ON THE HIGHLY ACCURATE SUMMATION OF CERTAIN SERIES OCCURRING IN PLATE CONTACT PROBLEMS

D.A. MACDONALD

Abstract. The infinite series \( R_p = \sum_{k=1}^{\infty} (2k - 1)^{-p} x^{2k-1}, 0 < 1 - x \ll 1, \) \( p = 2 \) or \( 3, \) and the related series

\[ C(x, b, 2) = \sum_{k=1}^{\infty} (2k - 1)^{-2} \frac{\cosh(2k - 1)x}{\cosh(2k - 1)b}, \quad 0 < 1 - x/b \ll 1, \]

\[ S(x, b, 3) = \sum_{k=1}^{\infty} (2k - 1)^{-3} \frac{\sinh(2k - 1)x}{\cosh(2k - 1)b}, \]

are of interest in problems concerning contact between plates and unilateral supports. This article will re-examine a previously published result of Baratella and Gabutti for \( R_p, \) and will present new, rapidly convergent, series for \( C(x, b, 2) \) and \( S(x, b, 3). \)

1. Introduction

When \( 0 < x \leq 1/2 \) the infinite series

\[ R_p = \sum_{k=1}^{\infty} (2k - 1)^{-p} x^{2k-1}, \quad p = 2 \text{ or } 3, \]

are easy to sum to high accuracy using direct summation and a suitable computer algebra package. For example, when either series is approximated by the sum of its first \( r \) terms the error, \( E_r, \) satisfies

\[ E_r(x) = \sum_{k=r+1}^{\infty} \frac{x^{2k-1}}{(2k - 1)^p} \leq \left( \frac{1}{2} \right)^{2r+1} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2r + 1 + 2k)^p} < \left( \frac{1}{2} \right)^{2r-1} \frac{1}{3 (2r + 1)^p}. \]

When, however, \( 0 < 1 - x << 1 \) the series \( R_p \) are not easy to sum. The series

\[ C(x, b, 2) = \sum_{k=1}^{\infty} (2k - 1)^{-2} \frac{\cosh(2k - 1)x}{\cosh(2k - 1)b}, \quad 0 < x/b < 1, \]

\[ S(x, b, 3) = \sum_{k=1}^{\infty} (2k - 1)^{-3} \frac{\sinh(2k - 1)x}{\cosh(2k - 1)b} \]

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are, likewise, easy to sum when \( b (1 - x/b) \) is not small. Suppose, for example, that the series \( C(x, b, 2) \) is truncated after \( r \) terms. The error is then

\[
\hat{E}_c(r) = \sum_{k=r+1}^{\infty} \frac{1}{(2k-1)^2} \cosh((2k-1)x)/\cosh((2k-1)b)
\]

\[
< \sum_{k=r+1}^{\infty} \frac{1}{(2k-1)^2} \left[ e^{-(2k-1)b(1-x/b)} + e^{-(2k-1)b(1+x/b)} \right]
\]

\[
< \sum_{k=r+1}^{\infty} \frac{(e^{-b(1-x/b)})^{2k-1}}{(2k-1)^2} + \sum_{k=r+1}^{\infty} \frac{(e^{-b})^{2k-1}}{(2k-1)^2}
\]

\[
< \frac{1}{(2r+1)^2} \left[ \frac{e^{-(2r+1)b(1-x/b)} + e^{-(2r+1)b}}{1 - e^{-2b(1-x/b)}} \right].
\]

Hence, when \( b(1 - x/b) \) is not small this series, too, is easy to sum to high accuracy by use of direct summation.

The series \( R_p(x), \ p = 2, 3, \ 0 < 1 - x << 1, \) and \( C(x, b, 2), S(x, b, 3), \ 0 < 1 - x/b << 1, \) are of interest in problems concerning contact between plates and unilateral supports, \([4]\), and the problem of their summation to high accuracy has attracted the attention of several numerical analysts \([2], [3], [5], [7]\). This article will address the same problem.

1.1. Previous work. The series were first discussed in \([5]\), where the method of summation is based on Plana’s summation formula (see, for example \([8]\), pp. 145-46). The accuracy of this initial attempt at the problem is inferior to later attempts.

