ON $\phi$–AMICABLE PAIRS

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Abstract. Let $\phi(n)$ denote Euler’s totient function, i.e., the number of positive integers $< n$ and prime to $n$. We study pairs of positive integers $(a_0, a_1)$ with $a_0 \leq a_1$ such that $\phi(a_0) = \phi(a_1) = (a_0 + a_1)/k$ for some integer $k \geq 1$. We call these numbers $\phi$–amicable pairs with multiplier $k$, analogously to Carmichael’s multiply amicable pairs for the $\sigma$–function (which sums all the divisors of $n$). We have computed all the $\phi$–amicable pairs with larger member $\leq 10^9$ and found 812 pairs for which the greatest common divisor is squarefree. With any such pair infinitely many other $\phi$–amicable pairs can be associated. Among these 812 pairs there are 499 so-called primitive $\phi$–amicable pairs. We present a table of the 58 primitive $\phi$–amicable pairs for which the larger member does not exceed $10^6$. Next, $\phi$–amicable pairs with a given prime structure are studied. It is proved that a relatively prime $\phi$–amicable pair has at least twelve distinct prime factors and that, with the exception of the pair $(4, 6)$, if one member of a $\phi$–amicable pair has two distinct prime factors, then the other has at least four distinct prime factors. Finally, analogies with construction methods for the classical amicable numbers are shown; application of these methods yields another 79 primitive $\phi$–amicable pairs with larger member $> 10^9$, the largest pair consisting of two 46-digit numbers.

1. Introduction

Let $\phi(n)$ be Euler’s totient function. The pair $(a_0, a_1)$ with $1 < a_0 \leq a_1$ is called $\phi$–amicable with multiplier $k$ if

$$\phi(a_0) = \phi(a_1) = \frac{a_0 + a_1}{k} \text{ for some integer } k \geq 1. \hspace{1cm} (1)$$

Since $\phi(n) < n$, we cannot have $k = 1$. To see that in fact $k > 2$, notice that if $k = 2$, then

$$\frac{a_0 + a_1}{2} > \frac{\phi(a_0) + \phi(a_1)}{2} = \phi(a_0).$$

If $a_0 = a_1 = a$, we have the equation $\phi(a) = 2a/k = a/l$ provided that $k$ is even. This is known [7] to have the (only) solutions $a = 2^5$ for $l = 2$ and $a = 2^\alpha 3^\beta$ for $l = 3$. If $k$ is odd, $k = pk'$ say, where $p$ is an odd prime, then $p \mid a$, $a = p^\gamma b$ with $\gcd(p, b) = 1$, and the equation easily reduces to the form $\phi(b) = b/l$ (where $l = k'(p-1)/2$). We assume from now that $a_0 < a_1$.

An analogous definition for the $\sigma$–function was given by Carmichael [4, p. 399], who called two positive integers $a_0$ and $a_1$ a multiply amicable pair if $\sigma(a_0) = \sigma(a_1)$.
Proposition 1. Let \((a_0, a_1; k)\) be given.

a. If \(p\) is a prime with \(p \mid \gcd(a_0, a_1)\), then we also have \((pa_0, pa_1; k)\).

b. If \(p\) is a prime with \(p^2 \mid \gcd(a_0, a_1)\), then we also have \((a_0/p, a_1/p; k)\).
Proposition 2. Let \((a_0, a_1; k)\) be given.

a. If \(p\) is a prime with \(p \nmid a_0a_1\) and \(p - 1 \mid k\), then we also have
\[(pa_0, pa_1; kp/(p - 1)).\]

b. If \(p\) is a prime with \(p \parallel \gcd(a_0, a_1)\) and \(p \mid k\), then we also have
\[(a_0/p, a_1/p; k(p - 1)/p).\]

Remark. When we reduce a \(\phi\)-amicable pair by applying (possibly repeatedly) Proposition 1b, we arrive at a pair with \(p \parallel \gcd(a_0, a_1)\). Now it is easy to see that if \(p^c \mid a_1\) (\(c \geq 2\)), then \(p^c - 1 \mid a_{1-c}\), and that, if \(p^2 \mid a_1\) and \(p \parallel a_{1-c}\), it is impossible to have \(p \mid k\) (\(i = 0\) or \(1\)). It follows that the condition \(p \parallel \gcd(a_0, a_1)\) in Proposition 2b effectively may be replaced by \(p \parallel a_0, p \parallel a_1\).

Propositions 1 and 2 suggest the definition of a minimal pair from which no smaller pairs can be generated. We call these pairs primitive \(\phi\)-amicable pairs.

