ON SOME INEQUALITIES FOR THE INCOMPLETE GAMMA FUNCTION

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Abstract. Let $p \neq 1$ be a positive real number. We determine all real numbers $\alpha = \alpha(p)$ and $\beta = \beta(p)$ such that the inequalities

\[ [1 - e^{-\beta x^p}]^{1/p} < \frac{1}{\Gamma(1 + 1/p)} \int_0^x e^{-t^p} \, dt < [1 - e^{-\alpha x^p}]^{1/p} \]

are valid for all $x > 0$. And, we determine all real numbers $a$ and $b$ such that

\[ -\log(1 - e^{-ax^p}) \leq \int_{x}^{\infty} \frac{e^{-t}}{t} \, dt \leq -\log(1 - e^{-bx^p}) \]

hold for all $x > 0$.

1. Introduction

In 1955, J. T. Chu [1] presented sharp upper and lower bounds for the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$. He proved that the inequalities

\[(1.1) \quad [1 - e^{-rx^2}]^{1/2} \leq \text{erf}(x) \leq [1 - e^{-sx^2}]^{1/2}\]

are valid for all $x \geq 0$ if and only if $0 \leq r \leq 1$ and $s \geq 4/\pi$. The right-hand inequality of (1.1) (with $s = 4/\pi$) was proved independently by J. D. Williams (1946) and G. Pólya (1949); see [1].

An interesting survey on inequalities involving the complementary error function $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} \, dt$ and related functions is given in [4, pp. 177–181]. In particular, one can find inequalities for Mills’ ratio $e^{x^2/2} \int_{x}^{\infty} e^{-t^2/2} \, dt$, derived by several authors.

In 1959, W. Gautschi [3] provided upper and lower bounds for the more general expression

\[(1.2) \quad I_p(x) = e^{x^p} \int_x^{\infty} e^{-t^p} \, dt.\]

He established that the double-inequality

\[(1.3) \quad \frac{1}{2} (x^p + 2)^{1/p} - x < I_p(x) \leq c_p [(x^p + 1/c_p)^{1/p} - x]\]

(with $c_p = \Gamma(1 + 1/p)^{p/(p-1)}$) holds for all real numbers $p > 1$ and $x \geq 0$. It has been pointed out in [3] that the integral in (1.2) for $p = 3$ occurs in heat transfer problems, and for $p = 4$ in the study of electrical discharge through gases. We note

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that the integral $\int_{x}^{\infty} e^{-t^p} dt$ can be expressed in terms of the incomplete gamma function
\[ \Gamma(a, x) = \int_{x}^{\infty} t^{a-1} e^{-t} dt, \]
namely,
\[ \int_{x}^{\infty} e^{-t^p} dt = \frac{1}{p} \Gamma \left( \frac{1}{p}, x^p \right). \]
Gautschi [3] showed that the inequalities (1.3) can be used to derive bounds for the exponential integral $E_1(x) = \Gamma(0, x)$. Indeed, if $p$ tends to $\infty$, then (1.3) leads to
\[ \frac{1}{2} \log \left( 1 + \frac{2}{x} \right) \leq e^x E_1(x) \leq \log \left( 1 + \frac{1}{x} \right) \quad (0 < x < \infty). \]

It is the main purpose of this paper to establish new inequalities for $\int_{0}^{x} e^{-t^p} dt$ and $\int_{x}^{\infty} e^{-t^p} dt$. In Section 2 we present sharp upper and lower bounds for
\[ \frac{1}{\Gamma(1 + 1/p)} \int_{0}^{x} e^{-t^p} dt \quad \text{and} \quad \frac{1}{\Gamma(1 + 1/p)} \int_{x}^{\infty} e^{-t^p} dt, \]
which are valid not only for $p > 1$, but also for $p \in (0, 1)$. In particular, we obtain an extension of Chu’s double-inequality (1.1). Moreover, we provide sharp inequalities for the exponential integral $E_1(x)$. Finally, in Section 3 we compare our bounds with those given in (1.3) and (1.4).

2. Main results

First, we generalize the inequalities (1.1).

