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ULTRACONVERGENCE OF THE PATCH RECOVERY TECHNIQUE

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Abstract. The ultraconvergence property of a derivative recovery technique recently proposed by Zienkiewicz and Zhu is analyzed for two-point boundary value problems. Under certain regularity assumptions on the exact solution, it is shown that the convergence rate of the recovered derivative at an internal nodal point is two orders higher than the optimal global convergence rate when even-order finite element spaces and local uniform meshes are used.

1. Introduction

In the finite element history, there have been many investigations on derivative recovery techniques. For the literature, the reader is referred to [2] and references therein. Recently, Zienkiewicz and Zhu proposed a patch recovery procedure in which the recovered derivative is obtained by discrete least squares fitting on an element patch. What distinguishes this new recovery technique from the others is its practical effectiveness. The recovered derivative at an internal nodal point is simply the weighted average at some Gauss points in the adjacent elements. The computational cost for this postprocessing is almost free.

A surprising observation from numerical tests is that we can actually get an ultraconvergence (two orders higher) result based on the superconvergence (one order higher than usual) data. An $O(h^4)$ convergence rate has been reported for the recovered derivative at the internal nodal points when quadratic elements and uniform meshes are used. It has been conjectured that ultraconvergence will occur for any even-order finite element space (quadratic element has order 2). The current work is devoted to the theoretical justification of this astonishing phenomenon. We shall prove, for a certain class of two-point boundary value problems, that the recovery procedure will result in ultraconvergence internal nodal recovery when local uniform meshes and even-order elements are used.

2. The patch recovery technique

Consider the following two-point boundary value problem:

$$-(a_2(x)u')' - (a_1(x)u)' + a_0(x)u = f \quad \text{in } I = (0,1), \tag{2.1}$$

$$u(0) = u(1) = 0. \tag{2.2}$$

We assume that a_i and f are sufficiently smooth for our analysis. We also assume that $a_2(x) \geq \alpha > 0$ for all $x \in \bar{I}$.

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The weak formulation of (2.1), (2.2) is to find $u \in H_0^1(I)$ such that

$$(a_2u', v') + (a_1u, v') + (a_0u, v) = (f, v), \quad \forall v \in H_0^1(I). \tag{2.3}$$

Let $\mathcal{T}_h$, $0 < h < 1/2$, be a sequence of subdivisions of $\bar{I}$,

$$\mathcal{T}_h = \{x_i\}_{i=0}^N, \quad 0 = x_0 < x_1 < \cdots < x_N = 1;$$

denote $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$, $h = \max_i h_i$, and set

$$S_h = \{v \in H^1(I), \quad v|_{I_i} \in P_r(I_i)\}, \quad S_h^0 = \{v \in H_0^1(I), \quad v|_{I_i} \in P_r(I_i)\}.$$

We see that S_h and S_h^0 are the spaces of continuous piecewise polynomials of degree not exceeding r on I under the subdivision $\mathcal{T}_h$.

The finite element solution of (2.3) is to find $u_h \in S_h^0$ such that

$$(a_2u_h', v') + (a_1u_h, v') + (a_0u_h, v) = (f, v), \quad \forall v \in S_h^0. \tag{2.4}$$

In order to define the recovered derivative, we introduce the Gauss points and the Lobatto points.

Let $L_r(x)$ be the Legendre polynomial of degree r on $[-1, 1]$. It is well known that $L_r(x)$ has r zeros and $L_r'(x)$ has $r-1$ zeros in $(-1, 1)$. Denote by $g_1^{(r)}, \ldots, g_r^{(r)}$ the zeros of $L_r(x)$, and by $l_1^{(r)}, \ldots, l_{r-1}^{(r)}$ the zeros of $L_r'(x)$ with $l_0^{(r)} = -1$, $l_r^{(r)} = 1$; then $g_j^{(r)}$, $j = 1, \ldots, r$, are called the Gauss points of order r, and $l_j^{(r)}$, $j = 0, 1, \ldots, r$, the Lobatto points of order r.