**Definition.** A \(\phi\)-amicable pair \((a_0, a_1; k)\) is called primitive if \(\gcd(a_0, a_1)\) is squarefree, and if \(\gcd(a_0, a_1, k) = 1\).

By applying Propositions 1b and 2b we see that any non-primitive pair can be reduced to a primitive pair. For example,
\[
\begin{align*}
\left\{ \begin{array}{l}
2 \cdot 3^2 \cdot 5 \cdot 11 \\
2 \cdot 3^2 \cdot 5 \cdot 17 \\
k = 8
\end{array} \right. & \xrightarrow{\text{Prop. 1b}} \left\{ \begin{array}{l}
2 \cdot 3 \cdot 5 \cdot 11 \\
2 \cdot 3^2 \cdot 11 \\
k = 8
\end{array} \right. & \xrightarrow{\text{Prop. 2b}} \left\{ \begin{array}{l}
3 \cdot 5 \cdot 31 \\
3 \cdot 5 \cdot 11 \\
k = 4
\end{array} \right.
\end{align*}
\]
where the third pair is a primitive \(\phi\)-amicable pair. In the other direction, starting with the primitive \(\phi\)-amicable pair \((3 \cdot 7 \cdot 71 \cdot 193, 3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17; 4)\):

\[
\begin{align*}
\left\{ \begin{array}{l}
3 \cdot 7 \cdot 71 \cdot 193 \\
3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\
k = 4
\end{array} \right. & \xrightarrow{\text{Prop. 2a}} \left\{ \begin{array}{l}
3 \cdot 5 \cdot 7 \cdot 71 \cdot 193 \\
3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\
k = 5
\end{array} \right. & \xrightarrow{\text{Prop. 1a}} \left\{ \begin{array}{l}
3^2 \cdot 5 \cdot 7 \cdot 71 \cdot 193 \\
3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\
k = 5
\end{array} \right.
\end{align*}
\]

By applying Proposition 2a with \(p = 2\) to the second pair in this chain, we obtain a pair with multiplier \(k = 10\) (which turns out to be the smallest pair with this multiplier).

In Tables 1–2 the 58 primitive \(\phi\)-amicable pairs \((a_0, a_1; k)\) with \(a_1 \leq 10^6\) are listed, for increasing values of \(a_1\). The last column gives pairs \(b, k\) for which \((ba_0, ba_1; k)\) is a non-primitive \(\phi\)-amicable pair—but with squarefree greatest common divisor—obtained by (possibly repeated) application of Proposition 2a.

The total number of primitive \(\phi\)-amicable pairs with larger member \(\leq 10^9\) is 499. They occur with multipliers 3, 4, 5, and 7, and frequencies 109, 158, 144, and 88, respectively. We have applied Proposition 2a to these pairs (where possible repeatedly), and generated 475 non-primitive pairs with multipliers 5, 6, 7, 8, 9, 10, and 11, and frequencies 34, 109, 20, 158, 109, 34, and 11, respectively. Of these 475 non-primitive pairs, 313 have larger member \(\leq 10^9\). So there are 812 \(\phi\)-amicable pairs \((a_0, a_1; k)\) with \(a_1 \leq 10^9\) for which \(\gcd(a_0, a_1)\) is squarefree.

\(^1\)Report [5] gives a slightly different definition which is equivalent to the one given here.
The smallest pair with multiplier 11 comes from primitive pair number 136:

\[13290459 = 3 \cdot 7 \cdot 13 \cdot 89 \cdot 547, \quad 14385189 = 3 \cdot 7 \cdot 13 \cdot 23 \cdot 29 \cdot 79, \quad k = 4,\]

by multiplication of both members by \(b = 110\) (i.e., by applying Proposition 2a successively with \(p = 2, 5, \) and 11). Furthermore, with the factors \(b = 2, 5, 10,\) pair number 136 gives three other pairs with multipliers 8, 5, and 10, respectively. The complete list of 499 primitive \(\phi\)-amicable pairs with larger member \(\leq 10^9\) is given in [5]. For each primitive pair \((a_0, a_1; k)\) the pairs \((b, k)\) are given for which \((bk_0, bk_1; k)\) is a \(\phi\)-amicable pair, obtained by (possibly repeated) application of Proposition 2a.