**Theorem 1.** Let $p \neq 1$ be a positive real number, and let $\alpha = \alpha(p)$ and $\beta = \beta(p)$ be given by
\[ \alpha = 1, \quad \beta = \left[ \Gamma(1 + 1/p) \right]^{-p}, \quad \text{if} \ 0 < p < 1, \]
and
\[ \alpha = \left[ \Gamma(1 + 1/p) \right]^{-p}, \quad \beta = 1, \quad \text{if} \ p > 1. \]
Then we have for all positive real $x$:
\[ [1 - e^{-\beta x^p}]^{1/p} \leq \frac{1}{\Gamma(1 + 1/p)} \int_{0}^{x} e^{-t^p} dt < [1 - e^{-\alpha x^p}]^{1/p}. \]

**Proof.** We have to show that the functions
\[ F_p(x) = \int_{0}^{x} e^{-t^p} dt - \Gamma(1 + 1/p)[1 - e^{-x^p}]^{1/p} \]
and
\[ G_p(x) = -\int_{0}^{x} e^{-t^p} dt + \Gamma(1 + 1/p)[1 - e^{-ax^p}]^{1/p} \quad (a = \left[ \Gamma(1 + 1/p) \right]^{-p}) \]
are both positive on $(0, \infty)$, if $p > 1$, and are both negative on $(0, \infty)$, if $0 < p < 1$.

First, we determine the sign of $F_p(x)$. Differentiation yields
\[ e^{x^p} \frac{\partial}{\partial x} F_p(x) = 1 - \Gamma(1 + 1/p)[L(z(x))]^{1-p/p}, \]
where
\[ L(z) = (z - 1)/\log(z) \quad \text{and} \quad z(x) = e^{-x^p}. \]

Setting \( f_p(x) = e^{xp} \frac{\partial}{\partial x} F_p(x) \), we obtain
\begin{equation}
\frac{\partial}{\partial x} f_p(x) = \frac{p-1}{p} \Gamma(1 + 1/p)(L(z(x)))^{-2+1/p} \frac{d}{dx} z(x) \frac{d}{dz} L(z) |_{z = z(x)}.
\end{equation}

Since
\[ \frac{d}{dz} L(z) = \frac{[\log(z) - 1 + 1/z]}{\log(z)^2} > 0 \quad (0 < z \neq 1) \]
\[ \frac{d}{dx} z(x) < 0, \]
we conclude from (2.2) that
\[ \frac{\partial}{\partial x} f_p(x) < 0, \quad \text{if } p > 1, \]
and
\[ \frac{\partial}{\partial x} f_p(x) > 0, \quad \text{if } 0 < p < 1. \]

If \( p > 1 \), then we have
\[ f_p(0) = 1 - \Gamma(1 + 1/p) > 0 \quad \text{and} \quad \lim_{x \to \infty} f_p(x) = -\infty, \]
which implies that there exists a number \( x_0 > 0 \) such that \( f_p(x) > 0 \) for \( x \in (0, x_0) \) and \( f_p(x) < 0 \) for \( x \in (x_0, \infty) \). Hence, the function \( x \mapsto F_p(x) \) is strictly increasing on \([0, x_0] \) and strictly decreasing on \([x_0, \infty) \). Since \( F_p(0) = \lim_{x \to \infty} F_p(x) = 0 \), we obtain \( F_p(x) > 0 \) for all \( x > 0 \).

If \( 0 < p < 1 \), then we have
\[ f_p(0) = 1 - \Gamma(1 + 1/p) < 0 \quad \text{and} \quad \lim_{x \to \infty} f_p(x) = 1. \]

This implies that there exists a number \( x_1 > 0 \) such that \( x \mapsto F_p(x) \) is strictly decreasing on \([0, x_1] \) and strictly increasing on \([x_1, \infty) \). From \( F_p(0) = \lim_{x \to \infty} F_p(x) = 0 \) we conclude that \( F_p(x) < 0 \) for all \( x > 0 \).

