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ITERATED SOLUTIONS OF LINEAR OPERATOR EQUATIONS WITH THE TAU METHOD

M. K. EL-DAOU AND H. G. KHAJAH

Abstract. The Tau Method produces polynomial approximations of solutions of differential equations. The purpose of this paper is (i) to extend the recursive formulation of this method to general linear operator equations defined in a separable Hilbert space, and (ii) to develop an iterative refinement procedure which improves on the accuracy of Tau approximations. Applications to Fredholm integral equations demonstrate the effectiveness of this technique.

1. Canonical polynomials of linear operators

Let X be a separable Hilbert space having a basis $\mathbf{x} := \{x_0, x_1, x_2, \dots\}$, and let $\mathcal{A}(X)$ denote the space of all linear operators on X. Then, for $A \in \mathcal{A}(X)$ and for all $i \in \mathbf{N}$, Ax_i is a linear combination of the basis elements:

$$Ax_i = \sum_j a_{ij} x_j \,, \qquad a_{ij} \in \mathbf{C}. \tag{1}$$

For $A \in \mathcal{A}(X)$ and $x, y \in X$, we define the equivalence relation

$$x \equiv_A y \quad \text{iff} \quad x - y \in \ker A \,.$$

An operator A is said to be *banded-from-above* if there exists an integer $k \in \mathbf{N}$ which satisfies the property $P(k)$ defined as:

$$a_{ij} = 0 \quad \forall i, j \geq 0 \ \text{ with } j - i \geq k + 1.$$

Let $\mathcal{A}_b(X) \subseteq \mathcal{A}(X)$ denote the subspace of all banded-from-above operators on X, and define an integer-valued function on the elements of $\mathcal{A}_b(X)$ by

$$h(A) := \min\{k \in \mathbf{N} : \ P(k)\}$$

which, following Ortiz [3, 4], is called the *height* of A. Then, if $A \in \mathcal{A}_b$ is of height $h(A)$ equation (1) becomes[1]

$$Ax_i = \sum_{j=0}^{h+i} a_{ij} x_j \,, \qquad i \geq 0 \,.$$

A *polynomial of degree* n in X is any finite linear combination of the form

$$p_n = \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_n x_n \,, \qquad \lambda_i \in \mathbf{C} \,, \ \lambda_n \neq 0.$$

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[1]When dealing with a particular operator A we write h instead of $h(A)$.

Given a linear operator A and an integer $n \geq 0$, we define an element $Q_n \in X$ in such a way that $AQ_n = x_n$; this Q_n is called the *n-th canonical polynomial* associated with A. The following proposition provides explicit expressions for canonical polynomials of orders $\geq h$:

Proposition 1. *Let $A \in \mathcal{A}_b(X)$ be an operator of height h and, with reference to* (1)*, assume that $a_{ij} = 1$ for all $i \geq 0$ and $j = h+i$. Then, the sequence of canonical polynomials $\{Q_k : k \geq h\}$ associated with A is determined by the recursive formulae*

$$Q_{h+k} \equiv_A x_k - \sum_{i=0}^{h+k-1} a_{k,i} Q_i \,, \quad k \geq 0,$$

where $Q_h \equiv_A x_0$.

Proof. We use the definition of Q_n to write x_n as AQ_n and employ the linearity of A. For $k \geq 0$ we have

$$\begin{aligned} Ax_k &= x_{h+k} + \sum_{i=0}^{h+k-1} a_{ki} x_i \\ &= A\,Q_{h+k} + \sum_{i=0}^{h+k-1} a_{ki} A\,Q_i \\ &= A\left[Q_{h+k} + \sum_{i=0}^{h+k-1} a_{ki} Q_i\right]. \end{aligned}$$

Therefore,

$$Q_{h+k} \equiv_A x_k - \sum_{i=0}^{h+k-1} a_{ki}\, Q_i$$

with $Q_h \equiv_A x_0$. □

Note that in the above proposition, if $h > 0$, the canonical polynomials $\{Q_n : n < h\}$ remain undefined. Furthermore, if A is invertible, i.e. $\ker A = \mathbf{0}$, then we have equalities in place of equivalences modulo $\ker A$.