In \([7]\), the method is to substitute the result

\[
\frac{1}{(k-1/2)^2} = \int_0^\infty \frac{e^{-kt} e^{t/2}}{(p-1)!} \frac{t^{p-1}}{p!} \, dt, \quad (k \geq 1)
\]

in \( R_p \), interchange the order of summation and integration, and sum the new (geometric) series in \( k \); hence the series \( R_p \) is transformed to the integral \(^1\)

\[
R_p(x) = \frac{x}{2^{p-1}(p-1)!} \int_0^1 \frac{[-\ln u^2]^{p-1}}{[1 - (xu)^2]} \, du.
\]

which is evaluated numerically by use of Gaussian quadrature.

In \([2]\) the identity \(- \ln u^2 \equiv \ln x^2 - \ln (xu)^2\) and the binomial theorem are used to transform (3) to

\[
R_p(x) = \frac{x}{2^{p-1}(p-1)!} \sum_{m=0}^{p-1} \binom{p-1}{m} \ln x^2 \ln^{p-1-m} T_m(x)
\]

where

\[
T_m(x) = \int_0^1 \frac{[-\ln (xu)^2]^{m}}{[1 - (xu)^2]} \, du.
\]

\(^1\)In \([2], [3]\) and \([7]\) the series are taken to be \( R_p(z) \), where \( z \) is a complex variable; in this article we take \( R_p = R_p(x) \) where \( x \) is a real variable.
By use of the expansion
\[
(-\ln u)^m = \sum_{k=0}^{\infty} (-1)^k S_k^{(m)} \frac{m!}{(m+k)!} (1-u)^{k+m}, \quad 0 < u \leq 1,
\]
where \(S_k^{(m)}\) are the Stirling numbers of the first kind, approximations, with accompanying error bounds, are obtained for the integrals \(T_m(x)\). These are valid for \(0 \leq x \leq 1\).

In [3] the series \(R_p, p = 2, 3\), are treated as special cases of Legendre’s Chi function and are essentially expressed as rapidly convergent alternating power series in \(\xi = -\ln(x)/\pi\); when these power series are truncated after \(r\) terms, where \(r\) is even, the error is close to the first term not taken, which is close to
\[
-\pi \xi^{2r+3} \frac{2b}{(2r+2)(2r+3)},
\]
in the case when \(p = 2\). As we shall show, these remarkable series provide the most efficient means of summing the series \(R_p\) when \(1/2 \leq x < 1\).

In the case of the series \(C(x, b, 2)\) and \(S(x, b, 3)\), the method given in [7] leads to infinite series (of integrals) which converge rapidly except when \(0 < b << 1\)—a deficiency which is resolved in [2] by use of a transformed series which converges like a geometric series with ratio \(1/3\) for all \(b \geq 0\).

In this article accurate bounds will be derived for the errors which occur when the series \(R_p, p = 2, 3\), as transformed in [3] are truncated after \(r\) terms and new series will be presented for \(C(x, b, 2)\) and \(S(x, b, 3)\); further, the errors which result when the latter series are truncated after \(r\) terms will be examined and it will be shown that the error is close to
\[
2b \frac{\left(1 - x/b\right)^{2r+3}}{(2r+2)(2r+3)}
\]
in the case of \(C(x, b, 2)\) and that it is smaller than this in the case of \(S(x, b, 3)\).

2. The series \(R_p(x), p = 2, 3\), as transformed by Boersma and Dempsey

As the series \(R_p(x)\) are easily summed when \(0 < x < 1/2\), intuition suggests that when they are transformed to series in the new variable \((1 - x)\) the transformed series will be easy to sum when \(1/2 \leq x < 1\). This is indeed the case, but the resulting series can be shown to converge more slowly in the region of interest than do the series presented in [3] .

\[\text{It can, for example, be shown that}
\]
\[R_2(x) = \frac{\pi^2}{8} + \frac{1}{2} \left( \ln x \ln \left(\frac{1+x}{1-x}\right) - \sum_{k=1}^{\infty} \frac{b_{k-1}}{k+1} \sum_{k=1}^{\infty} \frac{(1-x)^k}{k^2} \right),
\]
where \(0 < x < 1\) and
\[b_{k-1} = \sum_{i=0}^{k-1} \frac{1}{(k-i)^{2+i}}, \quad k = 1, 2, \ldots .
\]

