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ON SOME INEQUALITIES FOR THE INCOMPLETE GAMMA FUNCTION

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Abstract. Let $p \neq 1$ be a positive real number. We determine all real numbers $\alpha = \alpha(p)$ and $\beta = \beta(p)$ such that the inequalities

$$[1 - e^{-\beta x^p}]^{1/p} < \frac{1}{\Gamma(1+1/p)} \int_0^x e^{-t^p}\, dt < [1 - e^{-\alpha x^p}]^{1/p}$$

are valid for all $x > 0$. And, we determine all real numbers a and b such that

$$-\log(1 - e^{-ax}) \leq \int_x^\infty \frac{e^{-t}}{t}\, dt \leq -\log(1 - e^{-bx})$$

hold for all $x > 0$.

1. Introduction

In 1955, J. T. Chu [1] presented sharp upper and lower bounds for the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. He proved that the inequalities

$$[1 - e^{-rx^2}]^{1/2} \leq \operatorname{erf}(x) \leq [1 - e^{-sx^2}]^{1/2} \tag{1.1}$$

are valid for all $x \geq 0$ if and only if $0 \leq r \leq 1$ and $s \geq 4/\pi$. The right-hand inequality of (1.1) (with $s = 4/\pi$) was proved independently by J. D. Williams (1946) and G. Pólya (1949); see [1].

An interesting survey on inequalities involving the complementary error function $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$ and related functions is given in [4, pp. 177–181]. In particular, one can find inequalities for Mills' ratio $e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$, derived by several authors.

In 1959, W. Gautschi [3] provided upper and lower bounds for the more general expression

$$I_p(x) = e^{x^p} \int_x^\infty e^{-t^p}\, dt. \tag{1.2}$$

He established that the double-inequality

$$\frac{1}{2}[(x^p + 2)^{1/p} - x] < I_p(x) \leq c_p[(x^p + 1/c_p)^{1/p} - x] \tag{1.3}$$

(with $c_p = [\Gamma(1 + 1/p)]^{p/(p-1)}$) holds for all real numbers $p > 1$ and $x \geq 0$. It has been pointed out in [3] that the integral in (1.2) for $p = 3$ occurs in heat transfer problems, and for $p = 4$ in the study of electrical discharge through gases. We note

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that the integral $\int_x^\infty e^{-t^p}\,dt$ can be expressed in terms of the incomplete gamma function

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}\,dt,$$

namely,

$$\int_x^\infty e^{-t^p}\,dt=\frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right).$$

Gautschi [3] showed that the inequalities (1.3) can be used to derive bounds for the exponential integral $E_1(x)=\Gamma(0,x)$. Indeed, if p tends to ∞, then (1.3) leads to

$$\frac{1}{2}\log\left(1+\frac{2}{x}\right)\le e^x E_1(x)\le\log\left(1+\frac{1}{x}\right)\qquad(0<x<\infty). \tag{1.4}$$

It is the main purpose of this paper to establish new inequalities for $\int_0^x e^{-t^p}\,dt$ and $\int_x^\infty e^{-t^p}\,dt$. In Section 2 we present sharp upper and lower bounds for

$$\frac{1}{\Gamma(1+1/p)}\int_0^x e^{-t^p}\,dt \quad\text{and}\quad \frac{1}{\Gamma(1+1/p)}\int_x^\infty e^{-t^p}\,dt,$$

which are valid not only for $p>1$, but also for $p\in(0,1)$. In particular, we obtain an extension of Chu's double-inequality (1.1). Moreover, we provide sharp inequalities for the exponential integral $E_1(x)$. Finally, in Section 3 we compare our bounds with those given in (1.3) and (1.4).

2. Main results

First, we generalize the inequalities (1.1).

Theorem 1. *Let $p\ne1$ be a positive real number, and let $\alpha=\alpha(p)$ and $\beta=\beta(p)$ be given by*

$$\alpha=1,\quad \beta=[\Gamma(1+1/p)]^{-p},\quad \text{if } 0<p<1,$$

and

$$\alpha=[\Gamma(1+1/p)]^{-p},\quad \beta=1,\quad \text{if } p>1.$$

Then we have for all positive real x:

$$[1-e^{-\beta x^p}]^{1/p}<\frac{1}{\Gamma(1+1/p)}\int_0^x e^{-t^p}\,dt<[1-e^{-\alpha x^p}]^{1/p}. \tag{2.1}$$

Proof. We have to show that the functions

$$F_p(x)=\int_0^x e^{-t^p}\,dt-\Gamma(1+1/p)[1-e^{-x^p}]^{1/p}$$

and

$$G_p(x)=-\int_0^x e^{-t^p}\,dt+\Gamma(1+1/p)[1-e^{-ax^p}]^{1/p}\qquad(a=[\Gamma(1+1/p)]^{-p})$$

are both positive on $(0,\infty)$, if $p>1$, and are both negative on $(0,\infty)$, if $0<p<1$.