Corollary 2. *Under the assumptions of Proposition* 1*, if A is invertible, then the n-th canonical polynomial Q_n is of degree $n - h$, where $n \geq h$.*

2. Approximation of linear operators with the Tau Method

Let A be an invertible linear operator satisfying the conditions of Proposition 1 with $h(A) = h \geq 0$. Consider the following operator equation with its linear auxiliary conditions (if any):

$$\begin{aligned} Au &= f, \\ B_\mu u &= g_\mu \,, \qquad \mu = 1, 2, \ldots, m, \end{aligned} \tag{2}$$

where $g_\mu \in \mathbf{C}$ and $u \in X$ is the exact solution. The purpose of this section is to explain how the recursive formulation of the Tau Method, developed by Ortiz in [3, 4], can be applied to problem (2) in order to derive polynomial approximations of u. To this end, we proceed as follows: Assume f is of the form

$$f = \sum_{i=0}^{\alpha} f_i x_i \in X\,.$$

Then for some fixed $n \geq h+m$ we associate with (2) the following problem whose exact solution is denoted by u_n

$$\begin{aligned} Au_n &= f + \sum_{i=0}^{h+m-1} \tau_{n,i}\, x_{n-i}, \\ B_\mu u_n &= g_\mu, \end{aligned} \tag{3}$$

where $\{\tau_{n,i} : i = 0, 1, ..., h+m-1\}$ are unknown parameters to be fixed with n. Taking $\{Q_i\}$ to be the set of canonical polynomials of A, we set

$$U_n = \sum_{i=0}^{\alpha} f_i Q_i + \sum_{i=0}^{h+m-1} \tau_{n,i} Q_{n-i},$$

and find that $A(u_n - U_n) = 0$ by Proposition 1; since A is invertible, it follows that $u_n = U_n$. It is not difficult to determine the Tau parameters: Since the canonical polynomials $\{Q_i : 0 \leq i < h\}$ are undefined, as mentioned earlier, we equate their coefficients in U_n to zero and form a system of h linear algebraic equations which, when added to the m auxiliary conditions, will result in $h+m$ linear equations with an equal number of unknown τ's.

The polynomial u_n will be called the *n-th Tau Method approximation to u* and, by Corollary 2, its degree is n. The *n-th Tau error* is the difference $e_n := u - u_n$, and the term $\sum \tau_{n,i} x_{n-i}$ in (3), denoted by ρ_n, is called the *Tau perturbation term.* The next result assures the convegence of the sequence $\{e_n : n \geq h+m\}$ to zero.

Theorem 3. *If X is a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and if $A \in \mathcal{A}_b(X)$ is invertible, then for fixed $h \geq 0$ and $m \geq 0$ in equations* (2) *and* (3)*, we have*

i. $\lim_{n\to\infty} |\tau_{n,i}| = 0$ *for* $i = 0, 1, \ldots, h+m-1$.
ii. $\lim_{n\to\infty} \|e_n\| = 0$.

Proof. Let $\{x_i : i \geq 0\}$ be a maximal orthonormal basis for X. Subtracting (3) from (2) we obtain the error equation

$$Ae_n = -\sum_{i=0}^{h+m-1} \tau_{n,i}\, x_{n-i} = -\rho_n. \tag{4}$$

Since A is invertible, it is bounded and so is its inverse A^{-1}. It follows then that $\|e_n\| \leq \|A^{-1}\| \cdot \|\rho_n\|$. Let $X_n = \operatorname{span}\{x_0, x_1, ..., x_{n-h-m}\}$ and take $X_n^\perp$ to be its orthogonal complement. Then, for each $j = 0, 1, \ldots, n-h-m$, we have $\langle \rho_n, x_j \rangle = 0$ since $n \geq h+m$; thus $\rho_n \in X_n^\perp$. Since $\{X_n^\perp : n \geq h+m\}$ forms a decreasing sequence of closed linear subspaces of X satisfying

