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STIELTJES POLYNOMIALS AND LAGRANGE INTERPOLATION

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Abstract. Bounds are proved for the Stieltjes polynomial E_{n+1}, and lower bounds are proved for the distances of consecutive zeros of the Stieltjes polynomials and the Legendre polynomials P_n. This sharpens a known interlacing result of Szegö. As a byproduct, bounds are obtained for the Geronimus polynomials G_n. Applying these results, convergence theorems are proved for the Lagrange interpolation process with respect to the zeros of E_{n+1}, and for the extended Lagrange interpolation process with respect to the zeros of P_nE_{n+1} in the uniform and weighted L^p norms. The corresponding Lebesgue constants are of optimal order.

1. Introduction

Let P_n be the Legendre polynomial, normalized by $P_n(1) = 1$. The polynomials E_{n+1} defined (up to a multiplicative constant) by

$$\int_{-1}^{1} E_{n+1}(x)P_n(x)\,x^k\,dx = 0, \qquad k = 0, 1, \dots, n, \qquad n \geq 1,$$

were introduced by Stieltjes more than one hundred years ago. In 1934 Szegö [36], following Stieltjes idea, introduced the wider class of polynomials $E_{n+1}^{(\lambda)}$, defined by

$$\int_{-1}^{1} w_\lambda(x)E_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(x)\,x^k\,dx = 0, \qquad k = 0, 1, \dots, n, \qquad n \geq 1,$$

where $w_\lambda(x) = (1 - x^2)^{\lambda-1/2}$, $\lambda > -\frac{1}{2}$, and $P_n^{(\lambda)}$ is the Gegenbauer polynomial. In [36] Szegö proved, among other results, that for $0 \leq \lambda \leq 2$, the zeros of $E_{n+1}^{(\lambda)}$ interlace with those of $P_n^{(\lambda)}$. This proves and generalizes a conjecture of Stieltjes for the case $\lambda = \frac{1}{2}$. After Szegö's paper, Stieltjes' idea seemed to have had no further development for a long time. But in 1964 Kronrod, urged by the aim of estimating the error of the Gauss–Legendre quadrature formula, introduced the extended quadrature formula, now well known as the Gauss–Kronrod rule

$$\int_{-1}^{1} f(x)\,dx = \sum_{\nu=1}^{n} A_{\nu,n}^{GK} f(x_{\nu,n}) + \sum_{\mu=1}^{n+1} B_{\mu,n+1}^{GK} f(\xi_{\mu,n+1}) + R_{2n+1}^{GK}(f), \tag{1}$$

where $x_{\nu,n}$ are the zeros of P_n and the nodes $\xi_{\mu,n+1}$ as well as the weights $A_{\nu,n}^{GK}$ and $B_{\mu,n+1}^{GK}$ are chosen such that the formula has algebraic degree of precision $\geq 3n+1$,

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i.e. $R^{GK}_{2n+1}(p) = 0$ if $p \in \mathbb{P}_{3n+1}$ ($\mathbb{P}_k$ is the space of all algebraic polynomials of degree at most k). Some years later Barrucand [1] observed that $\xi_{\mu,n+1}$ are precisely the zeros of the Stieltjes polynomials E_{n+1}. In the second half of the seventies, G. Monegato in [25], proved that the interlacing property of the zeros of E_{n+1} with those of P_n is equivalent to the positivity of the coefficients $B^{GK}_{\mu,n+1}$, and then proved that the Gauss–Kronrod formula has positive weights even if it is constructed with respect to the weight w_λ, $0 < \lambda \le 1$ [26]. Kronrod's idea together with the results of Barrucand and Monegato urged a lot of mathematicians to consider Stieltjes polynomials for more general weight functions, to study the interlacing properties of the zeros and to construct extended positive quadrature formulas. Among them, we mention Gautschi and Notaris [16], Gautschi and Rivlin [17], and the recent papers of Peherstorfer [34] and of the first author of this paper. For a more complete history of the problem under consideration, the interested reader may consult the exhaustive surveys of Gautschi [15] and Monegato [28].

Nevertheless, the interpolation process based on the zeros of Stieltjes polynomials and/or the extended interpolation process that uses the zeros of the polynomials $K_{2n+1} = P_n E_{n+1}$ have received little attention. Recently several authors, following a different approach than Kronrod, constructed extended interpolation processes starting with the zeros of the product of two or three orthogonal polynomials with respect to different weights. By using the method of additional nodes they proved convergence theorems in uniform and weighted L^p norms (see for instance [4, 5, 6, 20, 21, 32]).

The reasons for the absence of results on interpolation processes based on the zeros of E_{n+1} and/or $P_n E_{n+1}$ are first of all the fact that in literature there are no accurate bounds available for the polynomials E_{n+1}, and in second place that information about the distribution of the zeros of E_{n+1} and/or $P_n E_{n+1}$ is very poor. The interlacing property of the zeros of E_{n+1} with those of P_n allows to obtain easily upper bounds on the distance between two consecutive zeros, while the respective lower bounds are harder to find.

The first result in this paper is an accurate pointwise bound of the polynomials E_{n+1}. This bound shows an "opposite" behaviour of E_{n+1} with respect to that of the Legendre polynomial P_n. In fact, in every closed subset of $(-1,1)$, E_{n+1} is unbounded (with respect to n), while it is bounded near the endpoints ± 1. As a consequence of this fact, the polynomial $P_n E_{n+1}$ results in being bounded in $[-1,1]$ and it seems to have a behaviour similar to that of the Chebyshev polynomials of the first kind.

Then we will prove that both the zeros $\xi_{\mu,n+1} = \cos\theta_{\mu,n+1}$ of E_{n+1} and those $y_{k,2n+1} = \cos\psi_{k,2n+1}$ of $K_{2n+1} = P_n E_{n+1}$ have an "arccos-type" distribution, i.e. their cosine arguments satisfy

$$|\theta_{\mu,n+1} - \theta_{\mu+1,n+1}| \ \sim\ |\psi_{k,2n+1} - \psi_{k+1,2n+1}| \ \sim\ n^{-1}.$$

These results are explained in §2 of this paper. In §3 we consider the behaviour of the Lagrange polynomial $L_{n+1}f$ which interpolates a preassigned function f at the zeros of E_{n+1}. We will prove that this interpolatory process is optimal in the sense that the n-th Lebesgue constant $\|L_n\| = \sup_{\|f\|=1} \|L_n f\|$, where $\|\cdot\|$ is the sup-norm, is $\sim \log n$. We also observe that this result seems surprising, since E_{n+1} is unbounded in $(-1,1)$, and on the other hand a "good" distribution of the zeros generally doesn't imply $\|L_n\| \sim \log n$.

We also prove some convergence theorems in weighted L^p norms by estimating the interpolation error by means of the best weighted one-sided approximation. Moreover, we prove that the Lagrange polynomials $\mathcal{L}_{2n+1}f$ interpolating a function f at the zeros of P_nE_{n+1} have optimal Lebesgue constants (i.e., $O(\log n)$). Therefore, the zeros of the Stieltjes polynomials E_{n+1} have the property of improving the interpolatory process based on the Legendre zeros, which, as well known, has Lebesgue constants $\sim \sqrt{n}$.

2. Inequalities for Stieltjes polynomials

For the Stieltjes polynomials E_{n+1}, we use the normalization (cf., e.g., [28])

$$\begin{aligned}\frac{\gamma_n}{2}E_{n+1}(\cos\theta) &= \alpha_{0,n}\cos(n+1)\theta + \alpha_{1,n}\cos(n-1)\theta \\ &\quad + \cdots + \begin{cases}\alpha_{\frac{n}{2},n}\cos\theta, & n \text{ even},\\ \frac{1}{2}\alpha_{\frac{n+1}{2},n}, & n \text{ odd},\end{cases}\end{aligned} \tag{2}$$

where

$$\alpha_{0,n} = f_{0,n} = 1, \qquad \sum_{\mu=0}^{\nu}\alpha_{\mu,n}f_{\nu-\mu,n} = 0, \quad \nu = 1,2,\ldots, \tag{3}$$

$$f_{\nu,n} = \left(1-\frac{1}{2\nu}\right)\left(1-\frac{1}{2n+2\nu+1}\right)f_{\nu-1,n}, \quad \nu = 1,2,\ldots, \tag{4}$$

and

$$\gamma_n = \sqrt{\pi}\,\frac{2^{2n+1}n!^2}{(2n+1)!}.$$

In the following we denote the zeros of the Legendre polynomials P_n by $x_{\nu,n} = \cos\phi_{\nu,n}$, $\nu = 1,\ldots,n$, and the zeros of the Stieltjes polynomials E_{n+1} by $\xi_{\mu,n+1} = \cos\theta_{\mu,n+1}$, $\mu = 1,\ldots,n+1$, ordered by increasing magnitude in both cases (we will frequently omit the index n where the meaning is clear from the context).