The Gauss and Lobatto points on I_i are defined as the affine transformations of $g_j^{(r)}$ and $l_j^{(r)}$ to I_i, respectively:

$$G_{ij} = \frac{1}{2}(x_{i-1} + x_i + h_i g_j^{(r)}), \quad j = 1, \ldots, r,$$

$$L_{ij} = \frac{1}{2}(x_{i-1} + x_i + h_i l_j^{(r)}), \quad j = 0, 1, \ldots, r.$$

Here the index r on G_{ij} and L_{ij} is dropped in order to simplify the notation.

In general, u_h' is a piecewise polynomial of degree $r-1$ and is discontinuous at the nodal points x_i, $1 \le i \le N-1$. The recovered derivative by the patch recovery is a continuous piecewise polynomial of degree r (as u_h), $Ru_h' \in S_h$, which is uniquely determined by its values at the Lobatto points. The values of the recovered derivative at the Lobatto points are obtained by the following least squares fitting procedure. On the element patch

$$J_i = I_i \cup \{x_i\} \cup I_{i+1},$$

consider a polynomial of degree r,

$$p_r^*(x) = (1, x, \ldots, x^r)\boldsymbol{a}.$$

The vector $\boldsymbol{a} = (a_0, a_1, \ldots, a_r)^T$ is computed by fitting, in the least squares sense, u_h' at $2r$ Gauss points $\{G_{ij}, G_{i+1,j}\}_{j=1}^r$ in J_i, $i = 1, \ldots, N-1$. Then the values of Ru_h' at the Lobatto points are the values of p_r^* at the same points, i.e.,

$$Ru_h'(L_{ij}) = p_r^*(L_{ij}), \quad j = 1, \ldots, r; \tag{2.5}$$

$$Ru_h'(L_{i+1,j}) = p_r^*(L_{i+1,j}), \quad j = 0, \ldots, r-1. \tag{2.6}$$

Note that there is an overlapping of adjacent element patches, i.e., $J_{i-1} \cap J_i = I_i$, $1 \le i \le N-1$. If different patches result in different recoveries on I_i, an averaging is applied (see [3] for more details). But if the exact solution is a polynomial of degree $r+1$, we shall show that the two recoveries from adjacent patches are the same.

3. Ultraconvergence analysis

The first step of our analysis is to reduce (2.4) to a simpler problem. Subtracting (2.4) from (2.3) yields

$$(a_2(u'-u_h'),v')+(a_1(u-u_h),v')+(a_0(u-u_h),v)=0, \quad \forall v\in S_h^0. \tag{3.1}$$

Let $\tilde{u}_h\in S_h^0$ be given by

$$(u'-\tilde{u}_h',v')=0, \quad \forall v\in S_h^0. \tag{3.2}$$

Then we have the following "superapproximation" and "ultra-approximation" results between u_h and $\tilde{u}_h$ (see [2, Theorem 1.3.1 and Remark 1.3.1]):

Lemma 3.1. *Let u_h, $\tilde{u}_h$ satisfy* (3.1), (3.2), *respectively. Then there exists a constant C, independent of h and u, such that*

$$\|u_h'-\tilde{u}_h'\|_{L_\infty(I)}\le Ch^{r+1}\|u\|_{W_\infty^{r+1}(I)}. \tag{3.3}$$

For the special case when $r\ge 2$, $a_2=1$, and $a_1=0$, we have

$$\|u_h'-\tilde{u}_h'\|_{L_\infty(I)}\le Ch^{r+2}\|u\|_{W_\infty^{r+1}(I)}. \tag{3.4}$$

By virtue of Lemma 3.1, we can reduce our discussion to a simple case:

$$-u''=f \quad \text{in } I=(0,1), \quad u(0)=u(1)=0;$$

or

$$(u',v')=(f,v), \quad \forall v\in H_0^1(I), \tag{3.5}$$

since the finite element solution of (3.5) satisfies (3.2).

In the following, we shall construct the finite element solution $u_h\in S_h^0$ for (3.5) and prove superconvergence and ultraconvergence properties of the recovered derivative.

