

REMARKS ON THE SCHOOF-ELKIES-ATKIN ALGORITHM

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ABSTRACT. Schoof's algorithm computes the number m of points on an elliptic curve E defined over a finite field \mathbb{F}_q . Schoof determines m modulo small primes ℓ using the characteristic equation of the Frobenius of E and polynomials of degree $O(\ell^2)$. With the works of Elkies and Atkin, we have just to compute, when ℓ is a "good" prime, an eigenvalue of the Frobenius using polynomials of degree $O(\ell)$. In this article, we compute the complexity of Müller's algorithm, which is the best known method for determining one eigenvalue and we improve the final step in some cases. Finally, when ℓ is "bad", we describe how to have polynomials of small degree and how to perform computations, in Schoof's algorithm, on x -values only.

1. INTRODUCTION

Let E be an elliptic curve defined over the finite field \mathbb{F}_q of large characteristic p . The set of \mathbb{F}_q -points of E , denoted $E(\mathbb{F}_q)$, is a finite abelian group [20].

In 1985, Schoof [17] gave a deterministic polynomial-time algorithm for computing $\#E(\mathbb{F}_q)$. The algorithm determines the characteristic equation of the Frobenius π of E , acting on the ℓ -torsion points $E[\ell]$ of E , for ℓ prime. But, working on $E[\ell]$ uses computations on polynomials modulo the ℓ -th division polynomial f_ℓ , and this is not practical, due to the size of f_ℓ .

In 1991, Elkies [10] showed how to perform computations in the kernel of an isogeny of degree ℓ , by computing a factor of degree $d = (\ell - 1)/2$ of f_ℓ . This idea works for nearly half the primes ℓ , called *Elkies primes*. For such an ℓ , the algorithm has just to compute an eigenvalue of π acting on $E[\ell]$.

Atkin [1] had given in 1988 the *sort and match* method used now for "bad" primes ℓ . Then he made the algorithm practical for very large finite fields [2] and the method became the SEA (for Schoof-Elkies-Atkin) algorithm.

For the last improvements in this scope, see [5], [6] and [12] and for the case p small, see [7] and the implementation in [13].

In this article we compute, for an Elkies prime ℓ , the complexity of the best asymptotic method used for computing an eigenvalue of π over $E[\ell]$ and we show then how to avoid, in some cases, the computation with y -coordinates of points. Finally, for a bad prime ℓ , we explain how to obtain a proper factor of f_ℓ and show then how to avoid again, in Schoof's algorithm, computations with y -coordinates of points.

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These results have enabled Morain [15] to compute $\#E(\mathbb{F}_p)$, for p prime of 500 digits (this is the actual record).

2. THE SEA ALGORITHM

2.1. Elliptic curve over \mathbb{F}_q . Let E be a non-supersingular elliptic curve given by an affine equation $\mathcal{F}(x, y) = 0$ where

$$\mathcal{F}(x, y) = y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6)$$

with the a_i 's in \mathbb{F}_q .

The set $E(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q, \mathcal{F}(x, y) = 0\} \cup \{O_E\}$ is an abelian group and the law, denoted \oplus , has $O_E = [0 : 1 : 0]$ as neutral element. We denote by f_n the n -th division polynomial in x . The degree of f_n is $(n^2 - 1)/2$ if n is odd. The group of n -torsion points, $E[n] = \{P \in E(\overline{\mathbb{F}}_q) \mid nP = O_E\}$ can be represented by $\mathbb{F}_q[x, y]/(f_n(x), \mathcal{F}(x, y))$ (see [18]).

The morphism $\pi : E(\overline{\mathbb{F}}_q) \rightarrow E(\overline{\mathbb{F}}_q)$, $(x, y) \mapsto (x^q, y^q)$ of E satisfies $\pi^2 - t\pi + q = 0$ over $E(\overline{\mathbb{F}}_q)$, with $t \in \mathbb{Z}$, satisfying $|t| \leq 2\sqrt{q}$. Recall : $\#E(\mathbb{F}_q) = q + 1 - t$. When ℓ is an odd prime number (see [6] for $\ell = 2$), we consider the restriction π_ℓ of π to $E[\ell]$, which satisfies $\pi_\ell^2 - t\pi_\ell + k = 0$ over $E[\ell]$ with $t \equiv \tau \pmod{\ell}$ and $q \equiv k \pmod{\ell}$. Now, if $\ell \neq p$, $E[\ell] \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$, so we can view $E[\ell]$ as a vector space over \mathbb{F}_ℓ and $x^2 - tx + k$ as the characteristic equation of π_ℓ . We denote by $G_1, G_2, \dots, G_{\ell+1}$ the $(\ell + 1)$ cyclic subgroups of $E(\overline{\mathbb{F}}_q)$, of order ℓ .

