

SPHERICAL BESSEL FUNCTIONS AND EXPLICIT QUADRATURE FORMULA

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ABSTRACT. An evaluation of the derivative of spherical Bessel functions of order $n + \frac{1}{2}$ at its zeros is obtained. Consequently, an explicit quadrature formula for entire functions of exponential type is given.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Given any complex number α , the function

$$\frac{J_\alpha(z)}{z^\alpha} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{\alpha+2k} k! \Gamma(k + \alpha + 1)}$$

is an even entire function of exponential type 1. Here $J_\alpha(z)$ is the Bessel function of the first kind of order α and is known as the spherical Bessel function when $\alpha = n + \frac{1}{2}$, $n \in \mathbb{Z}$. Let $j_k = j_k(\alpha)$, $k = \pm 1, \pm 2, \dots$, be the zeros of $\frac{J_\alpha(z)}{z^\alpha}$ ordered such that $j_{-k} = -j_k$ and $0 < |j_1| \leq |j_2| \leq \dots$.

An exact quadrature formula with zeros of Bessel functions as nodes has been recently given [1] as follows.

Theorem A. *Let $\Re(\alpha) > -1$. For all functions f of exponential type 2τ such that $f(x) = O(|x|^{-\delta})$, $x \rightarrow \pm\infty$, with $\delta > 2\Re(\alpha) + 2$, we have*

$$(1) \quad \int_0^\infty x^{2\alpha+1} (f(x) + f(-x)) dx = \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha}}{(J'_\alpha(j_k))^2} \left(f\left(\frac{j_k}{\tau}\right) + f\left(-\frac{j_k}{\tau}\right) \right).$$

The growth condition imposed on the functions has been relaxed by Grozev and Rahman.

Theorem B ([2]). *If $\alpha > -1$, then (1) holds for every entire function f of exponential type 2τ such that $x^{2\alpha+1}(f(x) + f(-x))$ belongs to $L^1[0, \infty)$.*

Since, in formula (1), $J'_\alpha(j_k)$ is not given explicitly, we find it interesting to evaluate it for the spherical Bessel functions. From now on, the notation j_k is used exclusively to denote $j_k(n + \frac{1}{2})$.

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Theorem 1. *Let n be a nonnegative integer and*

$$\lambda(j_k) := \left(\frac{\pi}{2} \sum_{r=0}^n \frac{(2n-r)! (2n-2r)!}{r! [2^{n-r} (n-r)!]^2} j_k^{2r} \right)^{-\frac{1}{2}}.$$

We have

$$(2) \quad J'_{n+\frac{1}{2}}(j_k) = (-1)^k j_k^{n-\frac{1}{2}} \lambda(j_k) \quad \text{for } k = \pm 1, \pm 2, \dots$$

Since (2) is not valid for negative integers, we give another result for these values.

We note that the zeros of $J_\alpha(z)$ are all real if $\alpha > -1$ and only a finite number of them are nonreal if $\alpha \leq -1$ [3, §15.27]. Let $\{l_k\}_{k=1}^\infty$ be the positive zeros of $\frac{J_\alpha(z)}{z^\alpha}$, $\alpha = n + \frac{1}{2}$, arranged in ascending order of magnitude and $l_k = -l_{-k}$ for $k = -1, -2, \dots$

Theorem 2. *Let n be a negative integer and*

$$\mu(l_k) := \left(\frac{\pi}{2} \sum_{r=0}^{-n-1} \frac{(-2n-r-2)! (-2n-2r-2)!}{r! [2^{-n-r-1} (-n-r-1)!]^2} l_k^{2r} \right)^{-\frac{1}{2}}.$$

We have

$$(3) \quad J'_{n+\frac{1}{2}}(l_k) = \begin{cases} (-1)^{n+k+1} l_k^{-n-\frac{3}{2}} \mu(l_k) & \text{for } k = 1, 2, \dots, \\ (-1)^{n+k} l_k^{-n-\frac{3}{2}} \mu(l_k) & \text{for } k = -1, -2, \dots \end{cases}$$

Remark 1. Using Theorems 1, 2 and the differential equation

$$z^2 y'' + z y' + (z^2 - \alpha^2) y = 0$$

satisfied by $J_\alpha(z)$, we can evaluate $J''_{n+\frac{1}{2}}(j_k), J'''_{n+\frac{1}{2}}(j_k)$, etc.

