COMPUTATION OF RELATIVE CLASS NUMBERS
OF CM-FIELDS

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Abstract. It was well known that it is easy to compute relative class numbers
of abelian CM-fields by using generalized Bernoulli numbers (see Theorem
4.17 in Introduction to cyclotomic fields by L. C. Washington, Grad. Texts
in Math., vol. 83, Springer-Verlag, 1982). Here, we provide a technique for
computing the relative class number of any CM-field.

1. Statement of the results

Proposition 1. Let \( n \geq 1 \) be an integer and \( \alpha > 1 \) be real. Set
\[
K_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} x^k,
\]
and
\[
\frac{A^2}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} f_n(s) ds.
\]
Then, it holds
\[
0 \leq K_n(A) \leq 2P_n(nA^2/n)e^{-nA^2/n} \leq 2n \exp(-A^2/n).
\]

Theorem 2. Let \( N \) be a totally imaginary number field of degree \( 2n \) which is a
quadratic extension of a totally real number field \( N^{+} \) of degree \( n \), i.e. \( N \) is a CM-
field. Let \( \omega_N \) be the number of roots of unity in \( N \), \( Q_N \in \{1, 2\} \) be the Hasse unit
index of \( N \), and \( d_N, \zeta_N \) and \( d_{N^+}, \zeta_{N^+} \) be the absolute values of the discriminants
and the Dedekind zeta functions of \( N \) and \( N^+ \), respectively. Let \( \chi_{N/N^+} \) be the
quadratic character associated with the quadratic extension \( N/N^+ \) and let \( \phi_k \) be
the coefficients of the Dirichlet series \( (\zeta_N/\zeta_{N^+})(s) = L(s, \chi_{N/N^+}) = \sum_{k \geq 1} \phi_k k^{-s}, \)
\( \Re(s) > 1 \). Set \( A_{N/N^+} = \sqrt{d_N/n} \log A_{N/N^+} \).

We have
\[
h_N = Q_N \omega_N (2\pi)^n \frac{d_N}{d_{N^+}} \sum_{k \geq 1} \phi_k \frac{K_n(k/A_{N/N^+})}{k},
\]
and according to (3) this series (4) is absolutely convergent. Moreover, set
\[
B(N) \overset{\text{def}}{=} A_{N/N^+} \left( \frac{\lambda}{n} \log A_{N/N^+} \right)^{n/2}.
\]
Then, if \( \lambda > 1 \) and \( n \) are given, then the limit of \(|h^{-N}_N - h^{-N}_N(M)|\) as \( A_{\mathbb{N}}/\mathbb{N}_+ \) approaches infinity is equal to 0, where \( h^{-N}_N(M) \) is the approximation of the relative class number obtained by disregarding in the series occurring in (4) the indices \( k > M \geq B(N) \).

For example, if \( N \) of degree \( m = 2n \) is the narrow Hilbert class field of a real quadratic number field \( L \) of discriminant \( d_L \), we have

\[
B(N) = \left( \frac{\lambda}{4\pi} \right)^{m/4} d_L^{m/8} \log^{m/4}(d_L/\pi^2).
\]

The following Proposition 3 explains how we compute the numerical values of the function \( A \mapsto K_n(A) \) according to its series expansion:

**Proposition 3.** Take \( A > 0 \). It holds

\[
K_n(A) = 1 + \pi^{n/2}A + 2A^2 \sum_{m \geq 0} \text{Res}_{s = -m}(f_n).
\]

This series is absolutely convergent and for any integer \( M \geq 0 \) we have

\[
\left| 2A^2 \sum_{m > M} \text{Res}_{s = -m}(f_n) \right| \leq \pi^{n/2}A^{2M+3} \frac{(M+1)(M!/2)^n}{(M+1)(M!/2)^n}.
\]

Finally, the following Proposition 4 explains how to compute recursively the values of the residues \( \text{Res}_{s = -m}(f_n) \) occurring in (6):