First, we determine the sign of $F_p(x)$. Differentiation yields

$$e^{x^p}\frac{\partial}{\partial x}F_p(x)=1-\Gamma(1+1/p)[L(z(x))]^{(1-p)/p},$$

where

$$L(z) = (z-1)/\log(z) \quad \text{and} \quad z(x) = e^{-x^p}.$$

Setting $f_p(x) = e^{x^p} \frac{\partial}{\partial x} F_p(x)$, we obtain

$$\frac{\partial}{\partial x} f_p(x) = \frac{p-1}{p} \Gamma(1+1/p)(L(z(x)))^{-2+1/p} \frac{d}{dx} z(x) \frac{d}{dz} L(z)|_{z=z(x)}. \tag{2.2}$$

Since

$$\frac{d}{dz} L(z) = [\log(z) - 1 + 1/z]/(\log(z))^2 > 0 \qquad (0 < z \neq 1)$$

and

$$\frac{d}{dx} z(x) < 0,$$

we conclude from (2.2) that

$$\frac{\partial}{\partial x} f_p(x) < 0, \quad \text{if } p > 1,$$

and

$$\frac{\partial}{\partial x} f_p(x) > 0, \quad \text{if } 0 < p < 1.$$

If $p > 1$, then we have

$$f_p(0) = 1 - \Gamma(1+1/p) > 0 \quad \text{and} \quad \lim_{x \to \infty} f_p(x) = -\infty,$$

which implies that there exists a number $x_0 > 0$ such that $f_p(x) > 0$ for $x \in (0, x_0)$ and $f_p(x) < 0$ for $x \in (x_0, \infty)$. Hence, the function $x \mapsto F_p(x)$ is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, \infty)$. Since $F_p(0) = \lim_{x \to \infty} F_p(x) = 0$, we obtain $F_p(x) > 0$ for all $x > 0$.

If $0 < p < 1$, then we have

$$f_p(0) = 1 - \Gamma(1+1/p) < 0 \quad \text{and} \quad \lim_{x \to \infty} f_p(x) = 1.$$

This implies that there exists a number $x_1 > 0$ such that $x \mapsto F_p(x)$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, \infty)$. From $F_p(0) = \lim_{x \to \infty} F_p(x) = 0$ we conclude that $F_p(x) < 0$ for all $x > 0$.

Next, we consider $G_p(x)$. Differentiation leads to

$$e^{x^p} \frac{\partial}{\partial x} G_p(x) = -1 + (y(x))^{1-1/a} [L(y(x))]^{(1-p)/p}, \tag{2.3}$$

where

$$L(y) = (y-1)/\log(y) \quad \text{and} \quad y(x) = e^{-ax^p}$$

with $a = a(p) = [\Gamma(1+1/p)]^{-p}$. To determine the sign of $\frac{\partial}{\partial x} G_p(x)$ we need the inequalities

$$0 < \left(1 - \frac{1}{a(p)}\right) \frac{p}{p-1} < \frac{1}{2} \quad \text{for } 0 < p \neq 1. \tag{2.4}$$

The left-hand inequality of (2.4) is obviously true. A simple calculation reveals that the second inequality of (2.4) is equivalent to

$$(1-x) \left[\Gamma(x+1) - \left(\frac{x+1}{2}\right)^x\right] > 0 \quad \text{for } 0 < x \neq 1. \tag{2.5}$$

To establish (2.5) we define for $x > 0$:

$$g(x) = \log \Gamma(x+1) - x \log \frac{x+1}{2}.$$

Then we have

$$\begin{aligned} \frac{d^2}{dx^2} g(x) &= \frac{d}{dx}\psi(x+1) - \frac{x+2}{(x+1)^2} = \sum_{n=2}^{\infty} \frac{1}{(x+n)^2} - \frac{1}{x+1} \\ &< \int_1^{\infty} \frac{dt}{(x+t)^2} - \frac{1}{x+1} = 0. \end{aligned}$$

Thus, g is strictly concave on $[0, \infty)$. Since $g(0) = g(1) = 0$, we conclude that g is positive on $(0,1)$ and negative on $(1, \infty)$. This implies (2.5).