$$\lim_{n\to\infty} \operatorname{diam}(X_n^\perp) = 0 \tag{5}$$

where $\operatorname{diam}(Y) = \sup\{\| a-b \| : a, b \in Y\}$—see the addendum and [1, 2]—it follows from Cantor's theorem that

$$\bigcap_{k=N}^{\infty} X_k^\perp = \{\mathbf{0}\},$$

where $N = h+m$. But since

$$\rho_n \in \bigcap_{k=N}^{n} X_k^\perp,$$

taking the limit as n tends to infinity yields

$$\lim_{n\to\infty} \rho_n \in \bigcap_{k=N}^{\infty} X_k^{\perp} = \{\mathbf{0}\},$$

from which it follows that

$$\lim_{n\to\infty} \| \rho_n \| = 0\,. \tag{6}$$

Furthermore, from Parseval's identity we see that

$$\| \rho_n \|^2 = \sum_{i=0}^{h+m-1} |\tau_{n,i}|^2 \tag{7}$$

Combining (6) and (7) we get (i)

$$\lim_{n\to\infty} |\tau_{n,i}| = 0\,, \qquad 0 \le i \le h+m-1,$$

and (ii)

$$\| e_n \| \le \| A^{-1} \| \cdot \| \rho_n \| \to 0\,. \quad \square$$

3. Iterated Tau Method solutions

The idea here is to start with the approximate solution u_n of (2) and generate more accurate approximations via an iterative process. We may assume that the operator A is of the form $A := T - \lambda$ where $T \in \mathcal{A}_b(X)$ and $\lambda \in \mathbf{C}$. Equation (3) becomes

$$Tu_n - \lambda u_n = f + \rho_n. \tag{8}$$

For $k \ge 0$, we define the k-th iterated Tau solution $\hat{u}_{n,k}$ associated with u_n as follows

$$\begin{aligned} &\hat{u}_{n,0} = u_n, \\ &\hat{u}_{n,k} = \lambda^{-1}(T\hat{u}_{n,k-1} - f), \qquad k \ge 1. \end{aligned} \tag{9}$$

Let the error $e_{n,k} = u - \hat{u}_{n,k}$. Then, for $k \ge 1$, we have

$$\begin{aligned} e_{n,k} &= \lambda^{-1}(Tu - f) - \lambda^{-1}(T\hat{u}_{n,k-1} - f) \\ &= \lambda^{-1}\,(Tu - T\hat{u}_{n,k-1}) \\ &= \lambda^{-k}T^k\,(u - u_n) \\ &= \lambda^{-k}T^k\,e_n \end{aligned}$$

and hence

$$\| e_{n,k} \| \le |\lambda|^{-k} \| T \|^k \cdot \| e_n \|.$$

Since e_n is independent of k, the right-hand side of this inequality will tend to zero as $k \to \infty$ if we assume $\| T \| < |\lambda|$. Thus, we have the main result of this paper:

Theorem 4. *If $A = T - \lambda$ is invertible and if $\| T \| < |\lambda|$, then, for a fixed n, the following assertions hold:*

i. $\| e_{n,k} \| \le |\lambda|^{-k} \| T \|^k \cdot \| e_n \|, k \ge 0,$
ii. $\lim_{k\to\infty} \| e_{n,k} \| = 0.$

Note that from equations (8) and (9) we obtain the following procedure for computing $\hat{u}_{n,k}$:

$$\hat{u}_{n,0} = u_n,$$
$$\hat{u}_{n,k} = \hat{u}_{n,k-1} + \lambda^{-k} T^{k-1} \rho_n, \qquad k \geq 1\,.$$

4. Numerical examples

We show, through the following two examples, how to improve the accuracy of Tau approximations using the procedure described above.