Next, we consider \( G_p(x) \). Differentiation leads to
\begin{equation}
e^{xp} \frac{\partial}{\partial x} G_p(x) = -1 + (y(x))^{1-1/a} [L(y(x))]^{(1-p)/p},
\end{equation}
where
\[ L(y) = (y - 1)/\log(y) \quad \text{and} \quad y(x) = e^{-ax^p} \]
with \( a = a(p) = [\Gamma(1 + 1/p)]^{-p} \). To determine the sign of \( \frac{\partial}{\partial x} G_p(x) \) we need the inequalities
\begin{equation}
0 < \left( 1 - \frac{1}{a(p)} \right) \frac{p}{p - 1} < \frac{1}{2} \quad \text{for } 0 < p \neq 1.
\end{equation}
The left-hand inequality of (2.4) is obviously true. A simple calculation reveals that the second inequality of (2.4) is equivalent to
\begin{equation}
(1 - x) \left[ \Gamma(x + 1) - \left( \frac{x + 1}{2} \right)^2 \right] > 0 \quad \text{for } 0 < x \neq 1.
\end{equation}
To establish (2.5) we define for $x > 0$:

$$g(x) = \log \Gamma(x + 1) - x \log \frac{x + 1}{2}.$$  

Then we have

$$\frac{d^2}{dx^2} g(x) = \frac{d}{dx} \psi(x + 1) - \frac{x + 2}{(x + 1)^2} = \sum_{n=2}^{\infty} \frac{1}{(x + n)^2} - \frac{1}{x + 1}$$

$$< \int_{1}^{\infty} \frac{dt}{(x + t)^2} - \frac{1}{x + 1} = 0.$$  

Thus, $g$ is strictly concave on $[0, \infty)$. Since $g(0) = g(1) = 0$, we conclude that $g$ is positive on $(0, 1)$ and negative on $(1, \infty)$. This implies (2.5).

Let $0 < r < 1/2$; we define for $y \in (0, 1)$:

$$h_r(y) = y^r \log(y)/(y - 1).$$

Then we get

$$(y - 1)^2 y^{1-r} \frac{\partial}{\partial y} h_r(y) = [(r - 1)y - y] \log(y) + y - 1$$

$$= \varphi_r(y), \text{ say.}$$

Since

$$\frac{\partial^2}{\partial y^2} \varphi_r(y) = \frac{r - 1}{y^2} \left[y - \frac{r}{1-r}\right],$$

it follows that $\varphi_r$ is strictly convex on $(0, \frac{r}{1-r})$ and strictly concave on $(\frac{r}{1-r}, 1)$. From $\lim_{y \to 0} \varphi_r(y) = \infty$,

$$\varphi_r(1) = \frac{\partial}{\partial y} \varphi_r(y)|_{y=1} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial y^2} \varphi_r(y)|_{y=1} = 2r - 1 < 0,$$

we conclude that there exists a number $y_0 \in (0, 1)$ such that $\varphi_r$ is positive on $(0, y_0)$ and negative on $(y_0, 1)$. This implies that $y \mapsto h_r(y)$ is strictly increasing on $(0, y_0)$ and strictly decreasing on $(y_0, 1)$. Since $\lim_{y \to 0} h_r(y) = 0$ and $\lim_{y \to 1} h_r(y) = 1$, we conclude that there exists a number $y_1 \in (0, 1)$ such that $h_r(y) < 1$ for $y \in (0, y_1)$ and $h_r(y) > 1$ for $y \in (y_1, 1)$. The function $y(x) = e^{-ax^p}$ is strictly decreasing on $[0, \infty)$. Since $y(0) = 1$ and $\lim_{x \to \infty} y(x) = 0$, there exists a number $x^* > 0$ such that

$$y_1 < y(x) < 1 \quad \text{for} \quad x \in (0, x^*),$$

and

$$0 < y(x) < y_1 \quad \text{for} \quad x \in (x^*, \infty).$$

Hence, we have:

If $0 < x < x^*$, then $h_r(y(x)) > 1$, and, if $x^* < x$, then $h_r(y(x)) < 1$. We set $r = (1 - \frac{1}{a(p)}) \frac{p}{p-1}$; then we obtain from (2.3) that

$$h_r(y(x)) = \left[1 + e^{x^*} \frac{\partial}{\partial x} G_p(x)\right]^{p/(p-1)}.$$  

Therefore, if $p > 1$, then

$$\frac{\partial}{\partial x} G_p(x) > 0 \quad \text{for} \quad x \in (0, x^*) \quad \text{and} \quad \frac{\partial}{\partial x} G_p(x) < 0 \quad \text{for} \quad x \in (x^*, \infty);$$
and, if $0 < p < 1$, then
\[ \frac{\partial}{\partial x} G_p(x) < 0 \quad \text{for } x \in (0, x^*) \quad \text{and} \quad \frac{\partial}{\partial x} G_p(x) > 0 \quad \text{for } x \in (x^*, \infty). \]

Since $G_p(0) = \lim_{x \to \infty} G_p(x) = 0$, we conclude that
\[ G_p(x) > 0 \quad \text{for } x \in (0, \infty), \quad \text{if } p > 1, \]
and
\[ G_p(x) < 0 \quad \text{for } x \in (x^*, \infty), \quad \text{if } 0 < p < 1. \]

This completes the proof of Theorem 1.

\textbf{Remark.} It is natural to ask whether the double-inequality (2.1) can be refined by replacing $\alpha$ by a positive number which is smaller than
\[ \max\{1, [\Gamma(1 + 1/p)]^{-p}\} = \begin{cases} 1, & \text{if } 0 < p < 1, \\ [\Gamma(1 + 1/p)]^{-p}, & \text{if } p > 1, \end{cases} \]
or by replacing $\beta$ by a number which is greater than
\[ \min\{1, [\Gamma(1 + 1/p)]^{-p}\} = \begin{cases} [\Gamma(1 + 1/p)]^{-p}, & \text{if } 0 < p < 1, \\ 1, & \text{if } p > 1. \end{cases} \]

We show that the answer is “no”! Let $\alpha > 0$ be a real number such that the right-hand inequality of (2.1) holds for all $x > 0$. This implies that the function
\[ \tilde{F}_p(x) = \int_0^x e^{-tp} \, dt - \Gamma(1 + 1/p)[1 - e^{-\alpha x^{1/p}}]^{1/p} \]
is negative on $(0, \infty)$. Since $\tilde{F}_p(0) = 0$, we obtain
\[ \frac{\partial}{\partial x} \tilde{F}_p(x)|_{x=0} = 1 - \alpha^{1/p} \Gamma(1 + 1/p) \leq 0, \]
which leads to $\alpha \geq [\Gamma(1 + 1/p)]^{-p}$. If $\alpha \in (0, 1)$, then we conclude from
\[ \lim_{x \to \infty} e^{x^{1/p}} \frac{\partial}{\partial x} \tilde{F}_p(x) = -\infty \]
that there exists a number $\overline{x} > 0$ such that $x \mapsto \tilde{F}_p(x)$ is negative and strictly decreasing on $[\overline{x}, \infty)$. This contradicts $\lim_{x \to \infty} \tilde{F}_p(x) = 0$. Thus, we have $\alpha \geq \max\{1, [\Gamma(1 + 1/p)]^{-p}\}$.

Next, we suppose that $\beta > 0$ is a real number such that the first inequality of (2.1) is valid for all $x > 0$. This implies
\[ \tilde{G}_p(x) = -\int_0^x e^{-tp} \, dt + \Gamma(1 + 1/p)[1 - e^{-\beta x^{1/p}}]^{1/p} < 0 \]
for all $x > 0$. Since $\tilde{G}_p(0) = 0$, we obtain
\[ \frac{\partial}{\partial x} \tilde{G}_p(x)|_{x=0} = \beta^{1/p} \Gamma(1 + 1/p) - 1 \leq 0, \]
which yields $\beta \leq [\Gamma(1 + 1/p)]^{-p}$. If $\beta > 1$, then we get
\[ \lim_{x \to \infty} e^{x^{1/p}} \frac{\partial}{\partial x} \tilde{G}_p(x) = -1. \]
This implies that there exists a number \( \tilde{x} > 0 \) such that \( x \mapsto \tilde{G}_p(x) \) is negative and strictly decreasing on \([\tilde{x}, \infty)\). This contradicts \( \lim_{x \to \infty} \tilde{G}_p(x) = 0 \). Hence, we get \( \beta \leq \min\{1, [\Gamma(1 + 1/p)]^{-p}\} \).