**Proposition 4.** We have

\[
\text{Res}_{s = -m}(f_n) = -(-1)^nm A^{2m} \sum_{i = -n}^{-1} 2^{-1-i} h_i(m)((2m + 1)^i + (2m + 2)^i)
\]

where the \( h_i(m) \)'s are computed recursively from the \( h_i(0) \)'s by using

\[
h_i(m+1) = \sum_{j = -n}^{i} h_j(m) \frac{b_{i-j}}{(m+1)^{i-j}} \quad \text{and} \quad \sum_{j = -n}^{-1} h_j(0) s^j + O(1) = \Gamma^n(s) A^{-2s},
\]

where \( b_k = C_{n+k-1}^m = ((k + n - 1)!/k!(n-1)!). \) Thus, if

\[
\Gamma^n(s + 1) = \sum_{i = 0}^{n-1} h_i s^i + O(s^n),
\]

then

\[
h_{j-n}(0) = \sum_{i = 0}^{j} \frac{(-2 \log A)^i}{i!} h_{j-i} \quad (0 \leq j \leq n-1).
\]

For proving these results, obvious questions of convergence of series and integrals, and questions of inversions of integrals and summations will not be gone into.

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1Note the misprint in the formula given in [Lou 2].
2. Introduction

Prior to the method we have developed here, the only general method for computing the relative class number of any CM-field was that developed by T. Shintani (see [Oka 1] and [Oka 2] for examples of actual relative class number computations using Shintani’s ideas). However, his method requires the knowledge of a great deal of information on the maximal totally real subfield $N^+$. In particular, it requires the knowledge of a system of fundamental units of the group of totally positive units of $N$. However, what makes the concept of CM-field an attractive one is that the relative class number formula

$$h_{N} = \frac{Q_{N}w_{N}}{(2\pi)^{n}} \sqrt{\frac{d_{N}}{d_{N^+}}} \text{Res}_{s=1}(\zeta_{N}) = \frac{Q_{N}w_{N}}{(2\pi)^{n}} \left( \sqrt{\frac{d_{N}}{d_{N^+}}} \right) L(1, \chi_{N/N^+})$$

enables us to get lower bounds on relative class numbers and solve class number and class group problems for CM-fields precisely because (12) does not involve any regulator (see [Lou-Oka] and [LOO]). Thus, the reader may possibly feel dissatisfied that he should have to know beforehand a good grasp of the unit group of $N^+$ before he can compute $h_{N}$, whereas (12) gives an expression for $h_{N}$ which does not involve units. The reader may now possibly feel satisfied that this paper shows how using (12) he indeed gets an efficient method for computing $h_{N}$ provided that he only knows how to compute the decomposition of any rational prime into a product of prime ideals of $N$. The key point of our method is to establish the holomorphic continuation of $s \mapsto (\zeta_{N}/\zeta_{N^+})(s) = L(s, \chi_{N/N^+})$ in the same way Riemann did in the case of the Riemann zeta function (by using Mellin transformation) and to evaluate the resulting series at $s = 1$ (see section 4).

Finally, we note that the results of this paper are better than those of [Lou 3]. Indeed, $B(N)$ in (5) is $n^{n/2}$-fold better than the one we gave in [Lou 3]. Moreover, our proof of (3) (in section 3) is more satisfactory and elegant than the one we gave in [Lou 3].

3. Proof of Proposition 1

We use:

Lemma 5. Let $\alpha > 1$ be real. We have

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{u^{s}}{2s-1} ds = \begin{cases} 0 & \text{if } u < 1, \\ i\pi \sqrt{u} & \text{if } u > 1; \end{cases} \quad \text{and} \quad \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{u^{s}}{2s-2} ds = \begin{cases} 0 & \text{if } u < 1, \\ i\pi u & \text{if } u > 1. \end{cases}$$

Now, using

$$\Gamma^n(s) = \int \int e^{-T(y)} y^{s} dy$$

where the multiple integral ranges over $(y_1, \cdots, y_n) \in (\mathbb{R}_+)^n$ and where we set $y = y_1 y_2 \cdots y_n$ and $\text{Tr}(y) = y_1 + y_2 + \cdots + y_n$, leads to

$$K_n(A) = \frac{A^2}{i\pi} \int \int e^{-T(y)} \left( \int_{\alpha-i\infty}^{\alpha+i\infty} (y/A^2)^{s} \left( \frac{1}{2s-1} + \frac{1}{2s-2} \right) \right) dy.$$
Using Lemma 5 yields
\[
K_n(A) = A^2 \int \int_{y \geq A^2} \left( \sqrt{y/A^2} + (y/A^2) \right) e^{-\text{Tr}(y)/y} \, dy \leq 2 \int \int_{y \geq A^2} e^{-\text{Tr}(y)/y} \, dy.
\]