Example A. Consider the integral equation

$$(F\,u)\,(t) := \int_0^1 K(t,s)\,u(s)\,ds - \lambda u(t) = \phi\,(t)\,, \qquad t \in [0,1], \tag{10}$$

where $\phi\,(t)$ is known. This defines an inhomogeneous Fredholm equation of the second kind with symmetric kernel

$$K(t,s) = \begin{cases} -t\,(1-s) & \text{if } s \geq t, \\ -s\,(1-t) & \text{if } s \leq t. \end{cases}$$

Taking the shifted Chebyshev polynomials $T_k^*(t)$ as basis elements, we approximate $u(t)$ in the subspace $X_n = \text{span}\{T_0^*, T_1^*, \ldots, T_n^*\}$. We follow the recursive formulation of the Tau Method described in Ortiz [3, 4] for linear differential operators: Define $Q_n(t)$ so that

$$(F\,Q_n)\,(t) = T_n^*(t)$$

is satisfied. This leads to

$$Q_2 = 16\,[T_0^* + (\frac{1}{16} + \lambda)\,Q_0\,],$$
$$Q_3 = 96\,[T_1^* + (\frac{1}{96} + \lambda)\,Q_1\,],$$
$$Q_4 = 192\,[T_2^* + (\frac{1}{24} + \lambda)\,Q_2 - \frac{7}{192}\,Q_0\,],$$

and for $n \geq 5$,

$$Q_n = 16\,n(n-1)\left[\,T_{n-2}^* + (\frac{1}{8(n-1)(n-3)} + \lambda)\,Q_{n-2} - \frac{1}{16(n-3)(n-4)}\,Q_{n-4} + \frac{3}{4n(n-1)(n-3)(n-4)}\,Q_\delta\right],$$

where $\delta = 0$ for even n and 1 for odd values of n . Note that the first two canonical polynomials, Q_0 and Q_1 , remain undefined.

If we take $\lambda = \frac{1}{4}$ and $\phi\,(t) = -\frac{\cosh 1}{4}$, the exact solution of (10) becomes $u(t) = \cosh\,(2t-1)$. For a given n , we compute the initial Tau approximation $u_n(t)$ by solving the perturbed problem:

$$\int_0^1 K(t,s)\,u_n(s)\,ds - \frac{1}{4}u_n(t) = -\frac{\cosh\,1}{4} + \tau_n\,T_n^*(t). \tag{11}$$

Then, using the above procedure, we generate the k-th iterated Tau solutions $\hat{u}_{n,k}$ and employ the supremum norm:

$$\|w\| = \sup\,\{|w(t)| : 0 \leq t \leq 1\}\,.$$

TABLE A. Error norms $\| e_{n,k} \|$ and Tau values

$k \setminus n$	2	4	6	8	10
1	3.09 E–1	5.65 E–3	4.56 E–5	2.01 E–7	5.53 E–10
2	7.41 E–2	4.59 E–4	1.28 E–6	3.38 E–9	5.58 E–12
3	2.87 E–2	1.06 E–4	4.97 E–8	1.22 E–10	9.37 E–14
4	1.16 E–2	3.94 E–5	7.74 E–9	2.52 E–11	2.42 E–14
5	4.69 E–3	1.58 E–5	3.46 E–9	9.60 E–12	1.13 E–14
6	1.90 E–3	6.40 E–6	1.43 E–9	3.87 E–12	3.77 E–15
7	7.70 E–4	2.59 E–6	5.81 E–10	1.57 E–12	2.44 E–15
8	3.12 E–4	1.05 E–6	2.35 E–10	6.35 E–13	8.88 E–16
9	1.26 E–4	4.26 E–7			
10	5.12 E–5	1.72 E–7			
11	2.07 E–5	6.99 E–8			
12	8.42 E–6	2.84 E–8			
13	3.41 E–6	1.15 E–8			
14	1.38 E–6	4.66 E–9			
15	5.60 E–7	1.89 E–9			
16	2.27 E–7	7.65 E–10			
17	9.20 E–8	3.10 E–10			
18	3.73 E–8				
19	1.51 E–8				
20	6.13 E–9				
τ_n	7.715 E–2	1.41 E–3	1.14 E–5	5.02 E–8	1.38 E–10

Numerical results for approximations of order $n = 2, 4, 6, 8, 10$ with various iterations k are presented in Table A, where the error norms $\| e_{n,k} \|$ are given. We note, for example, that an error of order $O(10^{-8})$ is obtained through Tau approximations u_2, u_4, u_6 and u_8 after 17, 11, 3 and 2 iterations, respectively.