Let $0 < r < 1/2$; we define for $y \in (0,1)$:

$$h_r(y) = y^r \log(y)/(y-1).$$

Then we get

$$\begin{aligned} (y-1)^2 y^{1-r} \frac{\partial}{\partial y} h_r(y) &= [(r-1)y - r]\log(y) + y - 1 \\ &= \varphi_r(y), \quad \text{say.} \end{aligned}$$

Since

$$\frac{\partial^2}{\partial y^2}\varphi_r(y) = \frac{r-1}{y^2}\left[y - \frac{r}{1-r}\right],$$

it follows that φ_r is strictly convex on $(0, \frac{r}{1-r})$ and strictly concave on $(\frac{r}{1-r}, 1)$. From $\lim_{y \to 0} \varphi_r(y) = \infty$,

$$\varphi_r(1) = \frac{\partial}{\partial y}\varphi_r(y)|_{y=1} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\varphi_r(y)|_{y=1} = 2r - 1 < 0,$$

we conclude that there exists a number $y_0 \in (0,1)$ such that φ_r is positive on $(0, y_0)$ and negative on $(y_0, 1)$. This implies that $y \mapsto h_r(y)$ is strictly increasing on $(0, y_0)$ and strictly decreasing on $(y_0, 1)$. Since $\lim_{y \to 0} h_r(y) = 0$ and $\lim_{y \to 1} h_r(y) = 1$, we conclude that there exists a number $y_1 \in (0,1)$ such that $h_r(y) < 1$ for $y \in (0, y_1)$ and $h_r(y) > 1$ for $y \in (y_1, 1)$. The function $y(x) = e^{-ax^p}$ is strictly decreasing on $[0, \infty)$. Since $y(0) = 1$ and $\lim_{x \to \infty} y(x) = 0$, there exists a number $x^* > 0$ such that

$$y_1 < y(x) < 1 \quad \text{for } x \in (0, x^*),$$

and

$$0 < y(x) < y_1 \quad \text{for } x \in (x^*, \infty).$$

Hence, we have:

If $0 < x < x^*$, then $h_r(y(x)) > 1$, and, if $x^* < x$, then $h_r(y(x)) < 1$. We set $r = (1 - \frac{1}{a(p)})\frac{p}{p-1}$; then we obtain from (2.3) that

$$h_r(y(x)) = \left[1 + e^{x^p} \frac{\partial}{\partial x} G_p(x)\right]^{p/(p-1)}.$$

Therefore, if $p > 1$, then

$$\frac{\partial}{\partial x} G_p(x) > 0 \quad \text{for } x \in (0, x^*) \quad \text{and} \quad \frac{\partial}{\partial x} G_p(x) < 0 \quad \text{for } x \in (x^*, \infty);$$

and, if $0 < p < 1$, then

$$\frac{\partial}{\partial x} G_p(x) < 0 \quad \text{for } x \in (0, x^*) \quad \text{and} \quad \frac{\partial}{\partial x} G_p(x) > 0 \quad \text{for } x \in (x^*, \infty).$$

Since $G_p(0) = \lim_{x\to\infty} G_p(x) = 0$, we conclude that

$$G_p(x) > 0 \quad \text{for } x \in (0, \infty), \text{ if } p > 1,$$

and

$$G_p(x) < 0 \quad \text{for } x \in (0, \infty), \text{ if } 0 < p < 1.$$

This completes the proof of Theorem 1. □

Remark. It is natural to ask whether the double-inequality (2.1) can be refined by replacing α by a positive number which is smaller than

$$\max\{1, [\Gamma(1 + 1/p)]^{-p}\} = \begin{cases} 1, & \text{if } 0 < p < 1, \\ [\Gamma(1 + 1/p)]^{-p}, & \text{if } p > 1, \end{cases}$$

or by replacing β by a number which is greater than

$$\min\{1, [\Gamma(1 + 1/p)]^{-p}\} = \begin{cases} [\Gamma(1 + 1/p)]^{-p}, & \text{if } 0 < p < 1, \\ 1, & \text{if } p > 1. \end{cases}$$