For example, we get \(K_1(A) \leq 2e^{-A^2}\). Now, using the arithmetic-geometric mean inequality yields that \(\{ (y_1, \cdots, y_n) ; y \geq A^2 \} \) is included in \(\{ (y_1, \cdots, y_n) ; \text{Tr}(y) \geq nA^{2/n} \}\), which yields
\[
K_n(A) \leq 2 \int \int_{\text{Tr}(y) \geq nA^{2/n}} e^{-\text{Tr}(y)/y} \, dy.
\]

Then, the following easily proved Lemma 6 provides us with the desired result.

We finally notice that we get a shorter and more satisfactory proof of [Lou 3, Proposition 1]:

**Lemma 6.** Set \(P_n(x) = \sum_{k=0}^{n-1} x^k/k!\). Then
\[
P_n(\alpha)e^{-\alpha} = \int \int_{(y_1, \cdots, y_n) \in \mathbb{R}^n_+} e^{-\text{Tr}(y)/y} \, dy \leq n \int_{\alpha/n}^{+\infty} e^{-y} \, dy = ne^{-\alpha/n}.
\]

**Proof.** Use
\[
\{ (y_1, \cdots, y_n) \in \mathbb{R}^n_+ ; \text{Tr}(y) \geq \alpha \}
\]
\[
\subseteq \bigcup_{i=1}^{n} \left\{ (y_1, \cdots, y_n) ; y_i \geq \frac{\alpha}{n} \text{ and } y_j \geq 0 \text{ for } j \neq i \right\}.
\]

\(\square\)

4. PROOF OF THEOREM 2

Let \(K\) be a number field of degree \(n = r_1 + 2r_2\), where \(r_1\) is the number of real places of \(K\) and \(r_2\) the number of complex places of \(K\). Let \(\zeta_K\) and \(\text{Reg}_K\) be the Dedekind zeta function and regulator of \(K\). We set
\[
A_K = 2^{-r_2} d^{1/2} \pi^{-(r_1+2r_2)/2},
\]
(13)
\[
\lambda_K = \frac{2^{r_1} \mu_K \text{Reg}_K}{w_K} \quad \text{where } w_K \text{ is the number of roots of unity in } K,
\]
\[
F_K(s) = A_K \Gamma \left( \frac{s}{2} \right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s).
\]

Hence, \(F_K\) a simple pole at \(s = 1\) with residue \(\lambda_K\), and \(F_K(s) = F_K(1-s)\).

From now on, we let \(N\) be a CM-field of degree \(2n\), i.e. \(N\) is a totally imaginary number field of degree \(2n\) which is a quadratic extension of a totally real number field \(N^+\) of degree \(n\). Define the \(\phi_k\)'s by :
\[
\Phi_{N/N^+}(s) = \frac{\zeta_N}{\zeta_{N^+}}(s) = \sum_{k \geq 1} \phi_k k^{-s} \quad (\Re(s) > 1).
\]

Then, \( (\zeta_N/\zeta_{N^+})(s) = L(s, \chi_{N/N^+}) \) yields
\[
(14) \quad \phi_k = \sum_{N_{N^+}^+ = k} \chi_N/N^+(I)
\]

where \(I\) ranges over the integral ideals of \(N^+\) of norm \(k\). Now,
\[
\Phi_{N/N^+} = \zeta_N/\zeta_{N^+} \quad \text{and} \quad \Psi_{N/N^+} = F_N/F_{N^+}
\]
are entire and $\Psi_{N/N^+}(s) = \Psi_{N/N^+}(1 - s)$. Notice that

$$
\Psi_{N/N^+}(1) = \frac{\lambda_N}{\lambda_{N^+}} = \frac{h_N^-}{Q_N w_N}
$$

(15)

where $Q_N \in \{1, 2\}$ is the Hasse unit index of $N$ (see [Wa, Th. 4.16]). Since

$$
\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right),
$$

using (13) for $N$ and $N^+$ leads to

$$
\Psi_{N/N^+}(s) = c_{N/N^+} A_{N/N^+} \Gamma^n \left(\frac{s+1}{2}\right) \Phi_{N/N^+}(s)
$$