It is easy to show that pairs with multipliers 3 and 4, and with squarefree greatest common divisor, must be primitive. All the pairs with multiplier 6, 8, 9, and 10 which we found are non-primitive, but we do not know whether there exist primitive

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<td>94666 = 2 \cdot 11 \cdot 13 \cdot 331</td>
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<td>29</td>
<td>87591 = 3 \cdot 7 \cdot 43 \cdot 97</td>
<td>105945 = 3 \cdot 5 \cdot 7 \cdot 1009</td>
<td>4, 2, 8</td>
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<tr>
<td>30</td>
<td>109298 = 2 \cdot 7 \cdot 37 \cdot 211</td>
<td>117502 = 2 \cdot 7^2 \cdot 11 \cdot 109</td>
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</table>
pairs with such a multiplier. Notice that we found both primitive and non-primitive pairs with multipliers 5 and 7. The smallest non-primitive pairs with multipliers 5 and 7 are

$$1438815 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 193, \quad 1786785 = 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17, \quad k = 5,$$

and

$$4905670 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 277, \quad 5295290 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23^2, \quad k = 7,$$

respectively; their “mother pairs” are numbers 37 and 38 in Table 2, respectively.

From each \(\phi\)-amicable pair \((a_0, a_1)\) with \(\gcd(a_0, a_1) > 1\) it is possible to generate infinitely many others with Proposition 1a. For example, from the primitive pair \((2 \cdot 5 \cdot 11 \cdot 23, 2 \cdot 3 \cdot 5 \cdot 11^2, 7)\) (number 7 in Table 1) we generate the non-primitive pairs:

\[(2i_1+15i_2+11i_3+123, 2i_1+13 \cdot 5i_2+11i_3+2, 7), \quad i_1, i_2, i_3 \geq 0, \quad i_1 + i_2 + i_3 > 0.\]
3. **Basic properties of primitive φ–amicable pairs**

In this section we list ten basic properties of primitive φ–amicable pairs. Most of the proofs are omitted. They are simple exercises, or see [5].

Let \((a_0, a_1)\) be a primitive φ-amicable pair with \(a_0 < a_1\), and let \(p\) be a prime.

**B1** We cannot have \(a_0 = 2\) or \(a_0 = 3\).

**B2** For \(i = 0\) or \(1\), if \(p^2 \parallel a_i\), then \(p^2 \parallel a_{1-i}\). Proof: If \(p^3 \mid a_i\), then \(k\phi(a_i) = k\frac{p^2\phi(a_i/p^2)}{p^2} = p^3(a_i/p^3) + a_{1-i}\), so \(p^2 \mid a_{1-i}\) which would make \((a_0, a_1)\) non-primitive. So \(p^2 \parallel a_i\). From this, it follows in a similar way that \(p \parallel a_{1-i}\).

**B3** For \(i = 0\) or \(1\), if \(4 \mid a_i\), then \((a_0, a_1) = (4, 6)\).

**B4** \(a_0\) is even if and only if \(a_1\) is even.

**B5** \(a_1\) is not prime. Proof: This is true for any \(a_0, a_1\) satisfying \(1 < a_0 < a_1\) and \(\phi(a_0) = \phi(a_1)\), for then \(\phi(a_1) < a_0\). If \(a_1 = p\), then \(\phi(a_1) = p - 1 < a_0 < p = a_1\), which is impossible.

**B6** For \(i = 0\) or \(1\), \(a_i\) cannot be a perfect square, or twice a perfect square, except if \((a_0, a_1) = (4, 6)\).

**B7** We cannot have \(a_0 \mid a_1\).

**B8** For at least one prime \(p\), \(p \parallel a_0a_1\). Proof: Let \(i = 0\) or \(1\). If the result is not true, then for all primes \(q\) dividing \(a_0a_1\) we have either \(q \parallel a_i\) and \(q \parallel a_{1-i}\), or \(q \parallel a_i\) and \(q \parallel a_{1-i}\). Cancellation of factors \((q - 1)\) from both sides of the equation \(\phi(a_0) = \phi(a_1)\) then leads to a denial of the Fundamental Theorem of Arithmetic, since \(a_0 \neq a_1\).

**B9** If \(a_0\) and \(a_1\) are squarefree, then \(3 \mid a_0\) if and only if \(3 \mid a_1\).

**B10** If \((a_0, a_1)\) has multiplier \(k = 3\), then the smallest prime divisor of \(a_0a_1\) is at least \(5\). If \((a_0, a_1)\) has multiplier \(k = 4\), then \(a_0a_1\) is odd. Proof: From (1) it follows that

\[
(3) \quad k = \frac{a_0}{\phi(a_0)} + \frac{a_1}{\phi(a_1)}.
\]

Let \(i = 0\) or \(1\). If \(a_i\) is even, then by **B4** also \(a_{1-i}\) is even, so that \(a_i/\phi(a_i) > 2\) for \(i = 0, 1\), which contradicts (3) if \(k \leq 4\). If \(3 \mid a_i\), then \((a_0 + a_1)/3\) can only be integral if \(3 \mid a_{1-i}\). This implies that \(a_i/\phi(a_i) > 3/2\) for \(i = 0, 1\), contradicting (3) if \(k = 3\).

