VORTEX METHOD
FOR TWO DIMENSIONAL EULER EQUATIONS
IN BOUNDED DOMAINS WITH BOUNDARY CORRECTION

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ABSTRACT. The vortex method for the initial-boundary value problems of the Euler equations for incompressible flow is studied. A boundary correction technique is introduced to generate second order accuracy. Convergence and error estimates are proved.

1. INTRODUCTION

The vortex methods are employed in the numerical simulation of incompressible flow. For viscous flow the Navier-Stokes equations are approximated by using an operator splitting between the inviscid Euler equations and the Stokes equations. One difficulty of the vortex methods is the treatment of boundary conditions. In the viscous step a vorticity creation operator is applied to maintain the no-slip condition. In the inviscid step the particle method is used to simulate the vorticity. The scheme is required to be consistent and stable. The purpose of this paper is to study this problem in the presence of boundaries. We restrict ourselves to the Euler equations only, because we think it is a necessary first step toward studying the problem for the entire scheme.

The consistency, stability and convergence of vortex methods for the initial value problems of the Euler equations was first obtained by Hald and DelPrete [9], [10], [11]; then the results were improved and different proofs were given by Anderson and Greengard [4], Beale and Majda [5], [6], and Raviart [13]. However there are some essential difficulties in the study of the vortex method for initial-boundary value problems. The high order consistency result depends on an application of Taylor's expansion which yields a superconvergence estimation. Therefore a uniform mesh is required for initial value problems, and it is difficult to get the same high order consistency for a bounded domain. Chiu and Nicolaides suggested some nonuniform meshes and introduced the derivatives of Dirac functions in the formula to get the necessary accuracy; however, the convergence result was proved for initial value problems.

To overcome the difficulty, the authors of [15] and [17] applied an extrapolation technique to make correction on the boundary which also gave high order convergence results for two dimensional initial-boundary value problems. Convergence of vortex methods for three dimensional initial-boundary value problems of the Euler

equations was proved in [16], where for simplicity the initial data was assumed compactly supported. For general initial data the same boundary correction technique is also needed.

There are boundary layers for viscous flows. The gradient of the velocity field near the boundary is large, so the extrapolation approach causes a large error. We introduce a simpler boundary correction technique in this paper, one for which extrapolation is no longer needed. We will prove that this scheme possesses second order accuracy.

In the next section we will state our scheme. In §3 we will prove our convergence results. We will apply some results of the theory for initial value problems which can be found in the paper of Raviart [13], or in the book by the author and Zhang [18]. Finally we will give a numerical example which shows by contrast the difference among different treatment of boundary conditions.

2. Scheme

For simplicity we assume that Ω is a bounded and convex domain in $\mathbb{R}^2$, the boundary of which, $\partial\Omega$, is sufficiently smooth. Let $x = (x_1, x_2)$ be the points in $\mathbb{R}^2$. We consider the following initial-boundary value problems:

$$(1) \qquad \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = 0,$$

$$(2) \qquad \nabla \cdot u = 0,$$

$$(3) \qquad u \cdot n|_{\partial\Omega} = 0,$$

$$(4) \qquad u|_{t=0} = u_0,$$

where $u = (u_1, u_2)$ is velocity, p is pressure, ρ is a constant density, $\nabla = (\partial_1, \partial_2)$, and n is the outward unit normal vector on the boundary. Let $\omega = -\nabla \wedge u$ be vorticity, where $\nabla\wedge = (\partial_2, -\partial_1)$, and ψ be the stream function such that $u = \nabla\wedge\psi$. Then the vorticity-stream function formulation for the problem is

$$(5) \qquad \frac{\partial \omega}{\partial t} + u \cdot \nabla\omega = 0,$$

$$(6) \qquad -\triangle\psi = \omega, \quad \psi|_{\partial\Omega} = 0, \quad u = \nabla \wedge \psi,$$

$$(7) \qquad \omega|_{t=0} = \omega_0 = -\nabla \wedge u_0.$$

If u_0 satisfies the consistency condition $u_0 \cdot n|_{\partial\Omega} = 0$, and if it is sufficiently smooth, then there is a unique smooth solution on $\bar{\Omega} \times [0, T]$ for any $T > 0$ [14].