2.2. The SEA algorithm. Schoof [17] determines $\#E(\mathbb{F}_q) = q + 1 - t$ by searching for a match among the ℓ equations $(x^{q^2}, y^{q^2}) \oplus k(x, y) = \tilde{\tau}(x^q, y^q)$, $0 \leq \tilde{\tau} \leq \ell - 1$, over $E[\ell]$.

Elkies works in the kernel G_i of one of the $\ell + 1$ isogenies $E \xrightarrow{E_i} E_i, 1 \leq i \leq \ell + 1$, of degree ℓ . When $D = \tau^2 - 4k$ is a square modulo ℓ the eigenvalues of π_ℓ are in \mathbb{F}_ℓ and ℓ is called an Elkies prime. Hence, in this case, the eigenspaces are \mathbb{F}_q -rational and the corresponding isogenies are defined over \mathbb{F}_q and if we let $E[\ell]_\lambda$ be an eigenspace with P_λ a generator, we have $h_\ell(x) = \prod_{i=1}^d (x - x(iP_\lambda)) \in \mathbb{F}_q[x]$ and

$$E[\ell]_\lambda = \langle P_\lambda \rangle = \mathbb{F}_q[x, y]/(h_\ell(x), \mathcal{F}(x, y)).$$

Let $\Phi_\ell(x, y) = 0$ be the canonical equation of the modular curve $X_0(\ell)$ (see [2], [15] for a simpler equation). We know that ℓ is an Elkies prime if and only if $\Phi_\ell(j(E), x) = 0$ has a root in \mathbb{F}_q .

For $p \neq 2, 3$ and ℓ an Elkies prime, the formulas of Atkin [2],[15] give, from a root of $\Phi_\ell(j(E), x) = 0$ in \mathbb{F}_q , the value of $p_1 = \sum_{i=1}^d x(iP_\lambda)$ and the coefficients of the corresponding E_i . So [10], one can compute the $p_k = \sum_{i=1}^d x^k(iP_\lambda)$ for $1 \leq k \leq d$ and hence h_ℓ by Newton's formula if $\ell \ll p$.

If $p = 2$ or 3 or $\ell \approx p$ see Couveignes' work [7] and also [13].

Once h_ℓ is known, we have to search a match among the $\ell - 1$ equations $(x^q, y^q) = \tilde{\lambda}(x, y)$, $1 \leq \tilde{\lambda} \leq \ell - 1$, over $E[\ell]_\lambda$.

If D is not a square modulo ℓ , then ℓ is called an *Atkin prime* and the G_i 's are \mathbb{F}_{p^e} -rational where e is the smallest integer n for which π_ℓ^n is in \mathbb{F}_ℓ .

3. LOOKING FOR ONE EIGENVALUE

3.1. Computing $\lambda \pmod{\ell}$. We compute the complexity of the algorithm of Müller [16] which computes $\lambda \pmod{\ell}$. Müller uses an integer $k_{opt} \approx \lceil \sqrt{d} \rceil$ such that for all

λ in \mathbb{F}_ℓ^* , there are integers i, j with $1 \leq i, j \leq k_{opt}$ such that $\lambda \equiv \pm i/j \pmod{\ell}$. So, he compares $j(x^q, y^q)$ and $i(x, y)$ using division polynomials, which means comparisons of rational functions.

The elementary operation is taken to be the cost of one multiplication of two elements in \mathbb{F}_q . Let $M(d)$ be the number of operations needed to compute the multiplication of two polynomials of degree d (see [11]).

Proposition 1. *Müller’s method takes $O(M(d) \log q) + O(\sqrt{d}M(d)) + O(d^2)$ operations and $O(d\sqrt{d})$ space.*

Proof. The computation of $x^q \pmod{h_\ell(x)}$ requires $O(M(d) \log q)$ operations and the computation of the k_{opt} first division polynomials requires $O(\sqrt{d}M(d))$ operations.

The $x(j(x^q, y^q))$ are computed using the recursive formulae of division polynomials in x^q (see [8]). This requires $O(\sqrt{d}M(d))$ operations, which is more efficient than modular compositions $x(j(x, y)) \circ x^q$ (see [4], [19]).