(16)

where

$$
c_{N/N^+} = \frac{1}{(4\pi)^{n/2}} \quad \text{and} \quad A_{N/N^+} = \sqrt{d_N / \pi^d_{N^+}}.
$$

Note that

$$
c_{N/N^+} A_{N/N^+} = \frac{1}{(2\pi)^n} \sqrt{\frac{d_N}{d_{N^+}}}
$$

(17)

Set

$$
\hat{\Psi}_{N/N^+}(x) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Psi_{N/N^+}(s)x^{-s}ds \quad (\alpha > 1),
$$

(18)

i.e., $\hat{\Psi}_{N/N^+}$ is the Mellin transform of the function $\Psi_{N/N^+}$. Using (18) and (16) yields

$$
\hat{\Psi}_{N/N^+}(x) = c_{N/N^+} \sum_{k \geq 1} \phi_k H_n \left(\frac{kx}{A_{N/N^+}}\right) \quad (x > 0),
$$

(19)

with

$$
H_n(x) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma^n \left(\frac{s+1}{2}\right) x^{-s}ds
$$

(20)

$$
= \frac{1}{i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma^n(S)x^{1-2S} dS \quad (x > 0 \text{ and } \alpha > 0).
$$

(21)

Now, we move the integral (18) to the line $\Re(s) = 1 - \alpha$. Since $\Psi_{N/N^+}$ is entire, we do not pick up any residue. Then, we use the functional equation

$$
\Psi_{N/N^+}(s) = \Psi_{N/N^+}(1 - s)
$$

satisfied by $\Psi_{N/N^+}$ to come back to the line $\Re(s) = \alpha$. We get

$$
x \Psi_{N/N^+}(x) = \hat{\Psi}_{N/N^+}(1/x) \quad (x > 0).
$$

Mellin’s inversion formula and (21) yield

$$
\Psi_{N/N^+}(s) = \int_0^\infty \hat{\Psi}_{N/N^+}(x) x^{s-1} dx = \int_1^\infty \hat{\Psi}_{N/N^+}(x) \left\{x^{s-1} + x^{-s}\right\} dx.
$$

(22)
Moreover, yield (23)

\[ g, \frac{d^2}{dx^2} \]

By using (22), (19) and (20) we thus get:

\[
\Psi_{N/N^+}(s) = c_{N/N^+} \sum_{k \geq 1} \phi_k \int_{i\pi}^{a+i\infty} \left( \frac{1}{2\pi i} \int_{a-i\infty}^{x} \frac{kx}{A_{N/N^+}} \Gamma^n(S) \left\{ x^{s-1} + x^{-s} \right\} dS \right) dx
\]

\[
= c_{N/N^+} \sum_{k \geq 1} \frac{1}{i\pi} \int_{a-i\infty}^{x} \Gamma^n(S) \left( \frac{kx}{A_{N/N^+}} \right)^{1-2S} \left\{ x^{s-1} + x^{-s} \right\} dS
\]

\[
= c_{N/N^+} \sum_{k \geq 1} \frac{1}{i\pi} \int_{a-i\infty}^{x} \Gamma^n(S) \left( k/A_{N/N^+} \right)^{1-2S} \left( \frac{1}{2S-2} + \frac{1}{2S+2} \right) dS,
\]

and the following yields (4):

\[
h^-_N = Q_{N/N^+} \Psi_{N/N^+}(1) = Q_{N/N^+} \Psi_{N/N^+}(0)
\]

\[
= Q_{N/N^+} \sum_{k \geq 1} \phi_k \frac{1}{i\pi} \int_{a-i\infty}^{x} \Gamma^n(S) \left( k/A_{N/N^+} \right)^{1-2S} \left( \frac{1}{2S-2} + \frac{1}{2S+2} \right) dS
\]

\[
= Q_{N/N^+} \sum_{k \geq 1} \frac{1}{i\pi} \int_{a-i\infty}^{x} \Gamma^n(S) \left( k/A_{N/N^+} \right)^{1-2S} \left( \frac{1}{2S-2} + \frac{1}{2S+2} \right) dS,
\]

