	0.992166

The bounds can be further improved by taking fractions of larger primes over smaller ranges, but the improvements become marginal very quickly.

Table 2 gives the best known upper and lower bounds for the density of sequences in GPF_k for $k \leq 7$. For $k = 3$ and 4 they are still far apart, but as k gets large they approach each other.

Theorem 6. *As $k \to \infty$, the optimal density for a sequence in GPF_k is $1 - 2^{-k}(1 - o(1))$.*

Proof. From Theorem 4, we have that the density is no greater than $1 - 2^{-k}$. Therefore, it suffices to show that the greedy sequence $G(A_k)$ has the stated density.

It is easy to see that the greedy APF_k sequence A_k starts off

$$\{0, 1, \ldots, k-2, k, k+1, \ldots, 2k-3, 2k-1\}$$

for k even and

$$\{0, 1, \ldots, k-2, k, k+1, \ldots, 2k-2, 2k\}$$

for $k > 3$ odd. For simplicity, we will handle the odd case (the even case is virtually identical). The density of $G(A_k)$ is the residue at $s = 1$ of

$$\prod_p \left(1 + p^{-s} + \cdots + p^{-(k-2)s} + p^{-ks} + \cdots + p^{-(2k-2)s} + p^{-2ks} + \cdots\right)$$
$$= \prod_p \frac{1}{1 - p^{-s}} \left(1 - p^{-(k-1)s} + p^{-ks} - p^{-(2k-1)s} + \cdots\right)$$
$$= \zeta(s) \prod_p \left(1 - p^{-(k-1)s} + p^{-ks} - p^{-(2k-1)s} + \cdots\right).$$

The residue of $\zeta(s)$ is one, so the density is

$$\prod_p \left(1 - p^{-(k-1)} + p^{-k} - p^{-(2k-1)} + \cdots\right)$$
$$\geq (1 - 2^{-k} - 2^{-(2k-1)}) \prod_{p>2} \left(1 - p^{-(k-1)}\right)$$
$$= \frac{1 - 2^{-k} - 2^{-(2k-1)}}{(1 - 2^{-(k-1)})\zeta(k-1)}.$$

For large k, we have $\zeta(k-1) \to 1 + 2^{-(k-1)}$, and the density becomes

$$1 - 2^{-k}(1 - o(1)). \quad \square$$

Acknowledgment

We would like to thank Carl Pomerance for suggesting Theorem 6.

References

1. P. Erdős and P. Turán, *On some sequences of integers*, J. London Math. Soc. **11** (1936), 261–264.
2. Joeseph L. Gerver and L. Thomas Ramsey, *Sets of integers with nonlong arithmetic progressions generated by the greedy algorithm*, Math. Comp. **33** (1979), 1353–1359. MR **80k:**10053
3. Joseph Gerver, James Propp, and Jamie Simpson, *Greedily partitioning the natural numbers into sets free of arithmetic progressions*, Proc. Amer. Math. Soc. **102** (1988), 765–772. MR **89f:**11026
4. Richard K. Guy, *Unsolved problems in number theory*, second ed., Springer–Verlag, 1994. CMP 95:02
5. A. M. Odlyzko and R. P. Stanley, *Some curious sequences constructed with the greedy algorithm*, Bell Labs internal memo, 1978.
6. R. A. Rankin, *Sets of integers containing not more than a given number of terms in arithmetical progression*, Proc. Roy. Soc. Edinburgh Sect. A **65** (1960/61), 332–344. MR **26:**95
7. K. F. Roth, *On certain sets of integers*, J. London Math. Soc. **28** (1953), 104–109. MR **14:**536g
8. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245. MR **51:**5547

9211 Mintwood Street, Silver Spring, Maryland 20901

Center for Communications Research, 4320 Westerra Court, San Diego, California 92121

E-mail address: gordon@ccrwest.org