As an immediate consequence of Theorem 1, the Remark, and the representation \( \int_x^\infty e^{-tv} \, dt = \Gamma(1 + 1/p) - \int_0^x e^{-tv} \, dt \), we obtain the following sharp bounds for the ratio \( \int_x^\infty e^{-tv} \, dt/ \int_0^\infty e^{-tv} \, dt \).

**Corollary.** Let \( p \neq 1 \) be a positive real number. The inequalities

\[
(2.6) \quad 1 - [1 - e^{-ax}]^{1/p} < \frac{1}{\Gamma(1 + 1/p)} \int_x^\infty e^{-vp} \, dt < 1 - [1 - e^{-bx}]^{1/p}
\]

are valid for all positive \( x \) if and only if

\[
\alpha \geq \max\{1, [\Gamma(1 + 1/p)]^{-p}\} \quad \text{and} \quad 0 \leq \beta \leq \min\{1, [\Gamma(1 + 1/p)]^{-p}\}.
\]

Now, we provide new upper and lower bounds for the exponential integral \( E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt \).

**Theorem 2.** The inequalities

\[
(2.7) \quad - \log(1 - e^{-ax}) \leq E_1(x) \leq - \log(1 - e^{-bx})
\]

are valid for all positive real \( x \) if and only if

\[
a \geq e^C \quad \text{and} \quad 0 < b \leq 1,
\]

where \( C = 0.5772\ldots \) is Euler’s constant.

**Proof.** The function \( t \mapsto - \log(1 - e^{-tx}) \) \( (x > 0) \) is strictly decreasing on \((0, \infty)\).

Therefore, it suffices to prove \((2.7)\) with \( a = e^C \) and \( b = 1 \). Let \( p > 1 \); from \((2.6)\) with \( \alpha = [\Gamma(1 + 1/p)]^{-p} \), \( \beta = 1 \), and \( x \) instead of \( x^p \), we obtain

\[
\Gamma(1/p)[1 - (1 - e^{-ax})^{1/p}] < \int_x^\infty t^{-1+1/p} e^{-t} \, dt < \Gamma(1/p)[1 - (1 - e^{-x})^{1/p}].
\]

If \( p \) tends to \( \infty \), then we get

\[
- \log(1 - e^{-ax}) \leq E_1(x) \leq - \log(1 - e^{-x})
\]

with \( a = e^C \).

We assume that there exists a real number \( b > 1 \) such that \( E_1(x) \leq - \log(1 - e^{-bx}) \) holds for all \( x > 0 \). Since

\[
e^x E_1(x) = \sum_{k=1}^n (-1)^{k-1} (k-1)! x^{-k} + r_n(x) \quad (x > 0)
\]

with

\[
|r_n(x)| < n! x^{-n-1}
\]

(see [2, pp. 673–674]), we obtain

\[
e^x x \log(1 - e^{-bx}) \leq -1 - x r_1(x).
\]

If we let \( x \) tend to \( \infty \), then inequality \((2.8)\) implies \( 0 \leq -1 \). Hence, we have \( b \leq 1 \).

Using the representation

\[
E_1(x) = -C - \log(x) - \sum_{n=1}^\infty (-1)^n \frac{x^n}{n! n} \quad (x > 0)
\]
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(see [2, p. 674]), we conclude from the left-hand inequality of (2.7) that

$$\log \frac{e}{1 - e^{-ax}} \leq -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n! n}.$$ 

If \( x \) tends to 0, then we obtain

$$\log(1/a) \leq -C$$ or \( a \geq e^C \).