4. **φ–amicable pairs with a given prime structure**

Property **B5** states that there exists no φ–amicable pair for which the larger member is a prime. Here, we shall study pairs with a given primitive structure more generally. First we have the following general finiteness result:

**Theorem 1.** There are only finitely many primitive φ–amicable pairs with a given number of different prime factors.

*Proof.* This proof is inspired by analogous results of Borho [1] for ordinary and unitary amicable pairs. First we notice that if the total number \(t\) of different prime factors of a primitive φ–amicable pair \((a_0, a_1)\) is prescribed, then there are only finitely many values of \(k, r, s\) with \(r + s = t\) for which

\[
k = \frac{a_0}{\phi(a_0)} + \frac{a_1}{\phi(a_1)} = \prod_{i=1}^{r} \frac{p_i}{p_i - 1} + \prod_{j=1}^{s} \frac{q_j}{q_j - 1}
\]

\[\text{In [5], B2 is given in the weaker form: if } p^2 \parallel a_i, \text{ then } p \parallel a_{1-i}, \text{ and similarly for B3.}\]
can hold. Therefore we are done if we can show that the equation
\[ k = \frac{x_1}{x_1 - 1} \cdots \frac{x_r}{x_r - 1} + \frac{y_1}{y_1 - 1} \cdots \frac{y_s}{y_s - 1} \]
has finitely many solutions in integers ≥ 2, for given \( k, r, \) and \( s \). Here, we should realize that the only solutions of (4) that can actually correspond to a primitive \( \phi \)-amicable pair are those for which all the \( x_i \)'s and \( y_j \)'s are prime. In those cases in which \( x_i = y_j = p \), say, with corresponding \( a_0 = \prod_{i=1}^r x_i \) and \( a_1 = \prod_{j=1}^s y_j \), also \((a_0 p, a_1), (a_0, a_1 p)\) are potential primitive \( \phi \)-amicable pairs (leading to the same equation (4)). However, only at most one of these three can actually be a primitive \( \phi \)-amicable pair (if one of them satisfies the first equality in (1), the others do not).

Suppose (4) has infinitely many solutions. Let \( z = (z_1, \ldots, z_t) \) be a solution of (4) and denote by \( z^{(1)}, z^{(2)}, \ldots \), an infinite sequence of different solutions. Then this contains an infinite subsequence in which the last component is non-decreasing, i.e., without loss of generality we may assume \( z_t^{(1)} \leq z_t^{(2)} \leq \ldots \). This reasoning can be repeated for the components \( t-1, t-2, \ldots, 1 \), so that after \( t \) subsequence-transitions we finally have a (still) infinite subsequence of different solutions ordered in such a way that each component of one solution is not greater than the corresponding component of the next solution. However, since the right-hand side of (4) is monotonically decreasing in each of its variables, if \( z^{(1)} \) is a solution, then \( z^{(2)} \) cannot be a solution. This is a contradiction.

The next theorem gives an upper bound for the smallest odd prime divisor of a \( \phi \)-amicable pair, as a function of the multiplier and the maximum of the numbers of different prime factors in the two members.

**Theorem 2.** Let \((a_0, a_1; k)\) be a \( \phi \)-amicable pair, where \( a_0 \) and \( a_1 \) have \( r \) and \( s \) different prime divisors, respectively. Let \( P \) be the smallest odd prime divisor of \( a_0 a_1 \) and \( m = \max\{r, s\} \). Then
\[ P \leq \frac{k + 4m - 8}{k - 4} \quad \text{with } k \geq 5, \text{ if the pair is even,} \]
and
\[ P \leq \frac{k + 2m - 2}{k - 2} \quad \text{with } k \geq 3, \text{ if the pair is odd.} \]

**Proof.** We have
\[ k = \prod_{p | a_0} \frac{p}{p - 1} + \prod_{q | a_1} \frac{q}{q - 1} \quad (p, q \text{ primes}). \]

If both \( a_0 \) and \( a_1 \) are even, \textbf{B10} implies that \( k \geq 5 \). Furthermore, since consecutive primes differ at least by 1, it follows that
\[ k \leq 2 \prod_{i=1}^{r-1} \frac{P + i - 1}{P + i - 2} + 2 \prod_{i=1}^{s-1} \frac{P + i - 1}{P + i - 2} \leq 4 \prod_{i=1}^{m-1} \frac{P + i - 1}{P + i - 2}. \]

Cancellation in the last product yields
\[ k \leq 4 \frac{P + m - 2}{P - 1}, \]
and the result follows. The proof is similar if both \( a_0 \) and \( a_1 \) are odd. \( \square \)
Theorem 3. If \((a_0, a_1; k)\) is a \(\phi\)-amicable pair with \(\gcd(a_0, a_1) = 1\), then
\[
(7) \quad k - 1 < \frac{a_0 a_1}{\phi(a_0 a_1)} < \frac{k^2}{4},
\]
and \(a_0 a_1\) has at least twelve different prime factors.