This result which, when each infinite series is truncated after \(r\) terms has an error, \(E_{R2}\), satisfying
\[0 > E_{R2} > -\frac{(1-x)^{r+1}}{4x} \left( \frac{1-x}{(r+2)} + \frac{2}{(r+1)^2} \right),
\]
should be compared to Procedure 3 of [2].
Summing all such pairs, and noting that from Leibnitz’s theorem for alternating series which, when the series is approximated by its first $p$ terms, the sum of each pair is less than zero and the terms in (4) alternate in sign and steadily decrease to zero as $\xi \to 0$, we see that

$$R_2(x) = \frac{\pi^2}{8} - \frac{\pi \xi}{2} \left[ 1 + \ln(2) - \ln(\xi) \right]$$

and

$$R_3(x) = \frac{7}{8} \zeta(3) - \frac{\pi^3}{8} \xi + \frac{\pi^2}{4} \xi^2 \left[ 3/2 + \ln(2) - \ln(\xi) \right]$$

where $(\xi) = \pi \xi / \ln(\xi)$.

The terms in (4) alternate in sign and steadily decrease to zero as $k \to \infty$. Bounds for the error when the series is approximated by its first $p$ terms are easily obtained from Leibnitz’s theorem for alternating series which, when $r$ is even, gives the result

$$E_{R2} = \pi \sum_{k=r+1}^{\infty} \left( -1 \right)^k \frac{(1 - 2^{-2k+1})}{(2k)(2k+1)} \zeta(2k) \xi^{2k+1}, \quad 0 \leq \xi \leq 1,$$

where $\xi = -\ln(x)/\pi$.

2.1. **Computation of $R_2(x)$ and $R_3(x)$**. When the infinite series in (4) is truncated after $r$ terms the error, $E_{R2}$, is given by

$$E_{R2} = \pi \sum_{k=r+1}^{\infty} \left( -1 \right)^k \frac{(1 - 2^{-2k+1})}{(2k)(2k+1)} \zeta(2k) \xi^{2k+1}, \quad 0 \leq \xi \leq 1,$$

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$$E_{R2} = \frac{-\pi \xi^{2r+3}}{(2r+2)(2r+3)} < E_{R2} < 0.$$

The upper bound in this result may, as will now be demonstrated, be improved on.

When $r$ is even and the terms of the series (6) are grouped in pairs, starting with the first two terms, the sum of each pair is less than zero and the $p$th pair, $T_p$ say, $p = 1, 2, \ldots$, satisfies the inequality

$$T_p < -\pi \xi^{2r+3+4p} \left[ \frac{1 - 2^{-2^{r+1}}}{(2r+4p)(2r+3+4p)} \right] \left[ \frac{\xi^2}{(2r+4+4p)(2r+5+4p)} \right].$$

Summing all such pairs, and noting that $T_p < 0$, we see that

$$E_{R2} < -\pi \xi^{2r+3} \sum_{k=1}^{\infty} \left( 1 - 2^{-2^r-1-4k} - \xi^2 \right) \xi^{4k-1}$$

$$= -\pi \xi^{2r+3} \sum_{k=1}^{\infty} \left( 1 - 2^{-2^r-5-4k} - \xi^2 \right) \xi^{4k}.$$

Further, use of results like

$$Q = \sum_{k=0}^{\infty} \frac{\xi^{2r+3+4k}}{(2r+2+4k)} = \xi \sum_{k=0}^{\infty} \int_{0}^{\xi} \xi^{2r+1+4k} d\xi = \xi \int_{0}^{\xi} \frac{\xi^{2r+1}}{(1-\xi^4)} d\xi$$

lead to

$$E_{R2} < -\pi \left( 1 - \xi^2 \right) \int_{0}^{\xi} \frac{u^{2r+1}}{(1-u^4)} \left( \xi - u \right) du + 4\pi \int_{0}^{\xi} \frac{u^{2r+1}}{(1-u^4)} \left( \xi/2 - u \right) du,$$
from which result and (8) we obtain the inequality

\[ \frac{-\pi \xi^{2r+3}}{(2r+2)(2r+3)} < E_{R2} < \frac{-\pi \xi^{2r+3}}{(2r+2)(2r+3)} \left[ 1 - \xi^2 - \frac{1}{(1-(\xi/2)^4)2^{2r+1}} \right] \]

which is valid for all \textbf{even} \( r > 1 \).

When \( r = 6 \) and \( x = 0.9 \) we find that

\[ -1.142486 \times 10^{-24} < E_{R2} < -1.141061 \times 10^{-24} \]

whereas calculation gives \( E_{R2} = -1.141425 \times 10^{-24} \).

