DENSITY OF CARMICHAEL NUMBERS WITH THREE PRIME FACTORS

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Abstract. We get an upper bound of $O(x^{5/14+o(1)})$ on the number of Carmichael numbers $\leq x$ with exactly three prime factors.

1. Introduction

A Carmichael number is a composite number $n$ which satisfies the condition $a^n \equiv a \mod n$ for every integer $a$. The smallest Carmichael number is 561. The Carmichael numbers have many interesting properties. For example, it is known that they are square-free and the product of at least three primes [5]. The reader may consult [4], [7], [8], [11] for more on Carmichael numbers.

The problem of proving the existence of infinitely many Carmichael numbers was a long-standing open problem until it was solved recently, by Alford, Granville and Pomerance [1]. They also gave a lower bound for the number of Carmichael numbers less than a given number $x$. Let $C(x)$ denote the number of Carmichael numbers up to $x$. They showed that $C(x) > x^{2/7}$ for all sufficiently large $x$.

Let $C_k(x)$ denote the number of Carmichael numbers up to $x$ with $k$ prime factors where $k \geq 3$. It is an open problem to show that the function $C_3(x)$ is unbounded. It is not known whether any of the functions $C_k(x)$ is unbounded. Pomerance et al. [9] proved that $C_3(x) = O(x^{2/3})$. Damgård et al. [3] improved this to $C_3(x) \leq (1/4)x^{1/2}(\log x)^{11/4}$ for all $x \geq 1$. An unpublished estimate of $O(x^{2/5+o(1)})$ for $C_3(x)$ was obtained by S. W. Graham. We show that for sufficiently large $x$, $C_3(x) = O(x^{5/14+o(1)})$. Granville (see [8]) has conjectured that $C_k(x) = x^{1/k+o_k(x)}$ for $x \to \infty$. Our upper bound for $C_3(x)$ comes very close to his conjectured value.

2. Proof of our bound

We state our result on the upper bound for $C_3(x)$ and give its proof. The proof is very similar to that in Damgård et al. [3].

Theorem 2.1. Let $C_3(x)$ denote the number of Carmichael numbers up to $x$ with exactly three prime factors. Then, for all sufficiently large $x$ we have $C_3(x) = O(x^{5/14+o(1)})$.

Proof. If $n$ is a Carmichael number with three prime factors $p, q, r$ with $2 < p < q < r$, then $n - 1 \equiv 0 \mod p - 1, n - 1 \equiv 0 \mod q - 1, n - 1 \equiv 0 \mod r - 1$. 

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Let \( g = \gcd(p - 1, q - 1, r - 1) \) and \( a, b, c \) be such that \( p - 1 = ga, q - 1 = gb, r - 1 = gc \); then \( a < b < c \). The congruences given above imply that \( gbc + b + c \equiv 0 \mod a, gac + a + c \equiv 0 \mod b \) and \( gab + a + b \equiv 0 \mod c \). These three congruences can be replaced by the single congruence \( g(ab + ac + bc) + a + b + c \equiv 0 \mod abc \) by observing that \( a, b, c \) are pair-wise coprime. This is true because \( \gcd(a, b, c) = 1 \) and \( c \equiv 0 \mod \gcd(a, b), b \equiv 0 \mod \gcd(a, c), a \equiv 0 \mod \gcd(b, c) \) implies that \( \gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1 \). Hence, if \( a, b, c \) are given, then \( g \) is determined modulo \( abc \).

We count the number \( N \) of quadruples \( (g, a, b, c) \) which satisfy the above conditions and \( g^3 abc \leq x \). Thus \( C_3(x) \leq N \). We write \( N = N_1 + N_2 + N_3 \) where \( N_1 \) is the number of quadruples \( (a, b, c) \) such that \( g > abc, N_2 \) is the number of quadruples \( (a, b, c) \) such that \( G < g \leq abc \) where \( G = x^{3/14}, N_3 \) is the number of quadruples \( (a, b, c) \) such that \( g \leq G \) and \( g \leq abc \) where \( G \) is as above.

**Estimate for \( N_1 \)**

If \( (a, b, c) \) are given, then the number of \( g \) with \( g^3 abc \leq x, g \) in a particular residue class modulo \( abc \) and \( g > abc \) is at most \( (x/abc)^{1/3}/abc \), which is \( x^{1/3}/(abc)^{4/3} \). Hence

\[
N_1 \leq \sum_{a < b < c} \frac{x^{1/3}}{(abc)^{4/3}} < \frac{\zeta^3(4/3)x^{1/3}}{6}
\]

where \( \zeta \) is the Riemann zeta function. Thus \( N_1 = O(x^{1/3}) \).

**Estimate for \( N_2 \)**

For each coprime triple \( (a, b, c) \) there is at most one \( g \) that satisfies the condition \( g(ab + ac + bc) + a + b + c \equiv 0 \mod abc \) and \( g \leq abc \). If \( g > G \) and \( g^3 abc \leq x \), then \( abc \leq x/G^3 \). Thus \( N_2 \) is at most the number of triples \( (a, b, c) \) with \( a < b < c \) and \( abc \leq x/G^3 \). Hence,

\[
N_2 \leq \sum_{1 \leq a < x^{1/3}/G} \sum_{a < b < (x/abc)^{1/2}} \sum_{b < x/abG^3} 1
\]

\[
< \sum_a \sum_b \frac{x}{abG^3} \left( 1 + \ln \left( \frac{x^{1/3}}{aG^3} \right) \right) \left( \ln \left( \frac{x}{G^3} \right) \right) \left( \frac{x}{6G^3} \ln(x) \right)^2 < O(x^{5/14+o(1)}), \text{ since } G = x^{3/14}.
\]

Thus \( N_2 = O(x^{5/14+o(1)}) \).