To compare two rational functions modulo $h_\ell(x)$ in $\mathbb{F}_q[x]$, one can test the match using a random linear map [16] and then verify polynomial equality. So, the comparisons of coordinates takes $O(2M(d)) + O(2d^2)$ operations. \square

3.2. The sign of $\lambda \pmod{\ell}$. Suppose we have integers i, j such that $j\pi_\ell = \pm i$ over $E[\ell]_\lambda$, where ℓ is an Elkies prime. We have $\lambda \equiv \pm\lambda_0 \pmod{\ell}$ with $\lambda_0 \equiv ij^{-1} \pmod{\ell}$. For $\mu \in \mathbb{F}_{\ell^2}^*$, we call *semi-order* of μ , denoted $s(\mu)$, the order of μ in $\mathbb{F}_{\ell^2}^*/(\pm 1)$.

- If $p \neq 2$, E has an equation of the form $y^2 = \mathcal{G}(x) := x^3 + a_2x^2 + a_4x + a_6$.

Theorem 1. *Let h_ℓ be the factor of f_ℓ corresponding to λ and g_ℓ be a factor of degree $s(\lambda_0)$ of h_ℓ and let r be $\text{Resultant}(g_\ell, \mathcal{G})$. Then $\lambda = \lambda_0^{s(\lambda_0)} (\frac{r}{q}) \lambda_0$. When $\ell \equiv 3 \pmod{4}$, one can take $g_\ell = h_\ell$.*

Proof. For $s(\lambda_0)$ odd, we have $\pi_\ell^{s(\lambda_0)} = \pm \lambda_0^{s(\lambda_0)} Id$ over $E[\ell]_\lambda$. If $\pi_\ell^{s(\lambda_0)} = Id$, then $E[\ell]_\lambda \subset E(\mathbb{F}_{q^{s(\lambda_0)}})$; hence, for all P in $E[\ell]_\lambda$, $\mathcal{G}(x(P))$ is a square in $\mathbb{F}_{q^{s(\lambda_0)}}$, and since $\prod_{i=1}^{s(\lambda_0)} (\mathcal{G}(x_i)) = r$, with x_i the roots of g_ℓ , r is a square in \mathbb{F}_q . Whereas, if $\pi_\ell^{s(\lambda_0)} = -Id$ over $E[\ell]_\lambda$, then $(\frac{r}{q}) = -1$. \square

Note that, if $\ell \equiv 1 \pmod{4}$, then one can compute λ_0 using h_ℓ and then determine λ , if $s(\lambda_0) = s(\pm\lambda_0)$ is odd, using a factor of h_ℓ .

- If $p = 2$, let $y^2 + xy = x^3 + B$ (with $B \in \mathbb{F}_{2^m}$) be an equation of E (see [14]).

Proposition 2. *Let h_ℓ be a factor of f_ℓ corresponding to $\lambda = \pm\lambda_0$. If h_ℓ has a factor $g_\ell = x^{s(\lambda_0)} - \tilde{s}_1 x^{s(\lambda_0)-1} + \dots + (-1)^{s(\lambda_0)} \tilde{s}_{s(\lambda_0)}$ of odd degree $s(\lambda_0)$, then*

$$\lambda = \begin{cases} \lambda_0^{s(\lambda_0)} \lambda_0 & \text{if } \text{Tr}(\tilde{s}_1 + B(\tilde{s}_{s(\lambda_0)-1}^2 - 2\tilde{s}_{s(\lambda_0)}\tilde{s}_{s(\lambda_0)-2})/\tilde{s}_{s(\lambda_0)}^2) = 0, \\ -\lambda_0^{s(\lambda_0)} \lambda_0, & \text{otherwise.} \end{cases}$$

When $\ell \equiv 3 \pmod{4}$, one can take $g_\ell = h_\ell$.

Proof. The equation $X^2 + X = \gamma$ has a root in an extension \mathbb{F}_{2^n} if and only if $\text{Tr}(\gamma) = 0$ (see [9]). Hence the points of $E[\ell]_\lambda = \langle P = (x, y) \rangle$ are in $\mathbb{F}_{q^{s(\lambda_0)}}$ if and only if $\text{Tr}(\gamma_i) = 0$, where $x_i = x(iP)$ and $\gamma_i = x_i + B/x_i^2$. Finally, computing $\sum_{i=1}^{s(\lambda_0)} \text{Tr}(\gamma_i)$ gives the desired result. \square

4. ELKIES' METHOD FOR ATKIN PRIMES

4.1. **Computing a factor of f_ℓ .** Assume that $q = p$ prime, $\neq 2, 3$ and that ℓ is an Atkin prime with $\ell \ll p$.