Proof. Recall that, for any real \(x > 0\), \(x + (1/x) \geq 2\), with equality only when \(x = 1\). Since \(\gcd(a_0, a_1) = 1\), we have, from (1),
\[
\frac{\phi(a_0 a_1)}{a_0 a_1} = \frac{\phi(a_0) \phi(a_1)}{a_0 a_1} = \frac{(a_0 + a_1)^2}{k^2 a_0 a_1} = \frac{1}{k^2} \left( \frac{a_0}{a_1} + 2 + \frac{a_1}{a_0} \right) \geq \frac{4}{k^2}.
\]
Since \(a_0 < a_1\), we have the right-hand inequality in (7). For the left-hand inequality, we note simply that \((a_i/\phi(a_i)) - 1 > 0\) for \(i = 0, 1\). Then
\[
k = \frac{a_0}{\phi(a_0)} + \frac{a_1}{\phi(a_1)} < \frac{a_0 a_1}{\phi(a_0) \phi(a_1)} + 1 = \frac{a_0 a_1}{\phi(a_0 a_1)} + 1.
\]

To prove that \(a_0 a_1\) has at least twelve different prime factors, we set \(A = a_0 a_1\), \(F(A) = A/\phi(A) = \prod_{p \mid A} p/(p - 1)\), and \(\omega\) equal to the number of different prime factors of \(A\). Let \(A = p_1 p_2 \ldots p_\omega\), with \(p_1 < p_2 < \cdots < p_\omega\). We require the following observations. (i) By B2, B4 and B9, \(A\) is squarefree and not divisible by 2 or 3. (ii) Using (1), if \(p\) and \(q\) are primes with \(p \mid A\) and \(q \mod p = 1\), then \(q \not\mid A\). (iii) Since \(\phi(a_0) = \phi(a_1)\) and \(\phi(a_0 a_1) = \phi(a_0) \phi(a_1)\), \(\phi(A)\) is a perfect square.

Suppose \(k \geq 4\). We have \(p_1 \geq 5\), \(p_2 \geq 7\), \ldots. If \(\omega \leq 32\), then
\[
F(A) \leq F(5 \cdot 7 \cdot 11 \cdot \ldots \cdot 139) < 2.994,
\]
(taking 32 primes from 5 to 139, inclusive) and we have a contradiction of the left-hand inequality in (7). So \(\omega \geq 33\) when \(k \geq 4\).

Now suppose that \(k = 3\). It is relatively easy to show that \(\omega \geq 11\). To show that in fact \(\omega \geq 12\), we assume that \(\omega = 11\) and show this to be untenable. The earlier calculations (omitted here) show that we must have \(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \mid A\).

We cannot have \(p_8 \geq 73\), for in that case \(p_1 = 5\), \ldots, \(p_7 = 37\), \(p_8 \geq 73\), \(p_9 \geq 83\), \(p_{10} \geq 89\), \(p_{11} \geq 97\), and
\[
F(A) \leq F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 83 \cdot 89 \cdot 97) < 1.997,
\]
contradicting the left-hand inequality in (7). Therefore, \(p_8 = 59\) or 67. If \(p_8 = 67\), then \(p_9 = 73\), since
\[
F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 83 \cdot 89 \cdot 97) < 1.9992,
\]
and \(p_{10} = 83\) or 89, since
\[
F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 73 \cdot 97 \cdot 107) < 1.999.
\]
If \(p_{10} = 89\), then \(p_{11} \in \{97, 107, 109\}\), since
\[
F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 89 \cdot 163) < 1.995.
\]
In all three cases, since $23 \cdot 67 \cdot 89 \mid A$, we have $11^3 \mid \phi(A)$ so $\phi(A)$ is not a perfect square. If $p_{10} = 83$, then $p_{11} \in \{89, 97, 107, 109\}$, since

$$F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 73 \cdot 83 \cdot 163) < 1.996.$$ 

In these cases, since $83 \mid A$, we have $41 \mid \phi(A)$.