**Estimate for \( N_3 \)**

In this case \( g \leq G \) and \( abc \) where \( G = x^{3/14} \). Let \( g(ab + bc + ac) + a + b + c = \lambda abc \) where \( \lambda \geq 1 \) is a positive integer. Then \( (\lambda a - g)bc = ga(b+c) + a + b + c \). We note that \( 6abc \geq g(ab + bc + ac) + a + b + c = \lambda abc \) implies that \( \lambda a \leq 6g \). We break the range for \( g, a, b \) as \( G_1 \leq g \leq 2G_1, A \leq a \leq 2A, B \leq b \leq 2B \). We consider two cases: \( B \geq Ax^{1/14} \) and \( B < Ax^{1/14} \).
The case \( B \geq Ax^{1/14} \)

We have,

\[
| \lambda a - g | = \frac{ga(b + c) + a + b + c}{bc} \\
= ga(1/c + 1/b) + a/bc + 1/c + 1/b \\
< 2ga/b + 3/b \quad \text{(since } 1/c < 1/b \text{ and } a < b < c) \\
= O(G_1 A/B) \quad \text{(since } g \leq 2G_1, a \leq 2A, B \leq b) \\
= O(x^{2/14}) \quad \text{(since } G_1 \leq G = x^{3/14} \text{ and } B \geq Ax^{1/14}).
\]

We can fix \( g \) in \( x^{3/14} \) ways since \( g \leq G = x^{3/14} \). For a given value of \( g, \lambda a \) has only \( O(x^{2/14}) \) choices since \( | \lambda a - g | = O(x^{2/14}) \). So we can fix \( g, a, \lambda \) in \( O(x^{5/14 + o(1)}) \) ways. Now \( b, c \) have only \( x^{o(1)} \) choices since \( (g - \lambda a)bc + (b + c)(ga + 1) + a = 0 \) implies \([g - \lambda a]b + 1 + ga\)[c + 1 + ga] = \([1 + ga]^2 - (g - \lambda a)a\). We must ensure that \( ga - \lambda a^2 \neq (ga + 1)^2 \). It is easily checked that this must be the case by looking, modulo \( a \), at both sides of this inequality.

The case \( B < Ax^{1/14} \)

Let \( AJ \leq B \leq 2AJ \); then \( J \leq x^{1/14} \). We consider the equality \( g(ab + bc + ca) + a + b + c = \lambda abc \). We fix \( \lambda, a, b \) first and show that \( g, c \) have \( x^{o(1)} \) choices by considering the equality \( gc(a+b)+c(1-\lambda ab)+gab+a+b = 0 \). This equality implies \([ab - 1 - (a + b)g]ab + (a + b)c = (\lambda ab - 1)ab + (a + b)^2 \) which is positive.

Thus, for fixed \( \lambda, a, b \) there are \( \leq x^{o(1)} \) choices for \( g, c \). Since \( \lambda a \leq 6g \leq 12G_1 \) there are \( O(G_1) \) choices for \( \lambda a \). Now if we consider \( G_1 \leq g \) and \( g^3 abc \leq x \) we get

\[
abc \leq x/g^3, \\
ab^2 \leq x/g^3 \text{ since } c > b, \\
A(J)^2 \leq x/G_1^3 \text{ since } A \leq a \leq 2A, B \leq b \leq 2B, AJ \leq B \leq 2AJ, G_1 \leq g, \\
A^3 J^2 = O(x/G_1^3), \\
A = O\left(\frac{x^{1/3}}{G_1 J^{1/3}}\right) \text{ and } B = O\left(\frac{x^{1/3} J^{1/3}}{G_1}\right).
\]

Then since \( B \leq b \leq 2B \) there are \( O(x^{1/3} J^{1/3}/G_1) \) choices for \( b \). Therefore to fix \( \lambda, a, b \) there are

\[
O(G_1^{1+o(1)}(x^{1/3} J^{1/3}/G_1)) = O(x^{1/3+o(1)} J^{1/3}) = O(x^{1/3+o(1)} x^{1/42}) = O(x^{5/14+o(1)})
\]

choices, since \( J \leq x^{1/14} \). Once we fix \( \lambda, a, b, \) then \( g, c \) have only \( x^{o(1)} \) choices. Therefore to fix \( \lambda, a, b, g, c \) there are \( O(x^{5/14+o(1)}) \) choices.

We let the \( A, B, J \) run over powers of \( 2 \) and this introduces a factor of \( x^{o(1)} \). Hence \( N_3 = O(x^{5/14+o(1)}) \). Hence \( N = N_1 + N_2 + N_3 = O(x^{1/3}) + O(x^{5/14+o(1)}) + O(x^{5/14+o(1)}) = O(x^{5/14+o(1)}) \).

Discussion. Our choices for parameters such as \( G \) were not arbitrary but optimal. We have used the optimal values for the parameters as this results in a shorter and clearer proof.

It would be best to make our bounds explicit and replace the \( x^{o(1)} \) with a power of \( \log x \). It is easy to see that these are two different problems. For the first problem
we could use a result of Ramanujan [10] that states that there is an explicit constant \( K_\alpha \) depending on \( \alpha \) such that the number of divisors of \( n \), \( d(n) < K_\alpha n^\alpha \) for any positive number \( 0 < \alpha < 1 \). For the second problem we need to consider the average of the divisor function over a polynomial on an interval. There are some results in this direction (see [6]), however, they depend on the coefficients of the polynomial in an unknown way.

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