The $(\ell + 1)$ curves E_i are defined over \mathbb{F}_{p^e} , hence f_ℓ has a factor of degree d over \mathbb{F}_{p^e} and so by conjugation we can find a factor of degree ed over \mathbb{F}_p .

First, we compute a monic irreducible factor $M_\ell(x)$ of degree e of $\Phi_\ell(j(E), x)$ in $\mathbb{F}_p[x]$. We denote by $x_i, i = 1, 2, \dots, e$, the roots of $M_\ell(x) = 0$ in \mathbb{F}_{p^e} . Then, in $\mathbb{F}_p[x]/M_\ell(x)$, we determine ed polynomials $p_k(x) = \sum_{j=0}^{e-1} a_{j,k}x^j$ of degree $e - 1$, (see [2], [10]) and since, for $1 \leq k \leq ed$, we have

$$p_k \stackrel{\text{def}}{=} \sum_{i=1}^e p_k(x_i) = \sum_{i=1}^e \left(\sum_{j=0}^{e-1} a_{j,k} x_i^j \right) = \sum_{j=0}^{e-1} a_{j,k} \left(\sum_{i=1}^e x_i^j \right) = \sum_{j=0}^{e-1} a_{j,k} \tilde{p}_j$$

with $\tilde{p}_j = \sum_{i=1}^e x_i^j$ computed from the symmetric functions of $M_\ell(x)$, a factor of degree ed of f_ℓ can be computed.

Example. We consider the elliptic curve $y^2 = x^3 + 2x + 41$ over \mathbb{F}_{59} with $j = 31$. We determine a factor of the division polynomial f_5 of E . Over \mathbb{F}_{59} , $x^3 + 41x^2 + 45x + 32$ is a factor of $\Phi_5(x, 31)$. We obtain

$p_1(x)$	$56x^2 + 31x + 41$	$p_4(x)$	$16x^2 + 11x + 6$
$p_2(x)$	$46x^2 + 22x + 26$	$p_5(x)$	$51x^2 + 41x + 17$
$p_3(x)$	$21x^2 + 20x + 39$	$p_6(x)$	$34x^2 + 41x$

And $p_0 = 2, p_1 = 38, p_2 = 28, p_3 = 22, p_4 = 7, p_5 = 38, p_6 = 21$, hence $x^6 + 21x^5 + 13x^3 + 10x^2 + 3x + 55$ is a factor of f_5 over \mathbb{F}_{59} .

4.2. **Computing $t \bmod \ell$.** We show how, when ℓ is an Atkin prime, we can test the equation $\pi_\ell^2 + k = \tilde{\tau}\pi_\ell$ in $\tilde{\tau}$ by computing only x -coordinates of points. We recall first that if

$$(x_1, y_1) \oplus (x_2, y_2) = (x_3, y_3) \quad \text{and} \quad (x_1, y_1) \ominus (x_2, y_2) = (x_4, y_4),$$

then we have

$$(x_3 + x_4)(x_1 - x_2)^2 = S(x_1, x_2) \quad \text{and} \quad x_3x_4(x_1 - x_2)^2 = P(x_1, x_2)$$

with

$$S(x_1, x_2) = (x_1 + x_2)(a_1a_3 + 2a_4 + 2x_1x_2) + x_1x_2(a_1^2 + 4a_2) + 4a_6 + a_3^2,$$

and

$$P(x_1, x_2) = (x_1x_2 - a_4)(x_1x_2 - a_4 - a_1a_3) - (x_1 + x_2 + a_2)(a_3^2 + 4a_6) - a_1^2a_6.$$

So the values x_3 and x_4 are solutions of the quadratic equation $E(X) = NX^2 - SX + P$ with $N(x_1, x_2) = (x_1 - x_2)^2$.

Following Müller's idea, we introduce the integers i, j and k_{opt} with the equation $i\pi_\ell^2 + ik = j\pi_\ell$. We search a value j for which $x(j\pi_\ell)$ is a root of $E(X) = 0$ given by $S(x_i^{q^2}, x_{ik}), P(x_i^{q^2}, x_{ik})$ and $N(x_i^{q^2}, x_{ik})$.