We show that the answer is "no"! Let $\alpha > 0$ be a real number such that the right-hand inequality of (2.1) holds for all $x > 0$. This implies that the function

$$\widetilde{F}_p(x) = \int_0^x e^{-t^p}\, dt - \Gamma(1 + 1/p)[1 - e^{-\alpha x^p}]^{1/p}$$

is negative on $(0, \infty)$. Since $\widetilde{F}_p(0) = 0$, we obtain

$$\frac{\partial}{\partial x} \widetilde{F}_p(x)|_{x=0} = 1 - \alpha^{1/p}\Gamma(1 + 1/p) \le 0,$$

which leads to $\alpha \ge [\Gamma(1 + 1/p)]^{-p}$. If $\alpha \in (0, 1)$, then we conclude from

$$\lim_{x\to\infty} e^{x^p} \frac{\partial}{\partial x} \widetilde{F}_p(x) = -\infty$$

that there exists a number $\overline{x} > 0$ such that $x \mapsto \widetilde{F}_p(x)$ is negative and strictly decreasing on $[\overline{x}, \infty)$. This contradicts $\lim_{x\to\infty} \widetilde{F}_p(x) = 0$. Thus, we have $\alpha \ge \max\{1, [\Gamma(1 + 1/p)]^{-p}\}$.

Next, we suppose that $\beta > 0$ is a real number such that the first inequality of (2.1) is valid for all $x > 0$. This implies

$$\widetilde{G}_p(x) = -\int_0^x e^{-t^p}\, dt + \Gamma(1 + 1/p)[1 - e^{-\beta x^p}]^{1/p} < 0$$

for all $x > 0$. Since $\widetilde{G}_p(0) = 0$, we obtain

$$\frac{\partial}{\partial x} \widetilde{G}_p(x)|_{x=0} = \beta^{1/p}\Gamma(1 + 1/p) - 1 \le 0,$$

which yields $\beta \le [\Gamma(1 + 1/p)]^{-p}$. If $\beta > 1$, then we get

$$\lim_{x\to\infty} e^{x^p} \frac{\partial}{\partial x} \widetilde{G}_p(x) = -1.$$

This implies that there exists a number $\tilde{x} > 0$ such that $x \mapsto \widetilde{G}_p(x)$ is negative and strictly decreasing on $[\tilde{x}, \infty)$. This contradicts $\lim_{x\to\infty} \widetilde{G}_p(x) = 0$. Hence, we get $\beta \le \min\{1, [\Gamma(1+1/p)]^{-p}\}$.

As an immediate consequence of Theorem 1, the Remark, and the representation $\int_x^\infty e^{-t^p}\,dt = \Gamma(1+1/p) - \int_0^x e^{-t^p}\,dt$, we obtain the following sharp bounds for the ratio $\int_x^\infty e^{-t^p}\,dt / \int_0^\infty e^{-t^p}\,dt$.

Corollary. *Let $p \neq 1$ be a positive real number. The inequalities*

$$1 - [1 - e^{-\alpha x^p}]^{1/p} < \frac{1}{\Gamma(1+1/p)} \int_x^\infty e^{-t^p}\,dt < 1 - [1 - e^{-\beta x^p}]^{1/p} \tag{2.6}$$

are valid for all positive x if and only if

$$\alpha \ge \max\{1, [\Gamma(1+1/p)]^{-p}\} \quad \textit{and} \quad 0 \le \beta \le \min\{1, [\Gamma(1+1/p)]^{-p}\}.$$

Now, we provide new upper and lower bounds for the exponential integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$.

Theorem 2. *The inequalities*

$$-\log(1 - e^{-ax}) \le E_1(x) \le -\log(1 - e^{-bx}) \tag{2.7}$$

are valid for all positive real x if and only if

$$a \ge e^C \quad \textit{and} \quad 0 < b \le 1,$$

where $C = 0.5772\ldots$ is Euler's constant.

Proof. The function $t \mapsto -\log(1 - e^{-tx})$ $(x > 0)$ is strictly decreasing on $(0, \infty)$. Therefore, it suffices to prove (2.7) with $a = e^C$ and $b = 1$. Let $p > 1$; from (2.6) with $\alpha = [\Gamma(1+1/p)]^{-p}$, $\beta = 1$, and x instead of x^p, we obtain

$$\Gamma(1/p)[1 - (1 - e^{-\alpha x})^{1/p}] < \int_x^\infty t^{-1+1/p} e^{-t}\,dt < \Gamma(1/p)[1 - (1 - e^{-x})^{1/p}].$$

If p tends to ∞, then we get

$$-\log(1 - e^{-ax}) \le E_1(x) \le -\log(1 - e^{-x})$$

with $a = e^C$.

We assume that there exists a real number $b > 1$ such that

$$E_1(x) \le -\log(1 - e^{-bx})$$

holds for all $x > 0$. Since

$$e^x E_1(x) = \sum_{k=1}^n (-1)^{k-1}(k-1)!\, x^{-k} + r_n(x) \qquad (x > 0)$$

with

$$|r_n(x)| < n!\, x^{-n-1}$$

(see [2, pp. 673–674]), we obtain

$$e^x x \log(1 - e^{-bx}) \le -1 - x r_1(x). \tag{2.8}$$

If we let x tend to ∞, then inequality (2.8) implies $0 \le -1$. Hence, we have $b \le 1$.