Theorem 2.1. *For $n \geq 1$, there holds*

$$|E_{n+1}(x)| \;\leq\; 2\,C^*\sqrt{\frac{2n+1}{\pi}}\,\sqrt[4]{1-x^2} \;+\; 2, \quad x_1 \leq x \leq x_n, \tag{5}$$

where $C^* = 1.0180\ldots$. For $x \in [-1,x_1]\cup[x_n,1]$, there holds

$$|E_{n+1}(x)| \;\leq\; 25 \;+\; \epsilon(n), \tag{6}$$

$\lim_{n\to\infty}\epsilon(n) \leq 0$, $\epsilon(n) < 30$. *Furthermore,*

$$E_{n+1}(1) \;\geq\; \frac{2}{3\sqrt{\pi}}, \qquad n \geq 1. \tag{7}$$

According to Theorem 2.1, rough bounds are

$$|E_{n+1}(x)| \;\leq\; 2\,C^*\sqrt{\frac{2n+1}{\pi}}\,\sqrt[4]{1-x^2} \;+\; 57, \quad -1 \leq x \leq 1, \tag{8}$$

and

$$|E_{n+1}(x)| \;\leq\; C\sqrt{n}\left(\sqrt{1-x^2}+\frac{1}{n}\right)^{\frac{1}{2}}, \qquad -1 \leq x \leq 1, \tag{9}$$

where C is a positive constant.

The best bound for Stieltjes polynomials in literature is (cf. [27])

$$|E_{n+1}(x)| \;\le\; \frac{4}{\gamma_n}, \qquad -1 \le x \le 1.$$

While this bound behaves uniformly on the whole interval $[-1,1]$, the bounds in Theorem 2.1 are smaller by a factor $\sim n^{-1/2}$ at the endpoints if compared to the interior of the interval. This conforms to recent results about the asymptotic behaviour of Stieltjes polynomials [10], namely that the formula

$$E_{n+1}(\cos\theta) = 2\sqrt{\frac{2n\sin\theta}{\pi}}\cos\left\{\left(n+\frac{1}{2}\right)\theta+\frac{\pi}{4}\right\}+o(\sqrt{n}) \tag{10}$$

holds uniformly for $\epsilon \le \theta \le \pi-\epsilon$. On the other hand, a comparison of (5) and (10) shows that in fixed closed subintervals of $(-1,1)$, the bound (5) can at most be improved by the factor $C = 1.0180\ldots$. It follows from (7) that the order in n of (6) and (8) is also unimprovable at the endpoints ± 1.

The associated sin-polynomial

$$\begin{aligned}\frac{\gamma_n}{2}e_n(\theta) \;&=\; \alpha_{0,n}\sin(n+1)\theta+\alpha_{1,n}\sin(n-1)\theta \\ &\qquad +\cdots+\begin{cases}\alpha_{\frac{n}{2}-1,n}\sin 3\theta+\alpha_{\frac{n}{2},n}\sin\theta, & n \text{ even},\\ \alpha_{\frac{n-1}{2},n}\sin 2\theta, & n \text{ odd},\end{cases}\end{aligned}$$

is important in connection with a class of polynomials G_n considered by Geronimus (cf. [18]; cf. also [28, 34, 36]). The connection is (cf. [28, 36])

$$\sin\theta\, G_n(\cos\theta) \;=\; e_n(\theta). \tag{11}$$

As a byproduct of the previous theorem, we also obtain bounds for the Geronimus polynomial G_n.

Theorem 2.2. *For $n \ge 1$,*

$$|G_n(x)| \;\le\; 2\,C^*\sqrt{\frac{2n+1}{n}}\,\frac{1}{\sqrt[4]{1-x^2}} \;+\; \frac{2}{\sqrt{1-x^2}}, \qquad x_1 \le x \le x_n,$$

where $C^* = 1.0180\ldots$. Moreover, there holds

$$|G_n(x)| \;\le\; C\,(n+1), \qquad -1 \le x \le 1,$$

where $C \le 35$.

With regard to the application for extended interpolation in §3, it is important to obtain accurate upper bounds also for the product P_nE_{n+1}. Recalling classical results about the Legendre polynomials P_n, we observe from (7) and (10) that E_{n+1} has an "opposite" behaviour with respect to the term $\sqrt[4]{1-x^2}$. Thus, P_nE_{n+1} is very similar to the Chebyshev polynomial T_{2n+1} of the first kind (see [27] for related numerical results). More precisely, we have the following corollary.

Corollary 2.3. *For $n \ge 1$, there holds*

$$|P_n(x)E_{n+1}(x)| \;\le\; C, \qquad -1 \le x \le 1,$$

where $C \le 55$. In particular, we have

$$|P_n(x)E_{n+1}(x)| \;\le\; 7, \qquad x_1 \le x \le x_n.$$

Let the zeros of $K_{2n+1} = P_n E_{n+1}$ be denoted by $y_\nu = \cos\psi_\nu$, $\nu = 1, \dots, 2n+1$, ordered in increasing magnitude, and let $y_0 = -y_{2n+2} = \xi_0 = -\xi_{n+2} = -1$. Sharpening the interlacing result of Szegö [36], we prove a lower bound for the distances of consecutive zeros of K_{2n+1}.

Theorem 2.4. *For $n \geq 1$,*

$$\liminf_{n\to\infty} (2n+1)(\psi_\nu - \psi_{\nu+1}) \ > \ \frac{1}{20}, \qquad \nu = 0, \dots, 2n+1,$$

and

$$\liminf_{n\to\infty} (n+1)(\theta_\mu - \theta_{\mu+1}) \ > \ \frac{1}{20}, \qquad \mu = 0, \dots, n+1.$$

3. Lagrange interpolation

We write $f \in L^p(E)$, $E \subseteq [-1,1]$, $1 \leq p < \infty$, if

$$\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

and we set $L^p = L^p([-1,1])$, $\|f\|_p = \|f\|_{L^p([-1,1])}$. In the case $p = \infty$, we keep the previous notation by setting $\|f\|_{L(E)} = \sup_{x\in E} |f(x)|$, $E \subseteq [-1,1]$. In the following, C denotes a positive constant which may be different in different formulas. With σ being a weight function and $1 \leq p \leq \infty$, we use the notations

$$\mathcal{E}_m(f)_{\sigma,p} = \inf_{q\in\mathbb{P}_m} \|[f-q]\sigma\|_p$$

and

$$\tilde{\mathcal{E}}_m(f)_{\sigma,p} = \inf\{\|[q^+ - q^-]\sigma\|_p, \quad q^\pm \in \mathbb{P}_m, \quad q^- \leq f \leq q^+\}$$

for the error of the best algebraic weighted approximation and the best one-sided weighted approximation. If $\sigma \equiv 1$ in $[-1,1]$, we write $\mathcal{E}_m(f)_p$ and $\tilde{\mathcal{E}}_m(f)_p$. Now let $L_{n+1}(f)$ be the $(n+1)$-th Lagrange polynomial interpolating f at the zeros of E_{n+1}. The following theorem holds.

Theorem 3.1. *For any continuous function f we have*

$$\|f - L_{n+1}f\|_\infty \ \leq \ C \log n \, \mathcal{E}_n(f)_\infty$$

where C is independent of n and f.

Let u be a Generalized Jacobi (GJ) weight, defined by

$$u(x) = \prod_{k=0}^{r} |t_k - x|^{\gamma_k}, \quad \gamma_k > -1, \quad -1 = t_0 < t_1 < \dots < t_{r-1} < t_r = 1, \quad |x| \leq 1.$$

We state some convergence theorems of $L_{n+1}f$ to f in the L^p norm with weight u.

Theorem 3.2. *Let $u \in L^p$ with $1 \leq p < \infty$. Then for any continuous function f we have*

$$\|[f - L_{n+1}f]u\|_p \ \leq \ C\,\mathcal{E}_n(f)_\infty, \tag{12}$$

where C is independent of n and f. Furthermore, if $u\sqrt{\varphi} \in L^p$ and $(u\sqrt{\varphi})^{-1} \in L^q$, $\varphi(x) = \sqrt{1-x^2}$, $p^{-1} + q^{-1} = 1$, $1 < p < \infty$, then, for any function $f : [-1,1] \to \mathbb{R}$ which is bounded and measurable, we have

$$\|[f - L_{n+1}f]u\|_p \ \leq \ C\,\tilde{\mathcal{E}}_n(f)_{u,p}, \tag{13}$$

where C is independent of n and f.