2. LEMMAS

For the recurrence formulas satisfied by Bessel functions and used in this section we refer the reader to [3, §3.2]. We need the following property of spherical Bessel functions to prove formula (2).

Lemma 1. *Let n be an integer. For all nonnegative integers p , we have*

$$(4) \quad J_{n-p-\frac{1}{2}}(j_k) = \left\{ \sum_{r=0}^{[p/2]} (-1)^r \binom{p-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p-2r}}{\Gamma(n-p+r+\frac{1}{2}) j_k^{p-2r}} \right\} J'_{n+\frac{1}{2}}(j_k).$$

Proof. We prove (4) by induction on p . For $p = 0$, (4) is equivalent to

$$(5) \quad J_{n-\frac{1}{2}}(j_k) = J'_{n+\frac{1}{2}}(j_k),$$

which we obtain using the formula

$$(6) \quad z J'_\alpha(z) + \alpha J_\alpha(z) = z J_{\alpha-1}(z)$$

with $\alpha = n + \frac{1}{2}$ and $z = j_k$. For $p = 1$, (4) gives $J_{n-\frac{3}{2}}(j_k) = \frac{2n-1}{j_k} J'_{n+\frac{1}{2}}(j_k)$, which is true by the formula

$$(7) \quad J_{\alpha-1}(z) = \frac{2\alpha}{z} J_\alpha(z) - J_{\alpha+1}(z),$$

taking $\alpha = n - \frac{1}{2}$ and using (5). Suppose that (4) is true for p and $p + 1$, where p is an even integer, and let us prove it for $p + 2$ and $p + 3$.

When $\alpha = n - p - \frac{3}{2}$, (7) and the recurrence hypothesis give

$$\begin{aligned}
 J_{n-p-\frac{5}{2}}(j_k) &= \frac{2n - (2p + 3)}{j_k} J_{n-p-\frac{3}{2}}(j_k) - J_{n-p-\frac{1}{2}}(j_k) \\
 &= \left\{ (2n - 2p - 3) \sum_{r=0}^{p/2} (-1)^r \binom{p+1-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p+1-2r}}{\Gamma(n-p+r-\frac{1}{2}) j_k^{p+2-2r}} \right. \\
 &\quad \left. - \sum_{r=0}^{p/2} (-1)^r \binom{p-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p-2r}}{\Gamma(n-p+r+\frac{1}{2}) j_k^{p-2r}} \right\} J'_{n+\frac{1}{2}}(j_k) \\
 &= \left\{ (2n - 2p - 3) \sum_{r=0}^{p/2} (-1)^r \binom{p+1-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p+1-2r}}{\Gamma(n-p+r-\frac{1}{2}) j_k^{p+2-2r}} \right. \\
 &\quad \left. - \sum_{r=1}^{\frac{p}{2}+1} (-1)^{r-1} \binom{p+1-r}{r} \frac{\Gamma(n-r+\frac{3}{2}) 2^{p+2-2r}}{\Gamma(n-p+r-\frac{1}{2}) j_k^{p+2-2r}} \right\} J'_{n+\frac{1}{2}}(j_k) \\
 &= \left\{ \sum_{r=1}^{p/2} (-1)^r \binom{p+1-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p+2-2r}}{\Gamma(n-p+r-\frac{1}{2}) j_k^{p+2-2r}} \frac{1}{(p-2r+2)} \right. \\
 &\quad \times \left[\frac{1}{2}(2n-2p-3)(p-2r+2) + r(n-r+\frac{1}{2}) \right] \\
 &\quad \left. + \frac{(2n-2p-3)\Gamma(n+\frac{1}{2}) 2^{p+1}}{\Gamma(n-p-\frac{1}{2}) j_k^{p+2}} - (-1)^{\frac{p}{2}} \right\} J'_{n+\frac{1}{2}}(j_k).
 \end{aligned}$$