Now, we prove the assertion below (5).
To start with we quote some elementary facts we will need.
1) We have

\[ P_n(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} \leq \sum_{k=0}^{n-1} \frac{1}{k!} \leq e^{x-1} \quad (x \geq 1). \]

2) The derivative of

\[ g(x) = x^{2n-2} e^{-nx^{2/n}} \]

is

\[ g'(x) = \frac{1}{n} \left( (2n-2) - 2nx^{2/n} \right) x^{n-2} e^{-nx^{2/n}} \]

and we have \( g'(x) \leq 0 \) if \( x \geq 1 \) and

\[ |g'(x)| \leq 2xe^{-nx^{2/n}} \quad (x \geq 1). \]

Moreover,

\[ g''(x) = \frac{1}{n^2} \left( 4n^2x^{4/n} - (6n^2 - 4n)x^{2/n} + (2n^2 - 6n + 4) \right) x^{-2/n} e^{-nx^{2/n}}, \]

the second derivative of \( g \), satisfies \( g''(x) \geq 0 \) if \( x \geq 2^{n/2} \). Note that (3) and (24) yield

\[ K_n(x) \leq 2e^{n-1} g(x) \quad (x \geq 1). \]
3) If \( g(x) \geq 0, g'(x) \leq 0 \) and \( g''(x) \geq 0 \) on \([\alpha, +\infty],\) then \( \alpha \leq a \leq b \) implies
\[
0 \leq g(a) - g(b) \leq (a - b)g'(a).
\]

(27)

4) If \( A((n + 1)/2)^{n/2} \geq 1, \) then the derivative of
\[
h(x) = x \log^n(cx)e^{-n(x/A)^{2/n}}
\]
is
\[
h'(x) = (n - (2(x/A)^{2/n} - 1) \log(cx)) \log^{n-1}(cx)e^{-n(x/A)^{2/n}}
\]
and we have \( h'(x) \leq 0 \) provided that \( x \geq A((n + 1)/2)^{n/2} \) and \( x \geq 1, \) hence provided that \( x \geq A((n + 1)/2)^{n/2} \) if \( A((n + 1)/2)^{n/2} \geq 1. \)

Now, we set \( A = A_{N^+}, S_n(k) = \sum_{i=1}^{k} \frac{d_n(i)}{i} \) where \( d_n(i) \) is the number of ways of writing \( i \) as an ordered product of \( n \) positive integers, and
\[
R_M = \sum_{k > M} \frac{\phi_k}{k} K_n(k/A).
\]

We want an upper bound on \( R_M. \) We note that (14) yields \( |\phi_k| \leq d_n(k). \) Moreover,
\[
S_n(k) = \sum_{i=1}^{k} \frac{d_n(i)}{i} \leq \left( \sum_{i=1}^{k} \frac{1}{i} \right)^n \leq \log^n(ek).
\]

Thus, we have
\[
|R_M| \leq \sum_{k > M} \frac{d_n(k)}{k} K_n(k/A)
\]
\[
\leq 2en^{n-1} \sum_{k > M} (S_n(k) - S_n(k - 1))g(k/A) \quad \text{if } M \geq A
\]
(by using (26))

\[
\leq 2en^{n-1} \sum_{k > M} S_n(k)(g(k/A) - g((k + 1)/A))
\]
\[
\leq \frac{2en^{n-1}}{A} \sum_{k > M} S_n(k)g'(k/A) \quad \text{if } M \geq 2^{n/2} A
\]
(by using (27))

\[
\leq \frac{4en^{n-1}}{A^2} \sum_{k > M} k \log^n(ek)e^{-n(k/A)^{2/n}}
\]
(by using (25))