In the case of \( R_3(x) \), we can similarly show that when the infinite series in (5) is truncated after \( r \) terms the error, \( E_{R3} \), satisfies, for \textbf{even} \( r \), the inequality

\[ \frac{\pi^2 \xi^{2r+4}}{(2r+2)(2r+3)(2r+4)} \left[ 1 - \xi^2 - \frac{1}{(1-(\xi/2)^4)2^{2r+1}} \right] < E_{R3} < \frac{\pi^2 \xi^{2r+4}}{(2r+2)(2r+3)(2r+4)} \]

When \( r = 6 \) and \( x = 0.9 \) :

\[ 7.513925 \times 10^{-27} < E_{R3} < 7.523306 \times 10^{-27} \]

whereas calculation gives \( E_{R3} = 7.517046 \times 10^{-27} \).

3. The series \( C(x, b, 2) \) and \( S(x, b, 3) \)

These are the series specifically referred to in [5] (equations (13a) and (13b), respectively).

3.1. \( C(x, b, 2) \), \( 0 < b - x \ll b \). Consider the following boundary value problem: solve for \( T \)

\[
T_{xx} + T_{yy} = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq b;
\]

\[
T(x, 0) = x; \quad T_y(x, b) = 0;
\]

\[
T_x(0, y) = 0; \quad T_x(\pi, y) = 0.
\]

By expanding in terms of the eigenfunctions \( \cos nx \), it is easy to show that

\[ T(x, y) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cosh[(2n-1)(b-y)]}{\cosh[(2n-1)b]} \frac{\cos(2n-1)x}{(2n-1)^2}. \]

Hence

\[ T(\pi, y) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \frac{\cosh[(2n-1)(b-y)]}{\cosh[(2n-1)b]} \]

\[ = \frac{\pi}{2} - \frac{4}{\pi} C(b-y, b, 2). \]

Now put \( \hat{T} = T - x \) and solve

\[
\hat{T}_{xx} + \hat{T}_{yy} = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq b;
\]

\[
\hat{T}(x, 0) = 0; \quad \hat{T}_y(x, b) = 0;
\]

\[
\hat{T}_x(0, y) = -1; \quad \hat{T}_x(\pi, y) = -1.
\]
The eigenfunctions are now
\[ \sin[(2n - 1)\frac{\pi y}{2b}] \] and
\[ \hat{T}(x, y) = \sum_{n=1}^{\infty} \frac{8b}{\pi^2} \frac{\sin[(2n - 1)\frac{\pi y}{2b}]}{(2n - 1)^2} \cdot \left( \tanh[(2n - 1)\frac{\pi^2}{4b}] \cosh[(2n - 1)\frac{\pi x}{2b}] - \sinh[(2n - 1)\frac{\pi x}{2b}] \right). \]

Hence
\[ \hat{T}(\pi, y) = -\sum_{n=1}^{\infty} \frac{8b}{\pi^2} \frac{\sin[(2n - 1)\frac{\pi y}{2b}]}{(2n - 1)^2} \left( 1 - \tanh[(2n - 1)\frac{\pi}{4b}] \right) \frac{\sin[(2n - 1)\frac{\pi y}{2b}]}{(2n - 1)^2}. \]

If we now substitute for \( \sum_{n=1}^{\infty} \sin[(2n - 1)\frac{\pi y}{2b}]/(2n - 1)^2 \) from equation (3.4) of [3] we obtain, on eliminating \( T \) from equations (10) and (11), the result
\[ C(x, b, 2) = \frac{\pi^2}{8} - \left[ 1 - \ln\left(\frac{\mu_1}{2}\right) \right] \frac{b}{\pi} \mu_1 \]
\[ - \frac{2b}{\pi} \frac{B_{2k}}{\mu_1} \sum_{k=1}^{\infty} \frac{(-1)^k [2^{2k-1} - 1]}{2k(2k+1)!} \mu_1^{2k} \]
\[ + \frac{2b}{\pi} \sum_{k=1}^{\infty} \left( 1 - \tanh[(2k - 1)\frac{\pi}{4b}] \right) \frac{\sin[(2k - 1)\frac{\pi y}{2b}]}{(2k - 1)^2}, \]

where \( \mu_1 = (1 - x/b) \pi/2 \) and \( B_{2k} \) denotes the Bernoulli polynomial of order \((2k)\), as defined in [1]. The series for \( S(x, b, 3) \) can be similarly transformed by integrating (12) with respect to \( x \), from \( x \) to \( b \).