TABLE B. Error norms $\| e_{n,k} \|$ and Tau values

$k \setminus n$	1	2	3	4	5
0	1.00	1.33 E–1	1.03 E–2	5.48 E–4	2.84 E–5
1	3.86 E–1	6.04 E–2	2.72 E–3	9.67 E–5	3.57 E–6
2	2.83 E–1	2.24 E–2	1.47 E–3	3.78 E–5	1.14 E–6
3	1.15 E–1	3.94 E–3	5.83 E–4	1.95 E–5	4.96 E–7
4	3.17 E–2	2.37 E–4	1.40 E–4	6.49 E–6	1.71 E–7
5	6.72 E–3	7.94 E–5	2.43 E–5	1.49 E–6	4.52 E–8
6	1.16 E–3	2.79 E–5	3.28 E–6	2.62 E–7	9.20 E–9
7	1.70 E–4	5.81 E–6	3.52 E–7	3.72 E–8	1.50 E–9
8	2.17 E–5	9.15 E–7	2.97 E–8	4.44 E–9	2.04 E–10
9	2.45 E–6	1.19 E–7	1.82 E–9	4.57 E–10	2.37 E–11
10	2.48 E–7	1.34 E–8	4.35 E–11	4.14 E–11	2.42 E–12
11	2.28 E–8	1.33 E–9	7.61 E–12	3.33 E–12	2.19 E–13
12	1.91 E–9	1.19 E–10	1.52 E–12	2.34 E–13	1.78 E–14
τ_n	1.00	1.11 E–1	8.85 E–3	5.48 E–4	2.72 E–5

Example B. Our next example is concerned with the exponential function e^t which satisfies the following Volterra equation of the second kind

$$(Fu)(t) = \int_0^t u(s)\,ds - u(t) = -1\,, \qquad t \in [0,1]. \tag{12}$$

The perturbed equation becomes

$$\int_0^t u_n(s)\,ds - u_n(t) = -1 + \tau_n T_n^*(t) \tag{13}$$

and we list the numerical results in Table B.

Addendum

Here we prove the following result:

Let H be a separable Hilbert space with an orthonormal basis $\{x_k : k \geq 0\}$ and let $\{H_n : n \geq 0\}$ be a sequence of subspaces of H such that $H_n^\perp \subset H_{n-1}^\perp$ for all $n \geq 1$. Then we have

$$\operatorname{diam}(H_n^\perp) \to 0 \qquad \text{as } n \to \infty,$$

where $\operatorname{diam}(H_n^\perp) = \sup\{\| a - b \| : a, b \in H_n^\perp\}$.

To prove this, let $a = \sum_{i\geq 0}\langle a, x_i\rangle\, x_i$ and $b = \sum_{i\geq 0}\langle b, x_i\rangle\, x_i$ be two elements of $H_n^\perp$. Then

$$a - b = \sum_{i\geq 0}\big(\langle a, x_i\rangle - \langle b, x_i\rangle\big)\, x_i = \sum_{i\geq 0}\langle a-b, x_i\rangle x_i = \sum_{i>n}\langle a-b, x_i\rangle\, x_i.$$

Since the set $\{x_k : k \geq 0\}$ forms an orthonormal basis, we get

$$\| a - b \|^2 = \sum_{i>n} |\langle a-b, x_i\rangle|^2. \tag{14}$$

But

$$\sum_{i\geq 0} |\langle a-b, x_i\rangle|^2 < \infty. \tag{15}$$

Thus (14) is the residual of the absolutely convergent series (15) and therefore

$$\lim_{n\to\infty}\sum_{i>n} |\langle a-b, x_i\rangle|^2 = 0$$

from which it follows that $\operatorname{diam}(H_n^\perp) \to 0$.

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Applied Sciences Department, College of Technical Studies, Paaet, P. O. Box 42325, Shuwaikh 70654, Kuwait

E-mail address: hkhajah@kuc01.kuniv.edu.kw