The proof continues in this way, on the assumption next that $p_8 = 59$.  

**Corollary.** If $p$ is prime, $p < a$ and $(p, a)$ is a $\phi$–amicable pair, then $a$ has at least twelve different prime factors.

The proof makes use of the calculations in the proof of Theorem 3, and the fact that, when $k = 3$, $p - 1 = \phi(a) = (p + a)/3$ from which $2\phi(a) = a + 1$.

The odd numbers $> 1$ we know satisfying the equation $a + 1 = 2\phi(a)$ are $a = 3, 3 \cdot 5, 3 \cdot 5 \cdot 17, 3 \cdot 5 \cdot 17 \cdot 257, 3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537, 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929, \ldots$ and $3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 83623937$ (cf.
[6, Problem B37]), but for all of them $3 \mid a$, so that these cannot be a member of a $\phi$–amicable pair $(p, a)$ with multiplier 3. In Section 5 we will see that solutions of the equation $a + 1 = 2\phi(a)$ sometimes can help to generate new $\phi$–amicable pairs.

The following propositions show how from a number $a$ satisfying $a + 1 = 2\phi(a)$ other numbers with that property can be found.

**Proposition 3.** If $a + 1 = 2\phi(a)$ and if $q = a + 2$ is a prime number, then $aq + 1 = 2\phi(aq)$.

**Proposition 4.** Let $a + 1 = 2\phi(a)$ and write $a^2 + a + 1 = D_1D_2$ with $0 < D_1 < D_2$. If both $q = a + 1 + D_1$ and $r = a + 1 + D_2$ are prime numbers, then $aqr + 1 = 2\phi(aqr)$.

We have also proved the following theorem in which one of the members of a $\phi$–amicable pair has precisely two distinct prime factors.

**Theorem 4.** Except for the pair $(4, 6)$, if one member of a $\phi$–amicable pair has exactly two distinct prime factors, then the other member has at least four distinct prime factors.

The proof is largely computational.

**5. Analogies with amicable pairs**

We have determined some analogies with amicable pairs in the construction of $\phi$–amicable pairs with a given prime structure. Assume, for example, that $a_0$ and $a_1$ have the form $a_0 = ar$, $a_1 = apq$, where $p$, $q$, and $r$ are distinct primes not dividing $a$. Examples are the pairs numbered 4 and 9 in Table 1. Substitution in (1) yields after some simple calculations:

$$r = (p - 1)(q - 1) + 1 \text{ and } (cp - d)(cq - d) = a(c + d),$$

where

$$c = k\phi(a) - 2a \text{ and } d = k\phi(a) - a.$$ 

It is convenient now to choose $a$ and $k$ such that $c$ is a small positive number. For example, if we choose $a = 5 \cdot 7$ and $k = 3$, then $c = 2$ and $d = 37$. The second equation in (8) then reduces to:

$$2p - 37)(2q - 37) = 1365 = 3 \cdot 5 \cdot 7 \cdot 13.$$
Writing the right-hand side as $1 \times 1365$ and equating with the two factors in the left-hand side yields $p = 19$ and $q = 701$, and, from the first equation in (8), $r = 12601$, $p$, $q$ and $r$ being primes. This gives the $\phi$–amicable pair

\[(11) \quad 441035 = 5 \cdot 7 \cdot 12601, \quad 466165 = 5 \cdot 7 \cdot 19 \cdot 701, \quad k = 3,
\]

which is number 42 in Table 2. Other ways of writing the right-hand side of (10) as a product of two factors do not yield success. The two pairs of this form in Table 1 are obtained by choosing $a = 6$, $k = 7$ (number 4), and $a = 14$, $k = 5$ (number 9); both cases have $c = 2$.

For $c = 2, 4, 6, 8$, and 10, we have computed all the numbers $a$ with $2 \leq a \leq 10^5$ for which $(c + 2a)/\phi(a)$ is an integer (called $k$ in (9)). We found 76 solutions, and for each of them, we checked whether there are primes $p$, $q$, and $r$ satisfying (8). As a result we only found five primitive $\phi$–amicable pairs of the form $(ar, apq)$, and they all occur in our list of 499 primitive $\phi$–amicable pairs below $10^9$ (namely, as numbers 4, 9, 42, 109, and 148). In the case $c = 2, k = 4$, the first equation in (9) reduces to $1 = 2\phi(a) - a$, which occurs in the Corollary to Theorem 3. For the three largest solutions given below this Corollary (the other four have $a \leq 10^5$), we also checked (9), and we found the 24-digit primitive $\phi$–amicable pair with multiplier $k = 4$:

\[(12) \quad 64343053433705010822135 = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 7694364698739721,
\]

\[
46433068822434241053705 = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 64231061 \cdot 119791963.
\]