Since

$$(n-p-3/2)(p-2r+2) + r(n-r+1/2) = (n-p+r-3/2)(p-r+2),$$

$$\frac{(p-r+2)}{(p-2r+2)} \binom{p+1-r}{r} = \binom{p+2-r}{r}$$

and

$$\frac{(n-p+r-\frac{3}{2})}{\Gamma(n-p+r-\frac{1}{2})} = \frac{1}{\Gamma(n-p+r-\frac{3}{2})},$$

we have

$$\begin{aligned}
 J_{n-p-\frac{5}{2}}(j_k) &= \left\{ \sum_{r=1}^{p/2} (-1)^r \binom{p+2-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p+2-2r}}{\Gamma(n-p+r-\frac{3}{2}) j_k^{p+2-2r}} \right. \\
 &\quad \left. + \frac{\Gamma(n+\frac{1}{2}) 2^{p+2}}{\Gamma(n-p-\frac{3}{2}) j_k^{p+2}} + (-1)^{\frac{p+2}{2}} \right\} J'_{n+\frac{1}{2}}(j_k) \\
 &= \left\{ \sum_{r=0}^{(p+2)/2} (-1)^r \binom{p+2-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{p+2-2r}}{\Gamma(n-p+r-\frac{3}{2}) j_k^{p+2-2r}} \right\} J'_{n+\frac{1}{2}}(j_k).
 \end{aligned}$$

Thus, (4) is true for $p+2$. For $p+3$ we use (7), taking $\alpha = n - p + \frac{5}{2}$, and the remainder of the proof is similar. □

To establish (3), we need another property of spherical Bessel functions.

Lemma 2. *Let n be an integer. For all nonnegative integers p , we have*

$$(8) \quad J_{n+p+\frac{3}{2}}(j_k) = \left\{ \sum_{r=0}^{\lfloor p/2 \rfloor} (-1)^{r+1} \binom{p-r}{r} \frac{\Gamma(n+p-r+\frac{3}{2}) 2^{p-2r}}{\Gamma(n+r+\frac{3}{2}) j_k^{p-2r}} \right\} J'_{n+\frac{1}{2}}(j_k).$$

Proof. The proof is similar to that of Lemma 1 except for the next few changes. For $p = 0$, we use the formula

$$(9) \quad zJ'_\alpha(z) - \alpha J_\alpha(z) = -zJ_{\alpha+1}(z)$$

with $\alpha = n + 1/2$. For $p = 1$, we use (7) with $\alpha = n + 3/2$. For $p + 2, p + 3$, we use (7) respectively with $\alpha = n + p + \frac{5}{2}, n + p + \frac{7}{2}$. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Using Lemma 1 with $p = 2n$, we obtain

$$(10) \quad J_{-(n+\frac{1}{2})}(j_k) = \left\{ \sum_{r=0}^n (-1)^r \binom{2n-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{2n-2r}}{\Gamma(-n+r+\frac{1}{2}) j_k^{2n-2r}} \right\} J'_{n+\frac{1}{2}}(j_k).$$

But

$$(11) \quad \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi} (2m)!}{2^{2m} m!} \quad \text{for } m = 0, 1, 2, \dots$$

and

$$\Gamma\left(-m + \frac{1}{2}\right) = \frac{\sqrt{\pi} (-1)^m 2^{2m} m!}{(2m)!} \quad \text{for } m = 0, 1, 2, \dots,$$

so that

$$(12) \quad \binom{2n-r}{r} \frac{\Gamma(n-r+\frac{1}{2}) 2^{2n-2r}}{\Gamma(-n+r+\frac{1}{2})} = (-1)^{n+r} \frac{(2n-r)! (2n-2r)!}{r! [2^{n-r} (n-r)!]^2}.$$