The proof of Theorem 2 is complete. \( \square \)

3. Concluding remarks

In the final part of this paper we want to compare the bounds for the integrals \( \int_{x}^{\infty} e^{-\nu^p} \, dt \) \((p > 1)\) and \( \int_{x}^{\infty} \frac{1}{t} \, dt \) which are given in (1.3), (2.6) and (1.4), (2.7), respectively. First, we consider the bounds for \( \int_{x}^{\infty} e^{-\nu^p} \, dt \).

We define

$$R_p(x) = \Gamma(1 + 1/p) \left\{ 1 - (1 - e^{-\alpha x^p})^{1/p} \right\} - \frac{e^{-x^p}}{2} (x^p + 2)^{1/p} - x$$

with

$$\alpha = \left[ \Gamma(1 + 1/p) \right]^{-p} \quad \text{and} \quad p > 1.$$ 

Then we have

$$R_p(0) = \Gamma(1 + 1/p) - 2^{-1+1/p} > 0,$$

$$\lim_{x \to \infty} R_p(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} e^{x^p} \frac{\partial}{\partial x} R_p(x) = 1.$$ 

This implies

$$R_p(x) > 0 \quad \text{for all sufficiently small} \ x > 0,$$

and

$$R_p(x) < 0 \quad \text{for all sufficiently large} \ x.$$ 

Let

$$S_p(x) = \Gamma(1 + 1/p) \left\{ 1 - (1 - e^{-x^p})^{1/p} \right\} - ce^{-x^p} (x^p + 1/c)^{1/p} - x$$

with

$$c = \left[ \Gamma(1 + 1/p) \right]^{p/(p-1)} \quad \text{and} \quad p > 1.$$ 

From \( S_p(0) = 0 \),

$$\lim_{x \to 0} \frac{\partial}{\partial x} S_p(x) = \left[ \Gamma(1 + 1/p) \right]^{p/(p-1)} - \Gamma(1 + 1/p) < 0,$$

$$\lim_{x \to \infty} S_p(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} e^{x^p} \frac{\partial}{\partial x} S_p(x) = -\infty,$$

we conclude

$$S_p(x) < 0 \quad \text{for all sufficiently small} \ x > 0,$$

and

$$S_p(x) > 0 \quad \text{for all sufficiently large} \ x.$$ 

Hence, for small \( x > 0 \) the bounds for \( \int_{x}^{\infty} e^{-\nu^p} \, dt \) \((p > 1)\) which are given in (2.6) are better than those presented in (1.3), whereas for large values of \( x \) the opposite is true.
Next, we compare the bounds for the exponential integral $E_1(x)$. First, we show that for all $x > 0$ the upper bound given in (1.4) is better than the upper bound given in (2.7). This means, we have to prove that

$$(3.1) \quad e^{-x} \log(1 + 1/x) < -\log(1 - e^{-x})$$

for all $x > 0$. Using the extended Bernoulli inequality

$$(1 + z)^t \geq 1 + tz \quad (t > 1; z > -1)$$

(see [4, p. 34]), and the elementary inequality $e^t > 1 + t$ ($t \neq 0$), we obtain for $x > 0$:

$$\left(1 + \frac{1}{e^x - 1}\right)^{e^x} \geq 1 + \frac{e^x}{e^x - 1} = 1 + \frac{1}{1 - e^{-x}} > 1 + \frac{1}{x},$$

which leads immediately to (3.1).

Finally, we compare the lower bounds for $E_1(x)$ given in (2.7) and (1.4). Let

$$T(x) = \frac{e^{-x}}{2} \log(1 + 2/x) + \log(1 - e^{-ax})$$

with $a = e^C$. Since $\lim_{x \to 0} T(x) = -\infty$, we obtain $T(x) < 0$ for all sufficiently small $x$. And, from

$$\lim_{x \to \infty} T(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} e^{ax} \frac{d}{dx} T(x) = -\infty,$$

we conclude that $T(x) > 0$ for all sufficiently large $x$. Thus, for small $x > 0$ the lower bound for $E_1(x)$ which is given in (2.7) is better than the bound given in (1.4), while for large values of $x$ the latter is better.

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**References**


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