Indeed, if $x(i\pi_\ell^2 + ik) = x(j\pi_\ell)$, then, for some $\tau_0, \pi_\ell^2 + k = \pm\tau_0\pi_\ell$ over $E[\ell]$, so $\tau \equiv \pm\tau_0 \bmod \ell$. Whereas, if $x(i\pi_\ell^2 - ik) = x(j\pi_\ell)$, then $\pi_\ell^2 - k = \pm\tau_0\pi_\ell$ and $\pi_\ell = 2k/(\tau \pm \tau_0)$, which is impossible since ℓ is an Atkin prime.

Hence, we avoid the computation of y^{q^2} and y^q and obtain $t \equiv \pm\tau_0 \bmod \ell$.

4.3. **The sign of $t \bmod \ell$.** Since π_ℓ satisfies the equation $x^2 - \tau x + k = 0$, we have $\pi_\ell^n = Q_n \pi_\ell + P_n$ with P_n and Q_n some polynomials in τ and k . We have $P_n = -kQ_{n-1}$ and moreover the polynomial Q_n contains only even powers of τ if n is odd and only odd powers otherwise [3]. On the other hand, $\pi_\ell^e = P_e$ and the value of e does not depend on the sign of τ . Hence, when e is odd, we have $P_e(\pm\tilde{\tau}, k) = \pm P_e(\tilde{\tau}, k)$, so $\pi_\ell^e = \pm P_e(\tau_0, k)$. Let w_0 be $P_e(\tau_0, k)$.

Proposition 3. *Assume $p \neq 2$, e odd; let h_ℓ be a factor of degree ed of f_ℓ , g_ℓ be a factor of degree $es(w_0)$ of h_ℓ and r be $\text{Resultant}(g_\ell, \mathcal{G})$. Then, when $s(w_0)$ is odd, we have $t \equiv \binom{r}{q} w_0^{s(w_0)} \tau_0 \bmod \ell$. When $\ell \equiv 3 \bmod 4$, one can take $g_\ell = h_\ell$.*

Proof. We have $\pi_\ell^e = \pm w_0 \text{Id}$ over $E[\ell]$; hence, if $s(w_0)$ is odd, then $\pi_\ell^{es(w_0)} = \pm w_0^{s(w_0)} \text{Id}$ over $E[\ell]$ and, if d is odd, then $\pi_\ell^{ed} = \pm w_0^d \text{Id} = \pm \left(\frac{w_0}{\ell}\right) \text{Id}$ over $E[\ell]$. \square

From $\pi_\ell^2 = \tau_0 \pi_\ell + k$, we easily compute $w_0 = P_e(\tau_0, k)$. The decomposition type of h_ℓ is determined by computing $s(w_0)$.

Example. Let us consider the curve $y^2 = x^3 + 4312x + 9167$ over \mathbb{F}_{12853} . If $\ell = 19$, then we have $e = 5$ and using a factor h_{19} of degree 45 of f_{19} we obtain $t \equiv \pm 7 \bmod 19$. We compute $r = \text{Resultant}(x^3 + 4312x + 9167, h_{19}) = 11226$; since $\binom{r}{p} = 1$ and $w_0 = P_5(7, 9) = 4$, we have $t \equiv 7 \bmod 19$.

If $\ell = 13$, then $e = 7$ and $\tau_0 = 5$. Since $w_0 = P_7(5, 9) = 10$, and $s(10) = 3$, the polynomial h_{13} has an irreducible factor g_{13} of degree 21. We obtain $r = \text{Resultant}(x^3 + 4312x + 9167, g_{13}) = 9515$ and $\binom{r}{p} = -1$, so we have $t \equiv 5 \bmod 13$.

Proposition 4. *Let h_ℓ be a factor of degree ed of f_ℓ . If $p = 2$ and e is odd, then, when $s(w_0)$ is odd, we have*

$$\tau = \begin{cases} w_0^{s(w_0)} \tau_0 & \text{if } \text{Tr}(\tilde{s}_1 + B(\tilde{s}_{es(w_0)-1}^2 - 2\tilde{s}_{es(w_0)} \tilde{s}_{es(w_0)-2}) / \tilde{s}_{es(w_0)}^2) = 0, \\ -w_0^{s(w_0)} \tau_0 & \text{otherwise,} \end{cases}$$

with \tilde{s}_i the symmetric functions of a factor g_ℓ of h_ℓ of degree $es(w)$. When $\ell \equiv 3 \bmod 4$, one can take $g_\ell = h_\ell$.

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