The assumptions about u which were made in Theorem 3.2 to obtain (13) are stronger than those to obtain (12). But (13) is better than (12) (if f is continuous) because $\tilde{\mathcal{E}}_n(f)_{u,p} \le \|u\|_p \tilde{\mathcal{E}}_n(f)_\infty$ and $\tilde{\mathcal{E}}_n(f)_\infty = 2\,\mathcal{E}_n(f)_\infty$. Moreover, bounds of the type (13) are useful to estimate the error $\|[f - L_{n+1}f]u\|_p$ for an interesting function class, more precisely, the class of functions f which are locally absolutely continuous in $(-1,1)$ ($f \in AC_{\rm loc}$), which generally need not be bounded at the endpoints ± 1, as the example $\log(1+x)$ shows. For such functions, we cannot use (12), but the following theorem is useful.

Theorem 3.3. *Assume $u\sqrt{\varphi} \in L^p$ and $(u\sqrt{\varphi})^{-1} \in L^q$, $1 < p < \infty$, and $p^{-1} + q^{-1} = 1$. If $f \in AC_{\rm loc}$ and $f'\varphi^{2/p}u \in L^1$, then*

$$\|[f - L_{n+1}f]u\|_p \;\le\; \frac{C}{n}\,\|f'\varphi u\|_{L^p[(\xi_1,\xi_{n+1})]} \;+\; c\,\|f'\varphi^{2/p}u\|_{L^1(I_n')}, \tag{14}$$

where $I_n' = [-1,1]\backslash(\xi_1,\xi_{n+1})$ and the constants are independent of n and f. In particular if $f'\varphi u \in L^p$, then

$$\|[f - L_{n+1}f]u\|_p \;\le\; \frac{C}{n}\,\mathcal{E}_{n-1}(f')_{\varphi u,p},$$

where C is independent of n and f.

For example, if $p = 2$, $u(x) = \sqrt[4]{1-x^2}$ and $f(x) = \log(1+x)$, from (14) we obtain $\|[f - L_{n+1}(f)]u\|_2 = O(n^{-1})$. The interested reader may find estimates of $\mathcal{E}_n(g)_{u,p}$ with a GJ weight u and $g \in AC_{\rm loc}$ in [3].

The case $p = 1$ is interesting in the applications, because it is connected with the error of the product quadrature rule. Estimates of $\|[f - L_{n+1}]u\|_1$ in the L^1 norm and the same weight u are only possible under strong conditions on the weight u (see for instance [7, 24]). From the previous theorems, we can derive better estimates than (12) when $p = 1$ by some assumption on the weight u. For instance since

$$\|[f - L_{n+1}f]u\|_1 \;\le\; \sqrt{\|u\|_1}\,\|[f - L_{n+1}f]\sqrt{u}\|_2,$$

if $(u\varphi)^{\pm 1} \in L^1$, using (13) we obtain

$$\|[f - L_{n+1}f]u\|_1 \;\le\; C\,\tilde{\mathcal{E}}_n(f)_{\sqrt{u},2}.$$

If in addition the function f is locally absolutely continuous, then we can use Theorem 3.3.

Now we consider the behaviour of the Lagrange polynomial $\mathcal{L}_{2n+1}f$ interpolating the function f at the zeros of $K_{2n+1} = P_n E_{n+1}$. We state the following theorem.

Theorem 3.4. *For every continuous function f we have*

$$\|f - \mathcal{L}_{2n+1}f\|_\infty \;\le\; C\,\log n\,\mathcal{E}_{2n}(f)_\infty,$$

where C is independent of n and f.

For a GJ weight u, we set $u_-(x) = \prod_{\gamma_k<0} |t_k - x|^{\gamma_k}$ and $u_- \equiv 1$ if $\gamma_k \ge 0$, $k = 0,\dots,r$. With this notation, we state the following theorem.

Theorem 3.5. *Let f be a bounded and measurable function. If $u \in L^p$ with $1 < p < \infty$, then*

$$\|[f - \mathcal{L}_{2n+1}f]u\|_p \;\le\; C\,\tilde{\mathcal{E}}_{2n}(f)_{u_-,p}. \tag{15}$$

If f is continuous and $u \in L^1$, then

$$\|[f - \mathcal{L}_{2n+1}f]u\|_1 \;\le\; C\,\mathcal{E}_{2n}(f)_\infty. \tag{16}$$

Furthermore, if f is bounded and measurable, $u \in L^p$ and $u^{-1} \in L^q$, $p^{-1}+q^{-1}=1$ and $1<p<\infty$, then we have

$$\|[f-\mathcal{L}_{2n+1}f]u\|_p \;\le\; C\,\tilde{\mathcal{E}}_{2n}(f)_{u,p}, \tag{17}$$

where the constants are independent of n and f.

By comparison of Theorem 3.2 and Theorem 3.5 one can see that the behaviour of $\mathcal{L}_{2n+1}$ is better than that of L_{n+1}. While (16) is the analogue of (12), we can see from the proof that the estimate (15) for L_{n+1} is only possible if $1<p<4$. Moreover, the inequality (16) can be replaced by

$$\|[f-\mathcal{L}_{2n+1}f]u\|_1 \;\le\; C\,\tilde{\mathcal{E}}_{2n}(f)_{\sqrt{u},2}, \qquad u^{\pm 1}\in L^1,$$

using the previous argument. Finally, we state the analogue to Theorem 3.3.

Theorem 3.6. *Let $u \in L^p$ and $u^{-1}\in L^q$, $p^{-1}+q^{-1}=1$ and $1<p<\infty$. If $f \in AC_{\mathrm{loc}}$ and $f'\varphi^{2/p}u \in L^1$, then*

$$\|[f-\mathcal{L}_{2n+1}f]u\|_p \;\le\; \frac{C}{n}\,\|f'\varphi u\|_{L^p(y_1,y_{2n+1})} \;+\; c\,\|f'\varphi^{2/p}u\|_{L^1(I_n')},$$

where $I_n'=[-1,1]\setminus(y_1,y_{2n+1})$. If in addition $f'\varphi u\in L^p$, then

$$\|[f-\mathcal{L}_{2n+1}f]u\|_p \;\le\; \frac{C}{n}\,\mathcal{E}_{2n-1}(f')_{u\varphi,p},$$

where the constants are independent of n and f.

4. Proofs

In the sequel, we write $A_n \sim B_n$ for two expressions depending on a common parameter n if $0<C_1<|A_n/B_n|<C_2<\infty$, where C_1, C_2 are independent of n.

Let E_{n+1} be defined as in (2), with the coefficients $\alpha_{\nu,n}$ as defined in (3); let $m=m(n)=\lfloor (n+1)/2\rfloor$. First, note that the bounds in §2 can be verified easily in the cases $n=1,2$, such that we can assume $n\ge 3$.

Lemma 4.1. *Let the sequence $(\alpha_{\nu,n})$ be defined as in* (3)*. Then*

$$\frac{1}{3\sqrt{n}} \;\le\; {\sum_{\nu=0}^{m}}'\alpha_{\nu,n} \;\le\; \frac{5}{3\sqrt{n}}. \tag{18}$$

Here the prime means that $\alpha_{m,n}$ has to be replaced by $\frac{1}{2}\alpha_{m,n}$ if n is odd.

Proof of Lemma 4.1. Szegö [36] proved that

$$\alpha_{0,n}=1, \qquad \alpha_{\nu,n}<0, \quad \nu=1,2,\dots, \qquad \sum_{\nu=0}^{\infty}\alpha_{\nu,n}=0. \tag{19}$$

In [10], it was proved that for the product

$$(\alpha_{0,n}+\cdots+\alpha_{k,n})(f_{0,n}+\cdots+f_{k,n}) \;=\; 1+R_{k,n},$$

where $f_{\nu,n}$ is defined in (3), we have

$$R_{k,n} \;<\; 0. \tag{20}$$

Using a lower bound for $f_{\nu,n}$ following from [10, (40)],

$$f_{\nu,n} \;\ge\; \sqrt{\frac{2}{3}}\,\frac{(2\nu)!}{2^{2\nu}\nu!^2}, \qquad \nu\le m,$$

after some elementary estimates we find

$$f_{0,n} + \cdots + f_{m-1,n} \;\geq\; \frac{2}{\sqrt{3\pi}} \int_0^m \frac{dx}{\sqrt{2x+1}} \;\geq\; \frac{2}{\sqrt{3\pi}} (\sqrt{n+1} - 1), \tag{21}$$

and obtain the upper bound in (18).