This construction method is analogous to similar methods known for (ordinary) amicable numbers. The simplest example is the so-called rule of Thabit ibn Qurrah [3] which constructs amicable pairs of the form $(apq, ar)$ where $a$ is a power of 2:

\[\text{If the three numbers } p = 3 \cdot 2^{n-1} - 1, \quad q = 3 \cdot 2^n - 1, \quad \text{and } r = 9 \cdot 2^{2^{n-1} - 1} - 1 \text{ are prime numbers, then } 2^n pq \text{ and } 2^n r \text{ form an amicable pair.}\]

This rule yields amicable pairs for $n = 2, 4$, and 7, but for no other values of $n \leq 20,000$ [2].

The $\phi$–amicable pairs which we have constructed so far are squarefree (if we choose $a$ to be squarefree). We also tried to find pairs of the form $a_1 = as^2r$, $a_2 = aspq$, where $s$ is a prime not dividing $a$. Similarly to the above derivation, we found the two $\phi$–amicable pairs

\[(13) \quad 2609115 = 3 \cdot 5 \cdot 31^2181, \quad 2747685 = 3 \cdot 5 \cdot 31 \cdot 19 \cdot 311, \quad k = 4,
\]

\[(14) \quad 11085135 = 3 \cdot 5 \cdot 31^2769, \quad 11770545 = 3 \cdot 5 \cdot 31 \cdot 17 \cdot 1489, \quad k = 4.
\]

From the pair (11), two more pairs with squarefree greatest common divisor can be generated with the help of Proposition 2a, namely with $p = 2$, and (next) $p = 3$; from each of the pairs (12), (13) and (14), one more such pair can be generated with Proposition 2a, $p = 2$.

With amicable pairs of the form $(au, ap)$ where $p$ is a prime and $\gcd(a, p) = 1$, it is often possible to associate many other amicable pairs of the form $(auq, ars)$ with the following rule [10, Theorem 2]:

\[\text{Let } (au, ap) \text{ be a given amicable pair, where } p \text{ is a prime with } \gcd(a, p) = 1 \text{ and let } C = (p + 1)(p + u). \text{ Write } C = D_1D_2 \text{ with } 0 < D_1 < D_2. \text{ If the three integers } r = p + D_1, \quad s = p + D_2, \quad \text{and } q = u + r + s \text{ are primes not dividing } a, \text{ then } (auq, ars) \text{ is also an amicable pair.}\]

At present, we know 319 amicable pairs of the required form $(au, ap)$, and for almost all of them the number $C$ has extremely many divisors. Consequently, many
Table 3. The 79 primitive $\phi$–amicable pairs generated by applying Theorem 5 to the 38 primitive $\phi$–amicable pairs $\leq 10^9$ which are of the form $(ap, au)$ with $p$ a prime, $\gcd(p, a) = 1$.

<table>
<thead>
<tr>
<th>#</th>
<th>a</th>
<th>p</th>
<th>u</th>
<th>k</th>
<th>D₁</th>
<th>the $D₁$'s</th>
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<td>73</td>
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<td>67</td>
<td>77</td>
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<td>1</td>
<td>16</td>
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<td>9</td>
<td>14</td>
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<td>319</td>
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<tr>
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<td>39</td>
<td>313</td>
<td>455</td>
<td>4</td>
<td>1</td>
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<td>299</td>
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<td>54</td>
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<td>465</td>
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<td>33263</td>
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<td>285</td>
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<td>203699</td>
<td>4</td>
<td>10</td>
<td>240, 1536, 2128, 2880, 14250, 25536, 161280, 178752, 220500, 238336</td>
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<tr>
<td>225</td>
<td>6</td>
<td>10266913</td>
<td>13689215</td>
<td>7</td>
<td>2</td>
<td>653184, 1586304</td>
</tr>
<tr>
<td>267</td>
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<td>18196993</td>
<td>24262655</td>
<td>7</td>
<td>4</td>
<td>86016, 180936, 5677056, 17842176</td>
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<td>935</td>
<td>269281</td>
<td>283679</td>
<td>3</td>
<td>3</td>
<td>1350, 61440, 345600</td>
</tr>
<tr>
<td>332</td>
<td>6578</td>
<td>44851</td>
<td>45149</td>
<td>5</td>
<td>4</td>
<td>120, 720, 1170, 1656</td>
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<tr>
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<td>506</td>
<td>642529</td>
<td>754271</td>
<td>5</td>
<td>3</td>
<td>270, 133860, 451632</td>
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<tr>
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<td>5478</td>
<td>87649</td>
<td>96031</td>
<td>7</td>
<td>3</td>
<td>73920, 86592, 95284</td>
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<td>2485</td>
<td>365509</td>
<td>375803</td>
<td>3</td>
<td>2</td>
<td>139392, 274912</td>
</tr>
</tbody>
</table>

thousands of new amicable pairs have been found with this rule, including several large pairs [9, Lemma 1].