4. Computation of \( C(x, b, 2) \) and \( S(x, b, 3) \)

When each of the infinite series in (12) and in the corresponding result for \( S(x, b, 3) \) are approximated by the sum of their first \( r \) terms errors, \( E_c(r) \) and \( E_s(r) \), respectively, are introduced. In this section we shall present bounds for these.

4.1. Computation of \( C(x, b, 2) \). From equation (12),
\[ E_c(r) = \frac{2b}{\pi} \frac{B_{2k}}{\mu_1} \sum_{k=1}^{r} \frac{(-1)^k [2^{2k-1} - 1]}{2k(2k+1)!} \mu_1^{2k} \]
\[ + \frac{2b}{\pi} \sum_{k=r+1}^{\infty} \left( 1 - \tanh[(2k - 1)\frac{\pi}{4b}] \right) \frac{\sin[(2k - 1)\frac{\pi y}{2b}]}{(2k - 1)^2}, \]

where
\[ G = \sum_{k=r+1}^{\infty} (-1)^{k+1} \frac{[2^{2k-1} - 1]}{2k(2k+1)!} \frac{\pi}{2} \left( 1 - \frac{x}{b} \right)^{2k}, \]
\[ H = \sum_{k=r+1}^{\infty} \left( 1 - \tanh[(2k - 1)\frac{\pi}{4b}] \right) \frac{\sin[(2k - 1)(1 - x/b) \pi/2]}{(2k - 1)^2}. \]
Hence, by use of (7),
\[ \sum_{k=r+1}^{\infty} \frac{[(1-x/b)/2]^{2k}}{2k(2k+1)} - \sum_{k=r+1}^{\infty} \frac{[(1-x/b)/4]^{2k}}{k(2k+1)} < G < \sum_{k=r+1}^{\infty} \frac{[(1-x/b)/2]^{2k}}{2k(2k+1)}, \]
which leads to
\[ \frac{[(1-x/b)/2]^{2r+2}}{(2r+2)(2r+3)} \left[ 1 - \frac{(1/2)^{2r+1}}{(1-(1-x/b)^2/16)} \right] < G < \frac{[(1-x/b)/2]^{2r+2}}{(2r+2)(2r+3)} \left( 1 - \frac{(1-x/b)^2/4}{4} \right). \]

In the case of \( H \) we have
\[ |H| < \sum_{k=r+1}^{\infty} \frac{1 - \tanh[(2k-1)\pi^2/(4b)]}{(2k-1)^2} < \left( \frac{\pi^2}{8} - 1 \right) \left( 1 - \tanh[(2r+1)\pi^2/4b] \right). \]

Hence, when each of the infinite series in (12) are truncated after \( r \) terms the error, \( E_r(r) \), satisfies
\[ U < E_r(r) < V, \]
where
\[ U = \frac{2b[(1-x/b)/2]^{2r+3}}{(2r+2)(2r+3)} \left[ 1 - \frac{(1/2)^{2r+1}}{(1-(1-x/b)^2/16)} \right] - \alpha \left( 1 - \tanh[(2r+1)\pi^2/4b] \right) \]
\[ V = \frac{2b[(1-x/b)/2]^{2r+3}}{(2r+2)(2r+3)} \left[ \frac{1}{1-(1-x/b)^2/4} \right] + \alpha \left( 1 - \tanh[(2r+1)\pi^2/4b] \right) \]
and \( \alpha = (\pi^2/8 - 1)(2b)/\pi \). When \( r = 5 \), \( b = 0.1 \) and \( x = 0.09 \) we find that
\[ 1.564239 \times 10^{-20} < E_r(5) < 1.568926 \times 10^{-20}, \]
whereas calculation of \( E_r(5) \) gives \( E_r(5) = 1.57538663 \times 10^{-20} \).