For the lower bound, let ρ be an integer > 1. We consider the product, for arbitrary $k \in \mathbb{N}$,

$$\begin{aligned} &(\alpha_{0,n} + \cdots + \alpha_{k,n})\,(f_{0,n} + \cdots + f_{\rho k,n}) \\ &\quad = \sum_{\nu=0}^{k} \sum_{\mu=0}^{\nu} f_{\nu-\mu,n} \alpha_{\mu,n} + \sum_{\nu=k+1}^{\rho k} \sum_{\mu=0}^{k} f_{\nu-\mu,n} \alpha_{\mu,n} + \sum_{\nu=1}^{k} \alpha_{\nu,n} \sum_{\mu=1}^{\nu} f_{\rho k+1-\mu,n}. \end{aligned}$$

By (3), the first term is equal to 1, and the second term is greater than 0. Therefore,

$$\begin{aligned} &(\alpha_{0,n} + \cdots + \alpha_{k,n})\,(f_{0,n} + \cdots + f_{\rho k,n}) \\ &\qquad > \; 1 \; + \; \sum_{\nu=1}^{k} \alpha_{\nu,n} \sum_{\mu=1}^{\nu} f_{\rho k+1-\mu,n} \\ &\qquad > \; 1 \; - \; f_{(\rho-1)k+1,n}\, |\alpha_{1,n} + 2\alpha_{2,n} + \cdots + k\alpha_{k,n}|. \end{aligned} \tag{22}$$

Now let $k = m = \lfloor (n+1)/2 \rfloor$. We again obtain, by elementary estimates, from [10, (40)]

$$f_{(\rho-1)m+1,n} \;\leq\; \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{(\rho-1)n + \frac{5}{2}}} \sqrt{\frac{2n + \frac{17}{20}}{(\rho+1)n + \frac{3}{4}}}.$$

Analogously as in (21), we use [10, (40)] for upper bounds of f_ν and estimate the sum in an elementary way by an integral, which, after straightforward calculations, leads to

$$f_{0,n} + \cdots + f_{\rho m,n} \;\leq\; \frac{1}{\sqrt{\pi}} \sqrt{2\rho(n+1)+1} \;+\; \frac{1}{2}.$$

We need the following lemma, which will be proved later.

Lemma 4.2. *Let the sequence* $(\alpha_{\nu,n})$ *be defined as in* (3)*. Then, for* $k \leq m$*,*

$$|\alpha_{1,n} + 2\alpha_{2,n} + 3\alpha_{3,n} + \cdots + k\alpha_{k,n}| \;\leq\; \sqrt{\frac{3\pi}{2}} \left[\sqrt{k + \frac{1}{2}} + \frac{1}{4} \ln\left(k + \frac{1}{2}\right) + 3 \right].$$

We observe that $\sqrt{n} \sum_{\nu=0}^{m}{}' \alpha_{\nu,n}$, for sufficiently large n, is bounded from below by a constant less than

$$A(\rho) \;:=\; \sqrt{\frac{\pi}{2\rho}} - \sqrt{\frac{3\pi}{2\rho(\rho^2-1)}}.$$

The function A has a maximum for $\rho = 5$, and we continue the proof with this value. Plugging the explicit bounds derived above into (22), after lengthy but straightforward and elementary calculations we obtain the lower bound for $n \geq 2600$. For $1 \leq n < 2600$, we explicitly compute the values $\sqrt{n} \sum_{\nu=0}^{m}{}' \alpha_{\nu,n}$. $\square$

Proof of Lemma 4.2. Using (19), we obtain

$$\begin{aligned}&|\alpha_{1,n}+2\alpha_{2,n}+\cdots+k\alpha_{k,n}|\\ &\quad= -\sum_{\nu=1}^{k}\sum_{\mu=\nu}^{k}\alpha_{\mu,n} = -\sum_{\nu=1}^{k}\left(-\sum_{\mu=0}^{\nu-1}\alpha_{\mu,n}-\sum_{\mu=k+1}^{\infty}\alpha_{\mu,n}\right) < \sum_{\nu=1}^{k}\sum_{\mu=0}^{\nu-1}\alpha_{\mu,n},\end{aligned}$$

and the last inequality follows again from (19). For $k \le m$, we obtain

$$\sum_{\nu=1}^{k}\sum_{\mu=0}^{\nu-1}\alpha_{\mu,n} < \sum_{\nu=1}^{k}\left(\sum_{\mu=0}^{\nu-1}f_{\mu,n}\right)^{-1} < \frac{\sqrt{3\pi}}{2}\sum_{\nu=1}^{k}[\sqrt{2\nu+1}-1]^{-1},$$

where the first inequality follows by (20) and the second by the same method as in (21). We estimate the sum by

$$\sum_{\nu=1}^{k}[\sqrt{2\nu+1}-1]^{-1} \le (\sqrt{3}-1)^{-1}+\frac{1}{\sqrt{2}}\int_{\frac{3}{2}}^{k+\frac{1}{2}}\frac{dx}{\sqrt{x}-1},$$

and obtain the result by some straightforward computations. □

Lemma 4.3. *For $n \ge 0$,*

$$|E'_{n+1}(1)| \le (10+\epsilon(n))\,(n+1)^2,$$

$\lim_{n\to\infty}\epsilon(n) \le 0$, $\epsilon(n) < 12$.

Proof of Lemma 4.3. We obtain from (2) that

$$\begin{aligned}E'_{n+1}(1) &= \frac{2}{\gamma_n}\sum_{\nu=0}^{m}{}'(n+1-2\nu)^2\alpha_{\nu,n}\\ &< \frac{2}{\gamma_n}(n+1)^2\left\{\sum_{\nu=0}^{m}{}'\alpha_{\nu,n}-\frac{4}{n+1}\sum_{\nu=1}^{m}{}'\nu\alpha_{\nu,n}\right\}.\end{aligned}$$

For γ_n, we compute the lower bound

$$\gamma_n \ge 4\sqrt{\frac{\pi}{6n+3}}. \tag{23}$$

An application of Lemma 4.1 and Lemma 4.2 then leads to the result. □

Lemma 4.4. *For $n \ge 1$,*

$$0 < E''_{n+1}(1) < (6+\epsilon(n))\,(n+1)^4,$$

$\lim_{n\to\infty}\epsilon(n) \le 0$, $\epsilon(n) < 8$.

Proof of Lemma 4.4. Using (2) and [35, Exercise 1.5.6], we obtain

$$\begin{aligned}E''_{n+1}(1) &= \frac{2}{3\gamma_n}\sum_{\nu=0}^{m-1}\alpha_{\nu,n}\left\{(n+1-2\nu)^4-(n+1-2\nu)^2\right\}\\ &= \frac{2}{3\gamma_n}(n+1)^4\left\{\sum_{\nu=0}^{m-1}\left(1-\frac{2\nu}{n+1}\right)^4\alpha_{\nu,n}\right.\\ &\qquad\left.-\sum_{\nu=0}^{m-1}\frac{1}{(n+1-2\nu)^2}\left(1-\frac{2\nu}{n+1}\right)^4\alpha_{\nu,n}\right\}.\end{aligned}$$

For the first sum, we have

$$\left(1-\frac{2\nu}{n+1}\right)^4 \;=\; 1-4\left(\frac{2\nu}{n+1}\right)+6\left(\frac{2\nu}{n+1}\right)^2-4\left(\frac{2\nu}{n+1}\right)^3+\left(\frac{2\nu}{n+1}\right)^4,$$

and for $\nu < \frac{n+1}{2}$,

$$6\left(\frac{2\nu}{n+1}\right)^2-4\left(\frac{2\nu}{n+1}\right)^3 \;=\; \left(\frac{2\nu}{n+1}\right)^2\left\{6-\frac{8\nu}{n+1}\right\} \;>\; 0.$$

Therefore, in view of (19), we have

$$\begin{aligned}&\frac{2}{3\gamma_n}(n+1)^4\sum_{\nu=0}^{m-1}\left(1-\frac{2\nu}{n+1}\right)^4\alpha_{\nu,n}\\ &\qquad\le\;\frac{2}{3\gamma_n}(n+1)^4\left\{\sum_{\nu=0}^{m-1}\alpha_{\nu,n}-\frac{8}{n+1}\sum_{\nu=1}^{m-1}\nu\alpha_{\nu,n}\right\}.\end{aligned}$$

We further estimate this term in a straightforward way, using (23) as well as Lemmas 4.1 and 4.2. Furthermore, we obtain

$$\begin{aligned}\sum_{\nu=0}^{m-1}\frac{1}{(n+1-2\nu)^2}\left(1-\frac{2\nu}{n+1}\right)^4\alpha_{\nu,n} &= \frac{1}{(n+1)^2}\sum_{\nu=0}^{m-1}\alpha_{\nu,n}\left(1+\frac{2\nu}{n+1-2\nu}\right)^{-2}\\ &\ge \frac{1}{(n+1)^2}\sum_{\nu=0}^{m-1}\alpha_{\nu,n}\;>\;0,\end{aligned}$$

such that this term can be omitted in the bound for $E''_{n+1}(1)$. □

Proof of Theorem 2.1. Let n be even; for odd n, only minor modifications are necessary. Let $0<\theta<\pi$. We have

$$E_{n+1}(\cos\theta)=\frac{2}{\gamma_n}\;\mathrm{Re}\;\left\{e^{i(n+1)\theta}\sum_{\nu=0}^{\frac{n}{2}}\alpha_{\nu,n}e^{-2i\nu\theta}\right\}.$$