\[
= \frac{4en^{n-1}}{A^2} \sum_{k > M} h(k) \leq \frac{4en^{n-1}}{A^2} \int_{M}^{\infty} h(x)dx \quad \text{if } M \geq \left( \frac{n + 1}{2} \right)^{n/2} A \geq 1
\]
(by using (28)).
Now, we set \( B = (eA)^{2/n} \) and we change the variable by setting \( x = Ay^{n/2} \). We get
\[
|R_M| \leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^{\infty} y^n \log^n(By)e^{-ny} \frac{dy}{y}.
\]
Since \( H(y) = y^{n+1} \log^n(By)e^{-ny} \) decreases on \( [(M/A)^{2/n}, +\infty] \) if \( M \geq (2n+2)^{n/2} A \geq e^{(n/2)-1} \) (since its derivative
\[
H'(y) = ((n + 1 - ny) \log(By) + n)y^{n-1}(By)e^{-ny}
\]
satisfies \( H'(y) \leq 0 \) if \( y \geq (2n + 2)/n \) and \( B \geq e^{(n/2)-1} \), then we have the following explicit upper bound:
\[
|R_M| \leq 2e \left( \frac{n^2}{2} G((M/A)^{2/n}) \right)^n \text{ where } G(y) = y \log(By)e^{-y} \text{ and } B = (eA)^{2/n}.
\]

Now, we choose \( M \approx B(N) = A \left( \frac{\lambda}{n} \log A \right)^{n/2} \) and note that
\[
G \left( \frac{\lambda}{n} \log A \right) = O_n \left( A^{-\lambda/n} \log^2 A \right)
\]
yields the desired result:
\[
|h_M - h_N(M)| = \frac{Q(N)w_N}{2n_{\pi n/2}} A |R_M| = O_n \left( \frac{\log^2 A}{A^{\lambda-1}} \right).
\]

5. Proof of Proposition 3

Let \( M \geq 0 \) be a given integer. Shifting the integral (2) to the left to the line \( \Re(s) = -M - \frac{1}{2} \), we pick a residue at \( s = 1 \), a residue at \( s = 1/2 \), and a residue \( \text{Res}_{s=-m}(f_n) \) at each nonpositive integer \( -m \leq 0 \). Hence, by using \( \Gamma(1/2) = \sqrt{\pi} \) we get
\[
K_n(A) = 1 + \pi^{n/2} A + 2A^2 \sum_{m=0}^{M} \text{Res}_{s=-m}(f_n) + \frac{A^2}{i\pi} \int_{-M-\frac{1}{2}-i\infty}^{M-\frac{1}{2}+i\infty} f_n(s) ds.
\]

Now, it is well known that for any nonnegative integer \( l \geq 0 \) we have
\[
\left| \Gamma \left( \frac{2l + 1}{2} + it \right) \right|^2 = \frac{\pi}{\text{ch}(\pi t)} \prod_{k=0}^{l-1} \left| \frac{2k + 1}{2} + it \right|^2,
\]
where \( \text{ch}(x) = (e^x + e^{-x})/2 \). Hence, using the functional equation \( \Gamma(s)\Gamma(1 - s) = \pi/\sin(\pi s) \) leads to
\[
\left| \Gamma \left( \frac{-2l + 1}{2} + it \right) \right|^2 = \frac{\pi}{\text{ch}(\pi t)} \prod_{k=0}^{l} \left| \frac{2k + 1}{2} + it \right|^{-2},
\]
and
If we set
\[ f_n(-M - \frac{1}{2} + it) \]
then we get
\[ \left| f_n(-M - \frac{1}{2} + it) \right| \leq \left( \frac{\pi}{\text{ch}(\pi t)} \right)^{n/2} \frac{A^{2M+1}}{(\prod_{k=0}^{M} \left| \frac{2k+1}{2} + it \right|)^n} \left( \frac{1}{|2M + 2 + it|} + \frac{1}{|2M + 3 + it|} \right) \]
\[ \leq \frac{1}{(\text{ch}(\pi t))^{n/2}} \frac{\pi^{n/2} A^{2M+1}}{(M!2^n) 2M + 2}. \]

Set
\[ c_n = \int_{-\infty}^{+\infty} \frac{dt}{(\text{ch}(\pi t))^{n/2}} = \frac{2}{\pi} \int_{0}^{1} \frac{2}{(u + u^{-1})^{n/2}} \frac{du}{u} \]
(note that the sequence \((c_n)_{n \geq 0}\) decreases, that \(c_1 = \frac{4\sqrt{2}}{\pi} \int_{0}^{1} \frac{dt}{\sqrt{v^2 + 1}} \leq 4\sqrt{2}/\pi \) and \(c_2 = 1\). Then,
\[ \left| \frac{A^2}{i\pi} \int_{-M - \frac{1}{2} - i\infty}^{-M - \frac{1}{2} + i\infty} f_n(s) ds \right| \leq \frac{c_n}{\pi} \frac{\pi^{n/2} A^{2M+3}}{(M+1)(M!2^n)}. \]

Note that the greater the value of \(n\), the faster the series (6) converges.