4.2. Computation of \( S(x, b, 3) \). From equation (12) it can be shown that
\[ S(x, b, 3) = \frac{\pi^2 x}{8} - \frac{b x}{4} [2-x/b] \left( \frac{3}{2} - \ln \frac{\pi}{4} \right) - \frac{b^2}{4} \left( 1 - \frac{x}{b} \right)^2 \ln(1-x/b) \]
\[ + \frac{4b^2}{\pi^2} \sum_{k=1}^{r} (-1)^k \left[ \frac{[2^{2k-1}-1]}{2k(2k+2)} \right] B_{2k} \left( \frac{\pi}{2} \right)^{2k+2} \]
\[ + \frac{4b^2}{\pi^2} \sum_{k=1}^{r} \left( 1 - \tanh[(2k-1)\pi^2/4b] \right) \frac{\cos \mu_k}{(2k-1)^3} + b^2 A + E_r(r), \]
where, for \( k = 1, 2, \ldots \), \( \mu_k = (2k-1)(1-x/b)/\pi/2 \) and
\[ A = \sum_{k=1}^{\infty} (-1)^{k+1} \left( \frac{\pi}{2} \right)^{2k} \left[ \frac{[2^{2k-1}-1]}{2k(2k+2)} \right] B_{2k}. \]
The error term, \( E_s(r) \), satisfies
\[
E_s(r) = -b^2 (1 - x/b)^2 \bar{G} + 4b^2 \bar{H}/\pi^2,
\]
where
\[
\bar{G} = \sum_{k=r+1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} - 1}{2k(2k+2)!} \frac{\pi}{2} \left(1 - \frac{x}{b}\right)^{2k},
\]
\[
\bar{H} = \sum_{k=r+1}^{\infty} \left(1 - \tanh[(2k-1)\pi/4b]\right) \frac{\cos[(2k-1)(1-x/b)/2\pi]}{(2k-1)^3}.
\]
It can be shown that
\[
\bar{U} < E_s(r) < \bar{V}, \tag{18}
\]
where
\[
\bar{U} = -\frac{4b^2 [(1-x/b)/2]^{2r+4}}{(2r+2)(2r+3)(2r+4)} \left[\frac{1}{1 - (1-x/b)^2/4}\right] - \frac{b^2}{\pi^2} \left[1 - \tanh[(2r+1)\pi^2/4b]\right],
\]
\[
\bar{V} = -\frac{4b^2 [(1-x/b)/2]^{2r+4}}{(2r+2)(2r+3)(2r+4)} \left[1 - \frac{(1/2)^{2r+1}}{1 - (1-x/b)^2/16}\right] + \frac{b^2}{\pi^2} \left[1 - \tanh[(2r+1)\pi^2/4b]\right].
\]
When \( r = 5, \ b = 0.1, \ x = 0.09 \), we find that
\[-1.120662 \times 10^{-23} < E_s(5) < -1.117314 \times 10^{-23},
\]
whereas calculation gives \( E_s(5) = -1.119409 \times 10^{-23} \).

5. Conclusion

In this article we have shown that when either of the series (4), (5) are truncated after \( r \) terms, the magnitude of the resulting error in the sum of the series will be close to (and will not exceed)
\[
\frac{\pi}{(2r+2)(2r+3)} \left(-\frac{\ln(x)}{\pi}\right)^{2r+3} \approx \frac{\pi \epsilon_1^{2r+3}}{(2r+2)(2r+3)},
\]
where \( 0 < \epsilon_1 = (1-x)/\pi << 1 \).

Further, when the series in (12) are truncated after \( r \) terms the error in the sum of the series will be close to
\[
\frac{2b \epsilon_2^{2r+3}}{(2r+2)(2r+3)}
\]
when \( 0 < \epsilon_2 = (1-x/b)/2 << 1 \), and \( b = O(1) \), at most; and the error in \( S(x, b, 3) \) will be smaller than this.

Besides being of interest to the numerical analyst ([2], [3], [5], [7]) and to the specialist in unilateral plate contact problems, [4], the series \( C(x, b, 2) \) and \( S(x, b, 3) \) occur elsewhere in applied mathematics—for example, in connection with the torsion of a prismatic bar of rectangular cross-section where, with \( b^* = (\pi c)/(2b) \),

\[3\text{We note that the series for the number } A \text{ converges like } \sum_k 2^{-2k}/k^3, \]
$C(y(2b^*)/c, b^*, 2)$ occurs in the expression for the shear stress on the boundaries $x = \pm b/2$, of the bar, $-b/2 \leq x \leq b/2$, $-c/2 \leq y \leq c/2$ (see, for example, [6], pp. 244-45).

In any such application the use of the alternative series discussed in this article should prove worthwhile.

References


[5] K.M. Dempsey, D Liu, and J.P. Dempsey, *Plana’s summation formula for $\sum_{m=1,3}^{\infty} m^{-2}\sin(m\alpha), m^{-3}\cos(m\alpha), m^{-2}A^m, m^{-3}A^m$*, Math. Comp. 55 (1990) 693-704. MR 91b:65003


Department of Mathematical Sciences, P.O. Box 147, The University, Liverpool L69 3BX, United Kingdom

E-mail address: sx1001iv.uk