Since, in the following equation, both series are convergent, we can write

$$\sum_{\nu=0}^{\frac{n}{2}}\alpha_{\nu,n}e^{-2i\nu\theta}=\sum_{\nu=0}^{\infty}\alpha_{\nu,n}e^{-2i\nu\theta}-\sum_{\nu=\frac{n}{2}+1}^{\infty}\alpha_{\nu,n}e^{-2i\nu\theta}.$$

From [10], we recall

$$\frac{2}{\gamma_n}\mathrm{Re}\left\{e^{i(n+1)\theta}\sum_{\nu=0}^{\infty}\alpha_{\nu,n}e^{-2i\nu\theta}\right\}=2\,Q_n(\cos\theta)\;\left\{[Q_n(\cos\theta)]^2+\left[\frac{\pi}{2}P_n(\cos\theta)\right]^2\right\}^{-1}.$$

Here Q_n, $0<\theta<\pi$, is defined by (cf. [36])

$$\lim_{\epsilon\to 0}(Q_n(x+i\epsilon)+Q_n(x-i\epsilon))=2\,Q_n(x),\qquad x\in(-1,1),$$

where, for complex $z\notin[-1,1]$,

$$Q_n(z)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(t)}{z-t}\,dt$$

is the Legendre function of the second kind.

Durand [9] proved that the symmetric function

$$\sin\theta\left\{\left[\frac{2}{\pi}Q_n(\cos\theta)\right]^2+[P_n(\cos\theta)]^2\right\}$$

is monotonically increasing for $0<\theta\le\frac{\pi}{2}$, and that

$$\sin\theta\left\{\left[\frac{2}{\pi}Q_n(\cos\theta)\right]^2+[P_n(\cos\theta)]^2\right\}\ \le\ \left(\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)}\right)^2.$$

We obtain that for $0<\theta<\pi$,

$$\sqrt{\sin\theta}\,|Q_n(\cos\theta)|\ \le\ \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)},$$

and that for $\phi_n\le\theta\le\phi_1$ $(x_\nu=\cos\phi_\nu)$,

$$\sin\theta\left\{\left[\frac{2}{\pi}Q_n(\cos\theta)\right]^2+[P_n(\cos\theta)]^2\right\}\ \ge\ \frac{4}{\pi^2}\sin\phi_n\,[Q_n(\cos\phi_n)]^2.$$

Hence, we can estimate

$$\frac{2}{\gamma_n}\left|\mathrm{Re}\left\{e^{i(n+1)\theta}\sum_{\nu=0}^{\infty}\alpha_{\nu,n}e^{-2i\nu\theta}\right\}\right|\le\sqrt{\pi}\,\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\sqrt{\sin\theta}\,[\sqrt{\sin\phi_n}Q_n(\cos\phi_n)]^{-2}.$$

It is well known (cf. e.g. [2, p. 151]), that the nodes $x_{1,n},\dots,x_{n,n}$ of the Gaussian quadrature formula $Q_n^G[f]=\sum_{\nu=1}^n a_{\nu,n}^G f(x_{\nu,n})$, defined by $\int_{-1}^1 p(x)\,dx-Q_n^G[f]=0$ if p is a polynomial of degree $\le 2n-1$, are the zeros of P_n, and the weights are represented by

$$a_{\nu,n}^G=-\frac{2Q_n(x_{\nu,n})}{P_n'(x_{\nu,n})}=\frac{2}{(1-[x_{\nu,n}]^2)[P_n'(x_{\nu,n})]^2}. \tag{24}$$

A lower bound for $a_{n,n}^G$ has been proved by Förster [12, p. 130],

$$a_{n,n}^G\ \ge\ \frac{1}{C^*}\,\frac{\pi}{n+\frac{1}{2}}\sin\phi_n, \tag{25}$$

where $C^*=1.0180\dots$. Invoking (24) and (25), we obtain

$$[\sqrt{\sin\phi_n}\,Q_n(\cos\phi_n)]^2\ \ge\ \frac{1}{C^*}\,\frac{\pi}{2n+1},$$

which leads to the first term in the inequality (5).

For the second term, we estimate

$$\begin{aligned}&|\alpha_{\frac{n}{2}+1,n}e^{i\theta}+\alpha_{\frac{n}{2}+2,n}e^{3i\theta}+\cdots|\\&\qquad\le|\alpha_{\frac{n}{2}+1,n}|+|\alpha_{\frac{n}{2}+2,n}|+\cdots\ =\ \alpha_{0,n}+\alpha_{1,n}+\cdots+\alpha_{\frac{n}{2},n},\end{aligned}$$

where we have used (19). In view of Lemma 4.1 and (23), we obtain

$$\frac{2}{\gamma_n}(\alpha_{0,n}+\alpha_{1,n}+\cdots+\alpha_{\frac{n}{2},n})\ \le\ \frac{5}{\sqrt{6\pi}}\sqrt{1+\frac{1}{2n}}\ <2. \tag{26}$$

In the following, we prove the inequality (6). For symmetry reasons, we only need to look at $[x_n,1]$. Since the zeros of P_n and E_{n+1} interlace, there exists precisely one zero ξ_{n+1} of E_{n+1} in $[x_n,1]$. E_{n+1} is monotone in $[\xi_{n+1},1]$, negative

in $[x_n, \xi_{n+1}]$, has precisely one local minimum $x^* \in [x_n, 1]$ and is convex in $[x^*, 1]$. Therefore, the bounds

$$E_{n+1}(1) + E'_{n+1}(1)(x-1) \;\le\; E_{n+1}(x) \;\le\; E_{n+1}(1), \qquad x \in [x_n, 1],$$

are valid. Thus, we obtain

$$|E_{n+1}(x)| \;\le\; \max\{E_{n+1}(1), (1-x_n)E'_{n+1}(1) - E_{n+1}(1)\} \tag{27}$$

for $x \in [x_n, 1]$. By (26), we have

$$E_{n+1}(1) \;=\; \frac{2}{\gamma_n} \sum_{\nu=0}^{\frac{n}{2}} \alpha_{\nu,n} \;\le\; \frac{5}{\sqrt{6\pi}} \sqrt{1 + \frac{1}{2n}} \;< 2.$$

The other part in (27) is larger, namely we obtain

$$(1-x_n)E'_{n+1}(1) - E_{n+1}(1) \;\le\; (1-x_n)E'_{n+1}(1) \;\le\; 25 \;+ \bar{\epsilon}(n),$$

$\lim_{n\to\infty} \bar{\epsilon}(n) \le 0$, $\bar{\epsilon}(n) < 30$, from Lemma 4.3 and a classical estimate for $(1-x_n)$ (cf. [37, Thm. 6.21.3]).

The lower bound follows from the lower bound in Lemma 4.1. □

Remark. For the associated sin-polynomial e_n, we have $e_n(0) = 0$. By (11) and the interlacing property of the zeros of E_{n+1} and the Geronimus polynomial G_n (see [36]), we obtain that $e_n(\theta)$ is symmetric and has $n+2$ zeros in $[0, \pi]$, but none in $(0, \theta_{n+1}]$, where $\xi_{n+1} = \cos\theta_{n+1}$ is the largest zero of E_{n+1}. From the symmetry we obtain $|e_n(\theta_1)| = |e_n(\theta_{n+1})|$. The derivative $e'_n(\theta)$ (with respect to θ) is for $\theta = \arccos x$ an algebraic polynomial of degree $n+1$ in x with positive leading coefficient. We have for $m = \lfloor (n+1)/2 \rfloor$

$$\frac{\gamma_n}{2}\, e'_n(0) \;=\; \sum_{\nu=0}^{m}{}' (n+1-2\nu)\alpha_{\nu,n} \;=\; (n+1)\sum_{\nu=1}^{m}{}' \alpha_{\nu,n} - 2\sum_{\nu=0}^{m}{}' \nu\alpha_{\nu,n}, \tag{28}$$

with the aforementioned definitions of γ_n and $\alpha_{\nu,n}$. From the latter, it follows that the first sum is positive and the second is negative, hence

$$e'_n(0) \;>\; 0.$$

A simple argument shows that $e_n(\theta)$, for $\theta \in [0, \theta_{n+1}]$, is positive and lies under the tangent in the point 0,

$$|e_n(\theta)| \;\le\; \theta\, e'_n(0) \;\le\; \theta_{n+1}\, e'_n(0).$$

Now, denoting the largest zero of P_n by $x_n = \cos\phi_n$, we have

$$\theta_{n+1} \;<\; \phi_n \;<\; \frac{\pi}{n+1},$$

where the last inequality is from [37, p.139]. We estimate the first sum in (28) by Lemma 4.1, and the second by Lemma 4.2. After some elementary computations, we obtain for $\theta \in [0, \theta_{n+1}]$

$$|e_n(\theta)| \;\le\; 11.$$

Furthermore, starting from (for even n, the case n odd can be treated analogously)

$$e_n(\theta) = \frac{2}{\gamma_n}\, \mathrm{Im}\, \left\{ e^{i(n+1)\theta} \sum_{\nu=0}^{\frac{n}{2}} \alpha_{\nu,n} e^{-2i\nu\theta} \right\},$$

the same bound as in Theorem 2.1 can be proved for the associated sin-polynomial $e_n(\theta)$ (C^* as in Theorem 2.1),

$$|e_n(\theta)| \;\le\; 2\,C^* \sqrt{\frac{2n+1}{\pi}}\,\sqrt{\sin\theta} \;+\; 2, \qquad \phi_n \le \theta \le \phi_1.$$