6. Proof of Proposition 4

We have
\[ \text{Res}_{s=-m}(f_n) \]
\[ = -A^{2m} \text{Res}_{s=0} \left( s \mapsto \Gamma^n(-m + s)A^{-2s} \left( \frac{1}{2m + 1 - 2s} + \frac{1}{2m + 2 - 2s} \right) \right). \]

If we set
\[ \Gamma^n(-m + s) = \sum_{i=-n}^{-1} a_i(m) s^i + O(1), \]
then we get
\[ \text{Res}_{s=-m}(f_n) = -A^{2m} \sum_{i=-n}^{-1} a_i(m) 2^{-1-i} \left( (2m + 1)^i + (2m + 2)^i \right). \]

Now, \(\Gamma(s) = \frac{1}{s} \Gamma(s+1)\) yields
\[ \sum_{i=-n}^{-1} a_i(m+1) s^i + O(1) = (-1)^n (m + 1 - s)^{-n} \left( \sum_{j=-n}^{-1} a_j(m) s^j + O(1) \right) \]
and
\[ (m + 1 - s)^{-n} = \frac{1}{(m+1)^n} \sum_{k=0}^{n-1} C_{k+n-1}^{n-1} \frac{s^k}{(m+1)^k} + O(s^n) \]
yields
\[ a_i(m+1) = \frac{(-1)^n}{(m+1)^n} \sum_{j=-n}^{i} \frac{a_j(m)}{(m+1)^{i-j}} C_{i-j+n-1}^{n-1}. \]

Thus, in order to simplify the recursion relation (36), we define
\[ h_i(m) = (-1)^n (m!)^n a_i(m). \]
Then, using (35) yields (8), and using (34) and (36) yields (9). Note that (10) makes it easy to compute the numerical values of the $h_i$’s by using Maple, for example.

7. Examples of relative class numbers computations

In order to use (4) to compute relative class numbers, it remains to explain how we compute the $\phi_k$’s. Since

$$\phi_k = \sum_{N_{N+N}(I) = k} \chi_{N/N\uparrow}(I)$$

(see (14)), then $k \mapsto \phi_k$ is multiplicative and we only have to explain how we compute the $\phi_{p^m}$ where $p$ is prime and $m \geq 1$. We will only explain this when $N$ is normal over $\mathbb{Q}$. In that case, let $e$ and $f$ be the inertia and residual degrees of $p$ in $N_+$. Set $g = n/(ef)$. Then in $\mathbb{N}^+$ we have $(p) = (P_1 \cdots P_g)^e$ and

$$\chi_{N/N\uparrow}(P_1) = \cdots = \chi_{N/N\uparrow}(P_g),$$

and we let $\epsilon_p$ be the common value of these $g$ symbols. Now, $N_{N+N}(I) = p^m$ if and only if $I = \prod_{i=1}^g P_i^{e_i}$ with $\sum_{i=1}^g e_i = m$. Set

$$C_j^i = \frac{i!}{j!(i-j)!}.$$

Since the equation $\sum_{i=1}^g e_i = K$ has $C_{g-1}^{g-1} K + g - 1$ solutions in nonnegative integers $e_i$, we easily get

$$(37) \quad \phi_{p^m} = \begin{cases} 0 & \text{if } f \text{ does not divide } m, \\ \epsilon_p^k C_{k+g-1}^{g-1} & \text{if } f \text{ divides } m \text{ and } m = kf. \end{cases}$$

This formula (37) makes it easy to compute the $\phi_{p^m}$. We refer the reader to [Lou 1], [Lou 2], [Lou 3], [Lou-Oka] and [LOO] for actual computations of relative class numbers.

References