Using the representation

$$E_1(x) = -C - \log(x) - \sum_{n=1}^\infty (-1)^n \frac{x^n}{n!\, n} \qquad (x > 0)$$

(see [2, p. 674]), we conclude from the left-hand inequality of (2.7) that

$$\log \frac{x}{1-e^{-ax}} \le -C - \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!\, n}.$$

If x tends to 0, then we obtain

$$\log(1/a) \le -C \quad \text{or} \quad a \ge e^C.$$

The proof of Theorem 2 is complete. □

3. Concluding remarks

In the final part of this paper we want to compare the bounds for the integrals $\int_x^\infty e^{-t^p}\,dt$ $(p>1)$ and $\int_x^\infty \frac{e^{-t}}{t}\,dt$ which are given in (1.3), (2.6) and (1.4), (2.7), respectively. First, we consider the bounds for $\int_x^\infty e^{-t^p}\,dt$. We define

$$R_p(x) = \Gamma(1+1/p)\{1-[1-e^{-\alpha x^p}]^{1/p}\} - \frac{e^{-x^p}}{2}[(x^p+2)^{1/p} - x]$$

with

$$\alpha = [\Gamma(1+1/p)]^{-p} \quad \text{and} \quad p>1.$$

Then we have

$$R_p(0) = \Gamma(1+1/p) - 2^{-1+1/p} > 0,$$

$$\lim_{x\to\infty} R_p(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} e^{x^p}\frac{\partial}{\partial x}R_p(x) = 1.$$

This implies

$$R_p(x) > 0 \quad \text{for all sufficiently small } x>0,$$

and

$$R_p(x) < 0 \quad \text{for all sufficiently large } x.$$

Let

$$S_p(x) = \Gamma(1+1/p)\{1-[1-e^{-x^p}]^{1/p}\} - ce^{-x^p}[(x^p+1/c)^{1/p} - x]$$

with

$$c = [\Gamma(1+1/p)]^{p/(p-1)} \quad \text{and} \quad p>1.$$

From $S_p(0)=0$,

$$\lim_{x\to 0}\frac{\partial}{\partial x}S_p(x) = [\Gamma(1+1/p)]^{p/(p-1)} - \Gamma(1+1/p) < 0,$$

$$\lim_{x\to\infty} S_p(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} e^{x^p}\frac{\partial}{\partial x}S_p(x) = -\infty,$$

we conclude

$$S_p(x) < 0 \quad \text{for all sufficiently small } x>0,$$

and

$$S_p(x) > 0 \quad \text{for all sufficiently large } x.$$

Hence, for small $x>0$ the bounds for $\int_x^\infty e^{-t^p}\,dt$ $(p>1)$ which are given in (2.6) are better than those presented in (1.3), whereas for large values of x the opposite is true.

Next, we compare the bounds for the exponential integral $E_1(x)$. First, we show that for all $x > 0$ the upper bound given in (1.4) is better than the upper bound given in (2.7). This means, we have to prove that

$$e^{-x}\log(1+1/x) < -\log(1-e^{-x}) \tag{3.1}$$

for all $x > 0$. Using the extended Bernoulli inequality

$$(1+z)^t \geq 1+tz \qquad (t>1; z>-1)$$

(see [4, p. 34]), and the elementary inequality $e^t > 1+t$ $(t \neq 0)$, we obtain for $x > 0$:

$$\left(1+\frac{1}{e^x-1}\right)^{e^x} \geq 1+\frac{e^x}{e^x-1} = 1+\frac{1}{1-e^{-x}} > 1+\frac{1}{x},$$

which leads immediately to (3.1).

Finally, we compare the lower bounds for $E_1(x)$ given in (2.7) and (1.4). Let

$$T(x) = \frac{e^{-x}}{2}\log(1+2/x) + \log(1-e^{-ax})$$

with $a = e^C$. Since $\lim_{x\to 0} T(x) = -\infty$, we obtain $T(x) < 0$ for all sufficiently small x. And, from

$$\lim_{x\to\infty} T(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} e^{ax}\frac{d}{dx}T(x) = -\infty,$$

we conclude that $T(x) > 0$ for all sufficiently large x. Thus, for small $x > 0$ the lower bound for $E_1(x)$ which is given in (2.7) is better than the bound given in (1.4), while for large values of x the latter is better.

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