Proof of Theorem 2.2. We obtain the first inequality from the above remark and (11). For the second inequality, we use bounds for x_n [37, p. 122] and the same argument as in [30, (i)] to obtain

$$|G_n(x)| \;\le\; 12 \max_{x_1 \le x \le x_n} |G_n(x)| \;\le\; 35\,(n+1). \qquad \square$$

Proof of Corollary 2.3. For P_n, we use the well known bounds

$$|P_n(\cos\theta)| \;\le\; 1, \qquad |P_n(\cos\theta)| \;\le\; \sqrt{\frac{2}{\pi n \sin\theta}}, \qquad 0 < \theta < \pi, \tag{29}$$

and obtain the inequalities by using Theorem 2.1. $\square$

Proof of Theorem 2.4. We shall treat the following cases separately:

(i) $\liminf\limits_{n\to\infty} (2n+1)\,(\phi_\nu - \theta_{\nu+1}) \;>\; 0.16, \quad \nu = 1, \ldots, n-1,$

(ii) $\liminf\limits_{n\to\infty} (2n+1)\,(\theta_\nu - \phi_\nu) \;>\; 0.16, \quad \nu = 2, \ldots, n,$

(iii) $\liminf\limits_{n\to\infty} (2n+1)\,(\theta_1 - \phi_1) \;=\; \liminf\limits_{n\to\infty} (2n+1)\,(\phi_n - \theta_{n+1}) \;>\; 0.05,$

(iv) $\liminf\limits_{n\to\infty} (n+1)(\pi - \theta_1) \;=\; \liminf\limits_{n\to\infty} (n+1)\,\theta_{n+1} \;>\; 0.13.$

We will first prove (i); the proof of (ii) follows in an analogous way. We obtain from Taylor's theorem

$$\sqrt{\sin\theta_{\nu+1}}P_n(\cos\theta_{\nu+1}) = (\phi_\nu - \theta_{\nu+1})\left\{\sin^{\frac{3}{2}}\phi^* {P_n}'(\cos\phi^*) - \frac{\cos\phi^* P_n(\cos\phi^*)}{2\sqrt{\sin\phi^*}}\right\}, \tag{30}$$

where $\theta_{\nu+1} < \phi^* < \phi_\nu$. Now, using the bound (29), we obtain

$$\left|\frac{\cos\phi^* P_n(\cos\phi^*)}{2\sqrt{\sin\phi^*}}\right| \le \sqrt{\frac{1}{2\pi n}}\frac{1}{\sin\phi^*} < \frac{\sqrt{2n}}{\pi\sqrt{\pi}}\left(1 - \frac{\pi}{n+1}\right)^{-1},$$

since $\sin\phi^* > \sin\phi_{n,n} > \frac{\pi}{2n}\left(1 - \frac{\pi}{n+1}\right)$, (cf. [37, Thm. 6.21.3]). For the other term at the right-hand side in (30), we use the well known equality $P_n' = P_{n-1}^{(\frac{3}{2})}$ for the ultraspherical polynomial $P_{n-1}^{(\frac{3}{2})}$ (cf. [37, (4.7.17)]), and obtain from [13, Corollary 1.8], that

$$|\sin^{\frac{3}{2}}\phi^* P_{n-1}^{(\frac{3}{2})}(\cos\phi^*)| \le \frac{2}{\Gamma(\frac{3}{2})}\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{1}{2})}.$$

Szegö proved in [36] that

$$Q_n(\cos\theta)E_{n+1}(\cos\theta) + \frac{\pi}{2}P_n(\cos\theta)e_n(\theta) \;>\; 1, \qquad 0 < \theta < \pi,$$

and it follows that

$$\sqrt{\sin\theta_{\nu+1}}|P_n(\cos\theta_{\nu+1})| > \frac{2}{\pi}\frac{\sqrt{\sin\theta_{\nu+1}}}{|e_n(\theta_{\nu+1})|}.$$

Now, using the remark after the proof of Theorem 2.1, there follows after straightforward calculations

$$\begin{aligned} \theta_\nu - \phi_\nu &> \frac{0.08}{n}\,(1+\epsilon_1(n))^{-1}, \quad \nu = 1,\dots,n-1, \\ \phi_\nu - \theta_{\nu+1} &> \frac{0.08}{n}\,(1+\epsilon_1(n))^{-1}, \quad \nu = 2,\dots,n, \end{aligned}$$

where $\lim_{n\to\infty}\epsilon_1(n) \le 0$, $\epsilon_1(n) < 2.4$ $(n \ge 3)$.

For (iii), we can proceed similarly as in the proof of (i). Here we use

$$P_n(\cos\theta_1) \;=\; -(\theta_1-\phi_1)\,\sin\phi^*\,P_n'(\cos\phi^*)$$

where $\phi_1 < \phi^* < \theta_1$. Now

$$\theta_1 - \phi_1 \;>\; \frac{|P_n(\cos\theta_1)|}{(\pi-\phi^*)|P_n'(\cos\phi^*)|} \;>\; \frac{2}{\pi(\pi-\phi_1)}\frac{1}{|e_n(\theta_1)|}\frac{2}{n(n+1)} \;>\; \frac{0.025}{n}.$$

For the proof of (iv), for symmetry reasons again, we only have to consider $\liminf_{n\to\infty}(n+1)\theta_{n+1}$. Since E_{n+1} is monotone and convex in $[\xi_{n+1},1]$, it follows that

$$1-\xi_{n+1} \;>\; \frac{E_{n+1}(1)}{E_{n+1}'(1)} \;>\; 0.0376\,(n+1)^{-2}\,(1+\epsilon_2(n))^{-1},$$

$\lim_{n\to\infty}\epsilon_2(n) \le 0$, $\epsilon_2(n) < 72$ $(n \ge 3)$. Now it follows with some simple trigonometric calculations that

$$\theta_{n+1} \;>\; \frac{0.137}{n+1}\,(1+\epsilon_3(n))^{-1},$$

$\lim_{n\to\infty}\epsilon_3(n) \le 0$, $\epsilon_3(n) < 7.6$ $(n \ge 3)$. □

Lemma 4.5. *Let $y_\nu = y_{\nu,2n+1}$ be the zeros of $K_{2n+1} = P_nE_{n+1}$. Then*

$$\frac{1}{|K_{2n+1}'(y_\nu)|} \;\le\; \frac{C}{n}\sqrt{1-y_\nu^2}, \qquad \nu = 1,\dots,2n+1, \tag{31}$$

for a positive constant C. Furthermore,

$$\frac{1}{|E_{n+1}'(\xi_\nu)|} \;\le\; \frac{C}{n\sqrt{n}}\sqrt[4]{1-\xi_\nu^2}, \qquad \nu = 1,\dots,n+1. \tag{32}$$

Proof of Lemma 4.5. We recall that the zeros of E_{n+1} are used as additional nodes for the Gauss-Kronrod formulas. For their weights $A_{\nu,n}^{GK}$ and $B_{\mu,n+1}^{GK}$ in (1), we obtain from [10, (93), (94)]

$$A_{\nu,n}^{GK} \;=\; a_{\nu,n}^{G} + \frac{2}{P_n'(x_\nu)E_{n+1}(x_\nu)}, \qquad \nu = 1,\dots,n, \tag{33}$$

$$B_{\mu,n+1}^{GK} \;=\; \frac{2}{P_n(\xi_\mu)E_{n+1}'(\xi_\mu)}, \qquad \mu = 1,\dots,n+1, \tag{34}$$

where $a_{\nu,n}^{G}$, $\nu = 1,\dots,n$, are the Gaussian quadrature weights. Now, we use the positivity of the weights in (33) and the bound [14, Corollary 1] to obtain

$$\frac{\pi}{2n+1}\sqrt{1-x_\nu^2}.$$

Using [11, Theorem 2.1] for $\mu = 1,\dots,\lfloor\frac{n+1}{2}\rfloor$, we obtain

$$B_{\mu,n+1}^{GK} \;\le\; a_{\mu-1,n}^{G} + a_{\mu,n}^{G} \;\le\; 2\,a_{\mu,n}^{G} \;\le\; \frac{2\pi}{n+\frac{1}{2}}\sqrt{1-x_\mu^2}. \tag{35}$$