An application of the formula [3, §3.12]

$$(13) \quad J'_\alpha(z)J_{-\alpha}(z) - J_\alpha(z)J'_{-\alpha}(z) = \frac{2 \sin(\alpha\pi)}{\pi z}$$

gives

$$J_{-(n+\frac{1}{2})}(j_k) = \frac{2 (-1)^n}{\pi j_k J'_{n+\frac{1}{2}}(j_k)}.$$

Hence, in view of (10) and (12), we obtain

$$(14) \quad \left(J'_{n+\frac{1}{2}}(j_k) \right)^2 = \left(\frac{\pi \sum_{r=0}^n \frac{(2n-r)! (2n-2r)!}{r! [2^{n-r} (n-r)!]^2} j_k^{-2n+2r+1}}{2} \right)^{-1} = j_k^{2n-1} \lambda^2(j_k).$$

It remains to study the sign of $J'_{n+\frac{1}{2}}(j_k)$. We have (see [3, §15.22])

$$(15) \quad 0 < j_k < j_k(n + 3/2) < j_{k+1} \quad \text{for } k = 1, 2, \dots$$

Hence, the interval (j_k, j_{k+1}) contains only one zero of $J_{n+\frac{3}{2}}(z)$ for $k = 1, 2, \dots$, which implies

$$(16) \quad \operatorname{sgn} \left(J_{n+\frac{3}{2}}(j_k) \right) = -\operatorname{sgn} \left(J_{n+\frac{3}{2}}(j_{k+1}) \right) \quad \text{for } k = 1, 2, \dots$$

By (9) we have

$$J'_{n+\frac{1}{2}}(j_k) = -J_{n+\frac{3}{2}}(j_k) \quad \text{for } k = 1, 2, \dots,$$

and it follows from (16) that

$$(17) \quad \operatorname{sgn} \left(J'_{n+\frac{1}{2}}(j_k) \right) = -\operatorname{sgn} \left(J'_{n+\frac{1}{2}}(j_{k+1}) \right) \quad \text{for } k = 1, 2, \dots,$$

which implies, in view of (14), that

$$\begin{aligned} J'_{n+\frac{1}{2}}(j_k) &= \operatorname{sgn} \left(J'_{n+\frac{1}{2}}(j_k) \right) j_k^{n-\frac{1}{2}} \lambda(j_k) \\ &= (-1)^{k-1} \operatorname{sgn} \left(J'_{n+\frac{1}{2}}(j_1) \right) j_k^{n-\frac{1}{2}} \lambda(j_k) \quad \text{for } k = 1, 2, \dots \end{aligned}$$

So, in order to obtain (2) for $j_k > 0$, it suffices to prove that

$$(18) \quad J'_{p+1/2}(j_1(p+1/2)) < 0 \quad \text{for each nonnegative integer } p.$$

For $p = 0$, we have

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad j_k\left(\frac{1}{2}\right) = k\pi, \quad k = 1, 2, \dots,$$

whence

$$J'_{\frac{1}{2}}(j_1(1/2)) = J'_{\frac{1}{2}}(\pi) = -\frac{\sqrt{2}}{\pi} < 0.$$

Suppose that (18) is true for some positive integer p , which implies that

$$J'_{p+\frac{1}{2}}(x) < 0 \quad \text{for all } x \in (j_1(p+1/2), j_2(p+1/2));$$

in particular,

$$J_{p+\frac{1}{2}}(j_1(p+3/2)) < 0 \quad \text{since, by (15), } j_1(p+3/2) \in (j_1(p+1/2), j_2(p+1/2)).$$

But, using (6), we have

$$J'_{p+\frac{3}{2}}(j_1(p+3/2)) = J_{p+\frac{1}{2}}(j_1(p+3/2)) < 0,$$

so that (18) holds for $p + 1$ and consequently for all $p \geq 0$.