We characterize S_h^0 by the following basis functions (cf. [1, p. 38]):

$$S_h^0=\mathrm{Span}\{N_i(x), i=1,2,\dots,N-1;\quad \phi_{jk}(x), j=1,2,\dots,N, k=2,3,\dots,r\}.$$

Here,

$$N_i(x)=\begin{cases}1+(x-x_i)/h_i, & x\in I_i,\\ 1+(x_i-x)/h_{i+1}, & x\in I_{i+1},\\ 0, & \text{otherwise}\end{cases}$$

is the usual finite element "tent" basis function, ϕ_{jk} is a "bubble" function with support on I_j and its value on I_j is defined as follows:

$$\phi_{jk}(x)=\phi_{jk}(x_j-\frac{1-\xi}{2}h_j)=\phi_k(\xi), \quad \xi\in(-1,1),$$

where

$$\phi_k(\xi)=\sqrt{\frac{2k-1}{2}}\int_{-1}^{\xi}L_{k-1}(t)dt,$$

and L_{k-1} is the Legendre polynomial of degree $k-1$. Observe that

$$\phi_k(-1) = \phi_k(1) = 0, \quad \int_{-1}^{1} \phi_k'(\xi) d\xi = 0, \quad \int_{-1}^{1} \phi_k'(\xi)\phi_l'(\xi) d\xi = 0, \quad k \neq l.$$

We then have,

$$\phi_{jk}(x_{j-1}) = \phi_{jk}(x_j) = 0, \quad \int_0^1 N_i'(x)\phi_{jk}'(x) dx = 0,$$

$$\int_0^1 \phi_{jk}'(x)\phi_{il}'(x) dx = 0, \quad i \neq j \text{ or } k \neq l.$$

These orthogonality properties greatly simplify our analysis. We are able to express explicitly the finite element solution of (3.5) on I_i as

$$u_h(x) = u(x_{i-1})N_{i-1}(x) + u(x_i)N_i(x) + \sum_{k=2}^{r} c_{ik}\phi_{ik}(x), \tag{3.6}$$

where

$$c_{ik} = (f, \phi_{ik})/(\phi_{ik}', \phi_{ik}').$$

Theorem 3.1. *Let u be the solution of* (3.5), *and let u_h be its finite element approximation on S_h^0. Assume that u is a polynomial of degree not greater than $r+1$ on an element patch $J_i = (x_{i-1}, x_{i+1})$. Then $Ru_h' = u'$ on J_i.*

Proof. From (3.6), we have, on I_i,

$$u_h'(x) = c_{i1} + \sum_{k=2}^{r} c_{ik}\phi_{ik}'(x), \tag{3.7}$$

where

$$c_{i1} = (u', \chi_{I_i})/(\chi_{I_i}, \chi_{I_i}) = \frac{u(x_i) - u(x_{i-1})}{h_i},$$

and χ_{I_i} is the characteristic function of I_i. By the definition of ϕ_{ik}, we see that

$$\text{Span}\{\chi_{I_i}(x), \phi_{ik}'(x), k = 2, \ldots, r+1\} = P_r(I_i).$$

When $u \in P_{r+1}(J_i)$, we have $u' \in P_r(I_i)$, and therefore

$$\begin{aligned} u'(x) &= c_{i1} + \sum_{k=2}^{r+1} \frac{(u', \phi_{ik}')}{(\phi_{ik}', \phi_{ik}')}\phi_{ik}' \\ &= c_{i1} + \sum_{k=2}^{r+1} \frac{(-u'', \phi_{ik})}{(\phi_{ik}', \phi_{ik}')}\phi_{ik}' = u_h'(x) + c_{i,r+1}\phi_{i,r+1}'(x). \end{aligned} \tag{3.8}$$

Note that $\phi_{i,r+1}'(x)$ is linearly dependent on the rth-degree Legendre polynomial on I_i; therefore, it vanishes at the r Gauss points g_{ik} of I_i; i.e., $\phi_{i,r+1}'(g_{ik}) = 0$, $k = 1, \ldots, r$. Hence,

$$u_h'(g_{ik}) = u'(g_{ik}), \quad k = 1, \ldots, r. \tag{3.9}$$

Applying the same argument on I_{i+1}, we have

$$u_h'(g_{i+1,k}) = u'(g_{i+1,k}), \quad k = 1, \ldots, r. \tag{3.10}$$

Recall that Ru_h' is a polynomial of degree r on J_i and fits u', a polynomial of the same degree, in a least squares sense at the $2r$ ($r \geq 1$) Gauss points on the element patch J_i (since u_h' equals u' at these points). Therefore, $Ru_h' = u'$ on J_i. □

A direct consequence of Theorem 3.1 is the following superconvergence property.