We argue analogously for $\mu = \lfloor \frac{n+1}{2} \rfloor + 1, \dots, n+1$, and, observing $\sqrt{1-x_\mu^2} \sim \sqrt{1-\xi_\mu^2}$, we obtain (31). Using (34), (35) and a standard bound for P_n (cf. [37]), we also obtain (32). □

We now proceed with the proofs of the results in §3. In the following, we use some properties of the Hilbert transform $H(f)$, defined by

$$H(f,t) \;=\; \lim_{\epsilon\to 0} \int_{|x-t|>\epsilon} \frac{f(x)}{x-t}\,dx, \qquad f \in L^1.$$

We recall that if $G \in L^\infty$ and $F \log^+ F \in L^1$, where F and G have compact support K, then we have

$$\int_K GH(F) \;=\; -\int_K FH(G), \tag{36}$$

see, for instance, [31]. Moreover, let $u \in V$ and $v \in V$ be two GJ weights with $u \le v$, $u \in L^p$ and $v^{-1} \in L^q$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, then

$$\|H(f)u\|_p \;\le\; C\,\|fv\|_p, \tag{37}$$

see, for instance, [29, 38].

Proof of Theorem 3.1. It is sufficient to prove that

$$|L_{n+1}(f,x)| \;\le\; C\,\log n\,\|f\|_\infty, \qquad -1 \le x \le 1.$$

Let d be chosen such that $\xi_d \le x < \xi_{d+1}$. Let also $|x-\xi_d| \le |\xi_{d+1}-x|$ (the other case can be treated analogously). Now

$$|L_{n+1}(f,x)| \le \left|\frac{E_{n+1}(x)\,f(\xi_d)}{E'_{n+1}(\xi_d)(x-\xi_d)}\right| + \left|\sum_{\substack{\nu=1\\ \nu\ne d}}^{n+1} \frac{E_{n+1}(x)\,f(\xi_\nu)}{E'_{n+1}(\xi_\nu)(x-\xi_\nu)}\right| =: I_1 + I_2.$$

In view of Theorem 2.4, we can use [20, Lemma 4.1] and obtain

$$I_2 \;\le\; C\,\log n\,\|f\|_\infty \frac{|E_{n+1}(x)|}{\sqrt{n}} (\sqrt{1-x}+n^{-1})^{-1/2} (\sqrt{1+x}+n^{-1})^{-1/2}.$$

We invoke the bound (9) and obtain

$$I_2 \;\le\; C\,\log n\,\|f\|_\infty,$$

where C is a positive constant. Next, we use Lemma 4.5 to obtain

$$I_1 \;\le\; \left|\frac{E'_{n+1}(\xi)}{E'_{n+1}(\xi_d)}\right| \|f\|_\infty \;\le\; C \left|\frac{\sqrt[4]{1-\xi_d^2}}{n\sqrt{n}} E'_{n+1}(\xi)\right| \|f\|_\infty,$$

where $\xi_d < \xi < \xi_{d+1}$. Applying the weighted Bernstein inequality (cf., e.g., [30]), observing $\sqrt[4]{1-\xi_d^2} \sim \sqrt[4]{1-\xi^2}$, and using (9) we obtain $I_1 \le C\,\|f\|_\infty$ for a positive constant C. □

Proof of Theorem 3.2. Let $q^\pm \in \mathbb{P}_n$ such that $q^- \le f \le q^+$. Using [30, (25)], we have

$$\begin{aligned} \|[f - L_{n+1}f]u\|_p &\le \|[f-q^-]u\|_p \;+\; \|L_{n+1}(f-q^-)\,u\|_p \\ &\le \|(q^+-q^-)u\|_p \;+\; C\|L_{n+1}(f-q^-)\,u\|_{L^p(A_n)}, \end{aligned} \tag{38}$$

where

$$A_n = [-1 + an^{-2}, 1 - an^{-2}] \setminus \bigcup_{k=1}^{r-1} \left(t_k - \frac{a}{n}, t_k + \frac{a}{n}\right),$$

$a > 0$ fixed. We define

$$G = \operatorname{sgn} L_{n+1}(f - q^-), \qquad G_1 = |L_{n+1}(f - q^-)|, \qquad \Gamma = G G_1^{p-1},$$

and

$$\pi(t) = \int_{A_n} \frac{E_{n+1}(x) - E_{n+1}(t)}{x - t} u^p(x)\Gamma(x)\, dx.$$

Recalling Lemma 4.5, we obtain

$$I^p \;:=\; \|L_{n+1}(f - q^-)u\|^p_{L^p(A_n)} \;=\; \sum_{k=1}^{n+1} \frac{f(\xi_k) - q^-(\xi_k)}{E'_{n+1}(\xi_k)} \pi(\xi_k).$$

Using [19, Theorem 2.2], we have

$$I^p \;\le\; C \sum_{k=1}^{n+1} \frac{\sqrt[4]{1 - \xi_k^2}}{n\sqrt{n}} (q^+ - q^-)(\xi_k)|\pi(\xi_k)| \;\le\; C \int_{-1}^{1} \frac{(q^+ - q^-)(t)}{\sqrt{n\varphi(t)}} |\pi(t)|\, dt.$$

Using again [30, (25)], we obtain

$$I^p \;\le C \int_{A_n} \frac{(q^+ - q^-)(t)}{\sqrt{n\varphi(t)}} (|H(E_{n+1}u^p\Gamma, t)| + |E_{n+1}(t)H(u^p\Gamma, t)|)\, dt,$$

where H is the Hilbert transform.

Assume that f is a continuous function, $u \in L^p$, $1 < p < \infty$. We have

$$\begin{aligned} I^p &\le C\, \|q^+ - q^-\|_\infty \left[\int_{A_n} \frac{|H(E_{n+1}u^p\Gamma, t)|}{\sqrt{n\varphi(t)}}\, dt + \int_{A_n} \frac{|E_{n+1}(t)|}{\sqrt{n\varphi(t)}} |H(u^p\Gamma, t)|\, dt \right] \\ &=: C\, \|q^+ - q^-\|_\infty\, (I_1 + I_2). \end{aligned}$$

To estimate I_1, we observe that $E_{n+1}u^p\Gamma$ and $(\sqrt{\varphi})^{-1}$ are bounded functions with respect to $x \in A_n$, such that we can use (36). Then, setting $g_1 = \operatorname{sgn} H(E_{n+1}u^p\Gamma)$, we obtain

$$I_1 \;\le\; \frac{C}{\sqrt{n}} \int_{A_n} |E_{n+1}(t)u^p(t)\Gamma(t)|\, |H(g_1(\sqrt{\varphi})^{-1}, t)|\, dt. \tag{39}$$

Using (9) and the Hölder inequality, we obtain

$$\begin{aligned} I_1 &\le C\, \|u\sqrt{\varphi}H(g_1(\sqrt{\varphi})^{-1})\|_p\, \|u^{p-1}\Gamma\|_q \\ &\le C\, \|L_{n+1}(f - q^-)u\|_p^{p-1}, \end{aligned}$$

since $\|u\sqrt{\varphi}H(g_1(\sqrt{\varphi})^{-1})\|_p < \infty$ by [30, p. 676].

Similarly, we set $g_2 = \operatorname{sgn} H(u^p\Gamma)$ and estimate

$$\begin{aligned} I_2 &\le \int_{A_n} |H(u^p\Gamma, t)|\, dt \le \int_{A_n} |u^{p-1}(t)\Gamma(t)|\, |u(t)H(g_2, t)|\, dt \\ &\le C\, \|uH(g_2)\|_p\, \|L_{n+1}(f - q^-)u\|_p^{p-1} \;\le\; C\, \|L_{n+1}(f - q^-)u\|_p^{p-1}, \end{aligned}$$

since again $\|uH(g_2)\|_p < \infty$ by [30, p. 676]. In conclusion, we have

$$\|L_{n+1}(f - q^-)u\|_p \;\le\; C\, \|q^+ - q^-\|_\infty.$$

Recalling (38), taking the infimum of $q^{\pm}$ and by $\inf\{\,\|q^+ - q^-\|_\infty,\ q^- \le f \le q^+,\ q^{\pm} \in \mathbb{P}_n\,\} = 2\,\mathcal{E}_n(f)_\infty$, then (12) follows for $1 < p < \infty$.