For $j_k < 0$, we assume first that in the definition of z^α , $\arg(z)$ has its principal value, and we suppose, as in [3, 3.62], that $\arg(-z) = \pi + \arg(z)$. Then we have

$$\begin{aligned} J'_{n+\frac{1}{2}}(j_k) &= J'_{n+\frac{1}{2}}(-j_{-k}) \\ &= -e^{(n+\frac{1}{2})\pi i} J'_{n+\frac{1}{2}}(j_{-k}) \\ &= e^{(n-\frac{1}{2})\pi i} (-1)^k (j_{-k})^{n-\frac{1}{2}} \lambda(j_{-k}) \\ &= (-1)^k (-j_{-k})^{n-\frac{1}{2}} \lambda(-j_k) \\ &= (-1)^k j_k^{n-\frac{1}{2}} \lambda(j_k), \end{aligned}$$

since $\lambda(-j_k) = \lambda(j_k)$ and $J_\alpha(-z) = e^{\alpha\pi i} J_\alpha(z)$. □

Proof of Theorem 2. Several details of the proof are similar to that of Theorem 1, and we omit them.

We replace p by $-2n - 2$ in Lemma 2 to obtain

$$(19) \quad \left(J'_{n+\frac{1}{2}}(j_k) \right)^2 = \left(\frac{\pi}{2} \sum_{r=0}^{-n-1} \frac{(-2n-r-2)! (-2n-2r-2)!}{r! [2^{-n-r-1} (-n-r-1)!]^2} j_k^{2n+2r+3} \right)^{-1}.$$

We have [3, §15.22]

$$(20) \quad 0 < l_k < l_k(n - 1/2) < l_{k+1} \quad \text{for } k = 1, 2, \dots,$$

which by virtue of (5) implies (17), where j_k is replaced by l_k . So we have, by (19),

$$J'_{n+\frac{1}{2}}(l_k) = (-1)^{k-1} \operatorname{sgn} \left(J'_{n+\frac{1}{2}}(l_1) \right) l_k^{-n-\frac{3}{2}} \mu(l_k) \quad \text{for } k = 1, 2, \dots .$$

Thus, to establish (3) for $l_k > 0$, we have to show that

$$(21) \quad (-1)^{p+1} J'_{p+\frac{1}{2}}(j_1(p+1/2)) < 0 \quad \text{for each negative integer } p.$$

For $p = -1$, we have

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \quad l_k(-1/2) = (2k - 1)\pi/2, \quad k = 1, 2, \dots,$$

whence

$$J'_{-\frac{1}{2}}(l_1(-1/2)) = J'_{-\frac{1}{2}}(\pi/2) = -\frac{2}{\pi} < 0.$$

Assume that (21) is true for some negative integer p , which implies by (20) that

$$(-1)^{p+1} J_{p+\frac{1}{2}} \left(l_1 \left(p - \frac{1}{2} \right) \right) < 0,$$

and using (9), we obtain

$$(-1)^p J'_{p-\frac{1}{2}}(l_1(p-1/2)) = (-1)^{p+1} J_{p+\frac{1}{2}}(l_1(p-1/2)) < 0.$$

Therefore, (21) holds for $p - 1$ and consequently for all $p \leq -1$.

For $l_k < 0$, we have

$$\begin{aligned} J'_{n+\frac{1}{2}}(l_k) &= e^{(n-\frac{1}{2})\pi i} (-1)^{n+k+1} (l_{-k})^{-n-\frac{3}{2}} \mu(l_{-k}) \\ &= (-1)^{n+k} (-l_{-k})^{-n-\frac{3}{2}} \mu(-l_k) \\ &= (-1)^{n+k} l_k^{-n-\frac{3}{2}} \mu(l_k). \quad \square \end{aligned}$$

4. AN EXPLICIT QUADRATURE FORMULA

We are now ready to deduce the following result from Theorems B and 1.