Theorem 3.2. *Let u be the solution of* (2.3)*, and let u_h be its finite element approximation on S_h^0. Then there exists a constant C, independent of h and u, such that*

$$|u'(x_i) - Ru_h'(x_i)| \leq Ch^{r+1}(|u|_{W_\infty^{r+2}(J_i)} + \|u\|_{W_\infty^{r+1}(I)}). \tag{3.11}$$

For the special case $a_2 = 1$ and $a_1 = a_0 = 0$, we have

$$|u'(x_i) - Ru_h'(x_i)| \leq Ch^{r+1}|u|_{W_\infty^{r+2}(J_i)}. \tag{3.12}$$

Proof. The proof of (3.12) follows from Theorem 3.1 and the standard argument by applying the Bramble-Hilbert Lemma. The proof of (3.11) follows from Lemma 3.1 and (3.12) for the special case. □

Based on Theorem 3.1, we can further prove the ultraconvergence result.

Theorem 3.3. *Let u be the solution of* (2.3) *when $a_2 = 1$ and $a_1 = 0$, and let u_h be its finite element approximation on S_h^0 with r (≥ 2) an even number. If the two elements on the element patch J_i have the same length, i.e., $h_i = h_{i+1}$, then there exists a constant C, independent of h and u, such that*

$$|u'(x_i) - Ru_h'(x_i)| \leq Ch^{r+2}(|u|_{W_\infty^{r+3}(J_i)} + \|u\|_{W_\infty^{r+1}(I)}). \tag{3.13}$$

Assuming further that $a_0 = 0$, we have

$$|u'(x_i) - Ru_h'(x_i)| \leq Ch^{r+2}|u|_{W_\infty^{r+3}(J_i)}. \tag{3.14}$$

Proof. We first prove (3.14). Associated with any interior node x_i, $i = 1, \ldots, N-1$, there is an element patch $J_i = (x_i - h_i, x_i + h_i)$ (recall that $h_i = h_{i+1}$), and a linear mapping F_i from $\hat{I} = (-1, 1)$ onto J_i defined by $x = x_i + h_i\xi$. Given any function v on J_i, we define

$$\hat{v} = v \circ F_i, \quad \text{or } \hat{v}(\xi) = v(F_i(\xi)) = v(x_i + h_i\xi).$$

Now, consider

$$u'(x_i) - Ru_h'(x_i) = \langle u' - Ru_h', \delta_i\rangle = h_i\langle \widehat{u' - Ru_h'}, \hat{\delta}_i\rangle = h_i E(\hat{u}'). \tag{3.15}$$

Here, $\hat{\delta}_i = \delta \circ F_i$ and δ is the discrete delta function. Obviously, $E(\hat{u}')$ is a linear functional which is bounded in $W_\infty^{r+2}(\hat{I})$. We shall show that $E(\hat{u}')$ vanishes when $\hat{u}'$ is a polynomial of degree not greater than $r + 1$.

Let $r = 2s$; we examine the case when

$$u(x) = a(\frac{x - x_{i-1}}{h_i})^{s+1}(\frac{x_{i+1} - x}{h_i})^{s+1}, \quad a \neq 0, \tag{3.16}$$

on J_i. Note that $u'(x_i) = 0$, and $u(x)$ is symmetric with respect to x_i on J_i (so is $u''(x)$). By definition, ϕ_{ik} and $\phi_{i+1,k}$ are symmetric (antisymmetric) with respect to x_i when k is even (odd). Therefore,

$$c_{ik} = (-u'', \phi_{ik})/(\phi_{ik}', \phi_{ik}') = (-u'', \phi_{i+1,k})/(\phi_{i+1,k}', \phi_{i+1,k}') = c_{i+1,k}, \quad k = 2l;$$