Such a rule also exists for $\phi$–amicable pairs. We have the following result.

**Theorem 5.** Let $(ap, au; k)$ be a given $\phi$–amicable pair, where $p$ is a prime with $\gcd(a, p) = 1$ (notice that $a$ and $u$ need not be coprime), and let $C = (p−1)(p+u)$. Write $C = D₁D₂$ with $0 < D₁ < D₂$. If the three integers $r = p + D₁$, $s = p + D₂$, and $q = u + r + s$ are primes not dividing $a$, then $(ars, auq; k)$ is also a $\phi$–amicable pair.

This result has been applied to those primitive $\phi$–amicable pairs with larger member $\leq 10^9$ which are of the required form, and to the large pair (12).

From the pairs with larger member $\leq 10^9$, 79 primitive $\phi$–amicable pairs were generated. Table 3 gives the rank number of the “mother” pair of the form $(ap, au)$ in column 1, the values of $a$, $p$, $u$, and $k$ in columns 2–5, the number of primitive $\phi$–amicable pairs generated from this mother pair in column 6, and the “successful” values of $D₁$ in column 7. To any listed value of $D₁$ corresponds a primitive $\phi$–amicable pair of the form $(ars, auq)$ with $r = p + D₁$, $s = p + (p−1)(p+u)/D₁$, and $q = u + r + s$, with the same multiplier as the mother pair. The three pairs generated by mother pair number 4 have larger member $\leq 10^6$, and occur as numbers 36, 40,
Table 4. Six primitive φ–amicable pairs with larger member > 10^6 and ≤ 10^9, found with Theorem 5 from pairs 8, 9, 14, 17 and 26 in Table 1.

<table>
<thead>
<tr>
<th>from #</th>
<th>a₀</th>
<th>a₁</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3017465 = 5 · 11 · 83 · 661</td>
<td>3476935 = 5 · 7 · 11^2 · 821</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>979014 = 2 · 7 · 421 · 1481</td>
<td>9918986 = 2 · 7 · 11 · 29 · 2221</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>24236842 = 2 · 7 · 313 · 5531</td>
<td>27523958 = 2 · 7 · 11 · 29 · 6163</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>27716559 = 3 · 13 · 569 · 1249</td>
<td>40334385 = 3 · 5 · 7 · 13^2 · 273</td>
<td>4</td>
</tr>
<tr>
<td>26</td>
<td>61531118 = 2 · 11 · 23 · 277 · 439</td>
<td>71445682 = 2 · 7 · 11 · 23^2 · 877</td>
<td>5</td>
</tr>
<tr>
<td>17</td>
<td>90348545 = 5 · 7 · 1117 · 2311</td>
<td>95264575 = 5^2 · 7^2 · 19 · 47</td>
<td>3</td>
</tr>
</tbody>
</table>

and 47 (with $D_1 = 36$, 24, and 16, respectively) in Table 2. Furthermore, among the 79 pairs, there are six with larger member between 10^6 and 10^9 (also found with our exhaustive search). We list them in Table 4.

From pair (12) we found eight new large primitive φ–amicable pairs. The values
of $D_1$ in Theorem 5 leading to these eight pairs are:

\[ 73914531840, 76666855680, 7394851553280, 123635643997056, 193847579836416, 865200857636352, 3982965255818208, 4194831919218688. \]

All pairs have multiplier $k = 4$. In the largest pair (with $D_1 = 73914531840$) both members have 46 decimal digits:

\[ a_0 = 1030754714216455355643689856057807107041652655, \]
\[ a_1 = 1030754738868600773437177012460443629727125585, \]

and $a_0 = ars$, $a_1 = auq$, with $a = 3 · 5 · 17 · 353 · 929$,

\[ r = 7694438613271561, \quad s = 160194569014099815433, \]
\[ u = 64231061 · 119791963, \quad q = 1601961087817595849737. \]

Hence, in addition to the 499 primitive φ–amicable pairs with larger member ≤ 10^9 we have found with the help of the method described in the beginning of this section and with Theorem 5, another 79 primitive φ–amicable pairs with larger member > 10^9.

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