Theorem 3. *Let n be a nonnegative integer. For all functions f of exponential type 2τ such that*

$$(22) \quad x^{2n} f(x) \in L^1(\mathbb{R}),$$

we have

$$\begin{aligned} (23) \quad & \int_{-\infty}^{\infty} x^{2n} f(x) dx \\ &= \frac{\pi}{\tau^{2n+1}} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\sum_{r=0}^n \frac{(2n-r)! (2n-2r)!}{r! [2^{n-r} (n-r)!]^2} j_k^{2r} \right) f\left(\frac{jk}{\tau}\right) \\ &+ \frac{\pi}{\tau^{2n+1}} (2n+1) \left(\frac{(2n)!}{2^n n!} \right)^2 f(0). \end{aligned}$$

Proof. Without loss of generality we may assume that $f(z)$ is even. Let

$$g(x) := \frac{1}{x^2} \left[f(x) - \left(2^{n+\frac{1}{2}} \Gamma\left(n + \frac{3}{2}\right) \frac{J_{n+\frac{1}{2}}(\tau x)}{(\tau x)^{n+\frac{1}{2}}} \right)^2 f(0) \right].$$

Since $f(z)$ and $J_{n+\frac{1}{2}}(z)/z^{n+\frac{1}{2}}$ are even, their derivatives vanish at zero. Besides, we have $\lim_{z \rightarrow 0} J_\alpha(z)/z^\alpha = 1/(2^\alpha \Gamma(\alpha + 1))$. Thus $\lim_{z \rightarrow 0} g(z)$ exists, and consequently $g(z)$ is entire. According to the hypothesis and to the formula [3, p. 405], we have

$$(24) \quad \int_{-\infty}^{\infty} \frac{J_{n+\frac{1}{2}}^2(x)}{x} dx = \frac{2}{2n+1},$$

and $g(x)$ satisfies the conditions of Theorem B with $\alpha = n + \frac{1}{2}$. Therefore, we have

$$(25) \quad \int_{-\infty}^{\infty} x^{2n+2} g(x) dx = \frac{\pi}{\tau^{2n+3}} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\sum_{r=0}^n \frac{(2n-r)! (2n-2r)!}{r! [2^{n-r} (n-r)!]^2} j_k^{2r+2} \right) g\left(\frac{j_k}{\tau}\right).$$

Replacing $g(x)$ by its value and using (24), we readily obtain (23). □

Note that, in formula (54) of [1], which corresponds to (25) with $n = 1$, there is a superfluous factor 32. As a consequence of Theorem 3 we have the following

Corollary 1. *If n is a nonnegative integer, then for all functions f of exponential type τ such that*

$$x^n f(x) \in L^2(\mathbb{R}),$$

we have

$$(26) \quad \int_{-\infty}^{\infty} x^{2n} |f(x)|^2 dx = \frac{\pi}{\tau^{2n+3}} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\sum_{r=0}^n \frac{(2n-r)! (2n-2r)!}{r! [2^{n-r} (n-r)!]^2} j_k^{2r} \right) \left| f\left(\frac{j_k}{\tau}\right) \right|^2 \\ + \frac{\pi}{\tau^{2n+1}} (2n+1) \left(\frac{(2n)!}{2^n n!} \right)^2 |f(0)|^2.$$

Proof. Write $f(x) = f_1(x) + i f_2(x)$, where $f_1(x) = \Re(f(x))$ and $f_2(x) = \Im(f(x))$ when $x \in \mathbb{R}$. The functions $f_1^2(x)$, $f_2^2(x)$ satisfy the conditions of Theorem 3. Hence, by (23), formula (26) holds for $f_1(x)$ and $f_2(x)$. The result follows since $|f(x)|^2 = f_1^2(x) + f_2^2(x)$. □

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