$$c_{ik} = -c_{i+1,k}, \quad k = 2l - 1.$$

Recalling (3.6), we have, on J_i,

$$u_h(x) = aN_i(x) + \begin{cases} \sum_{l=1}^{s} c_{i,2l}\phi_{i,2l}(x) + \sum_{l=1}^{s} c_{i,2l-1}\phi_{i,2l-1}(x), & x \in I_i \\ \sum_{l=1}^{s} c_{i,2l}\phi_{i+1,2l}(x) - \sum_{l=1}^{s} c_{i,2l-1}\phi_{i+1,2l-1}(x), & x \in I_{i+1}, \end{cases}$$

$$u_h'(x) = \begin{cases} a/h_i + \sum_{l=1}^{s} c_{i,2l}\phi_{i,2l}'(x) + \sum_{l=1}^{s} c_{i,2l-1}\phi_{i,2l-1}'(x), & x \in I_i \\ -a/h_i + \sum_{l=1}^{s} c_{i,2l}\phi_{i+1,2l}'(x) - \sum_{l=1}^{s} c_{i,2l-1}\phi_{i+1,2l-1}'(x), & x \in I_{i+1}. \end{cases}$$

Observe that

$$\phi_{i,2l}'(x_i - \tau) = -\phi_{i+1,2l}'(x_i + \tau), \quad \phi_{i,2l-1}'(x_i - \tau) = \phi_{i+1,2l-1}'(x_i + \tau), \quad 0 \le \tau \le h_i,$$

so that

$$u_h'(x_i - \tau) = -u_h'(x_i + \tau), \quad 0 \le \tau \le h_i. \tag{3.17}$$

By the patch recovery procedure,

$$Ru_h'(x_i) = \sum_{j=1}^{r} \alpha_j [u_h'(G_{ij}) + u_h'(G_{i+1,r-j+1})], \tag{3.18}$$

where the α_j's are weights of the least squares fitting. Note that when $h_i = h_{i+1}$, the Gauss points and weights are distributed symmetrically on J_i with respect to x_i. By symmetry, we see that $x_i - G_{ij} = G_{i+1,r-j+1} - x_i$, and we set this value as τ in (3.17) to obtain

$$u_h'(G_{ij}) = -u_h'(G_{i+1,r-j+1}).$$

We then have from (3.18)

$$Ru_h'(x_i) = 0 = u'(x_i), \tag{3.19}$$

when u is given by (3.16).

Since any $u \in P_{r+2}(J_i)$ $(r = 2s)$ can be decomposed into

$$u(x) = a(\frac{x - x_{i-1}}{h})^{s+1}(\frac{x_{i+1} - x}{h})^{s+1} + w(x)$$

for some $a \in R^1$ and $w \in P_{r+1}(J_i)$, from Theorem 3.1 and (3.19) we see that

$$Ru_h'(x_i) = u'(x_i) \quad \forall u \in P_{r+2}(J_i), \tag{3.20}$$

i.e., the linear functional $E(\hat{u'})$ vanishes for all $\hat{u'} \in P_{r+1}(\hat{I})$. Therefore, by the Bramble-Hilbert Lemma, we have

$$\begin{aligned} |E(\hat{u'})| &\le C\|\hat{\delta}_i\|_{W_1^0(\hat{I})}|\hat{u'}|_{W_\infty^{r+2}(\hat{I})} \\ &\le Ch^{-1}\|\delta_i\|_{W_1^0(J_i)}h^{r+2}|u'|_{W_\infty^{r+2}(J_i)} = Ch^{r+1}|u|_{W_\infty^{r+3}(J_i)}. \end{aligned} \tag{3.21}$$

Note that $\|\delta_i\|_{W_1^0(J_i)} \le C(r)$, a constant depending on r only. Combining (3.15) and (3.21), we obtain (3.14). Finally, (3.13) follows from (3.14) and Lemma 3.1. □

Remark 3.1. The ultraconvergence recovery result is local with regard to the mesh. If we want the ultraconvergence recovery at the node x_i, we only need to use uniform meshes adjacent to x_i.

Remark 3.2. The ultraconvergence recovery for the general case where $a_1 \neq 0$ is not known since we have only the superapproximation result (3.3) instead of the ultra-approximation result (3.4) in general.

Remark 3.3. The generalization of the result to the higher-dimensional tensor product case is not straightforward.

References

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