If $p = 1$ and f is a continuous function, starting from (39) it is easy to prove

$$|H(g_1(\sqrt{\varphi})^{-1}, t)| \ \le\ \frac{C}{\sqrt{\varphi(t)}}\,|H(g_1, t)|, \qquad -1 < t < 1,$$

and, using [31, (4)],

$$I_1 \ \le\ \int_{A_n} u(t)|H(g_1, t)| \ \le\ C \int_{-1}^{1} u(t)\log^+ u(t)\,dt \ <\ \infty.$$

In a similar way, we obtain

$$I_2 \ \le\ C \int_{-1}^{1} u(t)\log^+ u(t)\,dt \ <\ \infty.$$

Assume now that f is a bounded and measurable function, $u\sqrt{\varphi} \in L^p$ and $(u\sqrt{\varphi})^{-1} \in L^q$, $p^{-1} + q^{-1} = 1$, $1 < p < \infty$. We have

$$\begin{aligned}
&\|L_{n+1}(f - q^-)u\|^p_{L^p(A_n)} \\
&\quad\le \int_{A_n} \frac{(q^+ - q^-)(t)}{\sqrt{n\varphi(t)}}|H(E_{n+1}u^p\Gamma, t)|\,dt \\
&\qquad + \int_{A_n} \frac{(q^+ - q^-)(t)}{\sqrt{n\varphi(t)}}|E_{n+1}(t)|\,|H(u^p\Gamma, t)|\,dt \\
&\quad =:\ I_1 + I_2.
\end{aligned}$$

First, we use the Hölder inequality and obtain

$$I_1 \ \le\ \|(q^+ - q^-)u\|_p \ \left\|\frac{1}{u\sqrt{n\varphi}}\,H(E_{n+1}u^p\Gamma)\right\|_q .$$

Using (37), thereby taking $(u\sqrt{\varphi})^{-1}$ for the weight function both times, we obtain

$$\left\|\frac{1}{u\sqrt{n\varphi}}\,H(E_{n+1}u^p\Gamma)\right\|_q \ \le\ C\ \left\|\frac{E_{n+1}}{\sqrt{n\varphi(t)}}u^{p-1}\Gamma\right\|_q \ \le\ C\,\|L_{n+1}(f - q^-)u\|_p^{p-1}.$$

Finally, we use the same argument for I_2 and obtain

$$\|L_{n+1}(f - q^-)u\|_p \ \le\ C\,\|(q^+ - q^-)u\|_p.$$

The inequality (13) then follows recalling (38) and taking the infimum with respect to $q^{\pm}$. □

Proof of Theorem 3.3. Let $q^{\pm}$ be defined as in the previous proof. Let

$$f_n(x) = \begin{cases} f(\xi_1), & x < \xi_1, \\ f(x), & \xi_1 \le x \le \xi_{n+1}, \\ f(\xi_{n+1}), & \xi_{n+1} < x. \end{cases}$$

We have

$$\|[f - L_{n+1}f]u\|_p \ \le\ \|[f - f_n]u\|_p \ +\ \|[f_n - L_{n+1}f_n]u\|_p.$$

By [23], if $f \in AC_{\mathrm{loc}}$ and $f'\varphi^{2/p}u \in L^1$, we obtain

$$\|[f - f_n]u\|_p \ \le\ C\,\|f'\varphi^{2/p}u\|_{L^1(I_n')}.$$

We observe that f_n is a bounded and measurable function, and, using (13), we obtain

$$\|[f_n - L_{n+1} f_n]u\|_p \;\le\; C\,\tilde{\mathcal{E}}_n(f_n)_{u,p}.$$

By [8, Theorem 2.1] (see also [23]), we have

$$\tilde{\mathcal{E}}(f_n)_{u,p} \;\le\; \frac{C}{n}\,\|f'\varphi u\|_{L^p[\xi_1,\xi_{n+1}]}.$$

If $f'\varphi u \in L^p$, then

$$\|f'\varphi^{2/p} u\|_{L^1(I_n')} \;\le\; \frac{C}{n}\,\|f'\varphi u\|_p$$

(see [23]). Then,

$$\|[f - L_{n+1} f]u\|_p \;\le\; \frac{C}{n}\,\|f'\varphi u\|_p.$$

Now the theorem follows in a standard way. □

Proof of Theorem 3.4. Recalling Lemma 4.5, the proof of Theorem 3.4 follows the same line as the proof of Theorem 3.1. □

Proof of Theorem 3.5. Let $q^{\pm} \in \mathbb{P}_{2n}$ such that $q^- \le f \le q^+$. We have

$$\|[f - \mathcal{L}_{2n+1} f]u\|_p \;\le\; \|[f - q^-]u\|_p \;+\; \|\mathcal{L}_{2n+1}(f - q^-)u\|_p.$$

Using [30, (25)], Lemma 4.5 and the same argument as in the proof of Theorem 3.2, we obtain

$$\|\mathcal{L}_{2n+1}(f - q^-)u\|^p_{L^p(A_n)} \;\le\; C \int_{A_n} (q^+ - q^-)(t)|\pi(t)|\,dt,$$

where

$$\pi(t) = \int_{A_n} \frac{K_{2n+1}(x) - K_{2n+1}(t)}{x - t}\, u^p(x)\,\Lambda^{p-1}(x)\,dx,$$

$K_{2n+1} = P_n E_{n+1}$ and $\Lambda = \operatorname{sgn}(\mathcal{L}_{2n+1}(f - q^-))\,\mathcal{L}_{2n+1}(f - q^-)$. Assume $u \in L^p$. We recall the definition $u_-(x) = \prod_{\gamma_k<0} |t_k - x|^{\gamma_k}$ and $u_- \equiv 1$ if $\gamma_k \ge 0$, $k = 0, 1, \ldots, r$. Using the Hölder inequality, we have

$$\int_{A_n} (q^+ - q^-)(t)|\pi(t)|\,dt \;\le\; \|(q^+ - q^-)u_-\|_{L^p(A_n)}\,\|u_-^{-1}\pi\|_{L^q(A_n)}.$$

Using the definition of π, we now estimate

$$\begin{aligned}\|u_-^{-1}\pi\|_{L^q(A_n)} &\le \|u_-^{-1} H(K_{2n+1} u^p \Lambda^{p-1})\|_{L^q(A_n)} + \|u_-^{-1} K_{2n+1} H(u^p \Lambda^{p-1})\|_{L^q(A_n)} \\ &=:\; I_1 + I_2.\end{aligned}$$

Using (37) with $u_-^{-1} \le u^{-1}$, $u_-^{-1} \in L^q$ and $u \in L^p$, we obtain

$$I_1 \;\le\; C\,\|K_{2n+1}(u\Lambda)^{p-1}\|_q \;\le\; C\,\|\mathcal{L}_{2n+1}(f - q^-)u\|_p^{p-1}.$$

Similarly, we obtain

$$I_2 \;\le\; C\,\|\mathcal{L}_{2n+1}(f - q^-)u\|_p^{p-1}.$$

Adding the inequalities, (15) is proved.

Similarly, we prove (17). In fact, we have

$$\int_{A_n} (q^+ - q^-)(t)\,|\pi(t)|\,dt \;\le\; \|(q^+ - q^-)u\|_p \,+\, \|u^{-1}\pi\|_q,$$

and using again (37) with $v = u$, we obtain the result.

Finally, we assume $p = 1$ and $u \in L^1$. We obtain

$$\begin{aligned}\|\mathcal{L}_{2n+1}(f - q^-)u\|_{L^1(A_n)} &\le C \int_{A_n} (q^+ - q^-)(t)|\pi(t)|\,dt \\ &\le \|q^+ - q^-\|_{L^\infty(A_n)}\,\|\pi\|_{L^1(A_n)},\end{aligned}$$

where

$$\pi(t) = \int_{A_n} \frac{K_{2n+1}(x) - K_{2n+1}(t)}{x - t}\,u(x)\,dx.$$

Now

$$\|\pi\|_{L^1(A_n)} \le \int_{A_n} |H(K_{2n+1}u, t)|\,dt \;+\; \int_{A_n} |K_{2n+1}(t)|\,|H(u,t)|\,dt \;=:\; I_1 + I_2.$$

Using [31, (4)], we have

$$I_2 \;\le\; C \int_{A_n} |H(u,t)|\,dt \;\le\; C \int_{A_n} u(t)\log^+ u(t)\,dt.$$

Furthermore, we have

$$I_1 \;\le\; C \int_{A_n} |K_{2n+1}(t)u(t)|\,|H(g,t)|\,dt,$$

where $g = \operatorname{sgn} H(K_{2n+1}u)$. Using [31, (2)], we obtain

$$I_1 \;\le\; C \int_{A_n} u(t)\log^+ u(t)\,dt.$$

Now (16) follows. □

Proof of Theorem 3.6. The proof is identical to the proof of Theorem 3.3. □

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