Let u^ε, ω^ε, ψ^ε be the approximate velocity, vorticity, and stream function respectively, and X_j^ε be the approximate characteristic curve, where $\varepsilon > 0$ is a parameter and j is an index corresponding to a particular particle. For each j, X_j^ε satifies

$$(8) \qquad \frac{dX_j^\varepsilon}{dt} = u^\varepsilon(X_j^\varepsilon(t), t), \qquad X_j^\varepsilon(0) = X_j.$$

For simplicity we assume that the support of "vortex blob function" ζ is in the unit disk, supp $\zeta \subset S(o, 1)$; then we define $\zeta_\varepsilon(x) = \frac{1}{\varepsilon^2}\zeta(\frac{x}{\varepsilon})$. As usual, we require that

$$\int \zeta(x)\,dx = 1.$$

For a given time t, ω^ε is a linear combination,

$$\omega^\varepsilon(x,t) = \sum_{j \in J \cup J(t)} \alpha_j^\varepsilon \zeta_\varepsilon(x - X_j^\varepsilon(t)), \tag{9}$$

where $J = \{j; X_j \in \bar{\Omega}\}$, and $J(t)$ is the set of additional particles. Let us explain it in detail.

For each particle near the boundary we fix one reference point, $X_{j_0}^\varepsilon$, on the boundary at $t = 0$. For definiteness we may take $X_{j_0}^\varepsilon$ to be the nearest point to X_j. According to (8) we get $X_{j_0}^\varepsilon$ for all t. We define

$$X_{j'}^\varepsilon(t) = 2X_{j_0}^\varepsilon(t) - X_j^\varepsilon(t), \quad \alpha_{j'}^\varepsilon = \alpha_j^\varepsilon,$$

which yields one term in (9) with $j' \in J(t)$. Because the equation (5) is homogeneous, α_j^ε does not change with time. If the distance from $X_{j'}^\varepsilon$ to $\partial\Omega$ is greater than ε, then this term has no contribution, so only those X_j^ε in a neighborhood of $\partial\Omega$ need to be taken into account.

The equations for u^ε and ψ^ε are the same:

$$-\triangle \psi^\varepsilon = \omega^\varepsilon, \qquad u^\varepsilon = \nabla \wedge \psi^\varepsilon, \tag{10}$$

$$\psi^\varepsilon|_{\partial\Omega} = 0. \tag{11}$$

To solve (8)–(11) further discretization is needed. However we restrict ourselves to consider semi-discrete problems in this paper. Some material on full discretization was presented in [17]. If this scheme is applied to the inviscid step for the Navier-Stokes equations, then $X_{j_0}^\varepsilon$ keeps stationary, and the scheme is even simpler.

To prove the consistency, stability, and convergence of the scheme (8)–(11) we need a regular mesh as usual. Let us construct the mesh as follows.

For each point $x \in \partial\Omega$, there is a neighborbood such that $\partial\Omega$ in this neighborhood can be expressed in terms of a smooth function in local coordinates. Let us take a finite set of them, $\{\Omega^{(k)}\}_{k=1}^K$, such that $\bigcup_{k=1}^K \Omega^{(k)}$ covers $\partial\Omega$. We thus take a subdomain $\Omega^{(0)} \subset \Omega$ such that $\bigcup_{k=0}^K \Omega^{(k)}$ covers $\bar{\Omega}$. For each $\Omega^{(k)}$, $k = 1, \cdots, K$, there is a domain $\tilde{\Omega}$ on the ξ-plane $\mathbb{R}^2$ such that there is a bijection $F^{(k)}$ defined on $\tilde{\Omega}$ such that $F^{(k)}$ and $(F^{(k)})^{-1}$ are smooth and $F^{(k)}\tilde{\Omega} = \Omega^{(k)}$. Moreover we require that the image of $\{\xi; \xi_2 = 0\} \cap \tilde{\Omega}$ is $\partial\Omega \cap \Omega^{(k)}$ and $\xi_2 > 0$ corresponds to the interior of Ω, where $\xi = (\xi_1, \xi_2)$. We subdivide $\tilde{\Omega}$ into squares with side length h in the following way: Let $j = (j_1, j_2, k)$, $j_1, j_2 \in \mathbb{Z}$, $\tilde{X}_j = (j_1 h, j_2 h)$, $\tilde{B}_j = \{\xi; j_1 h - \frac{1}{2}h < \xi_1 < j_1 h + \frac{1}{2}h, j_2 h - \frac{1}{2}h < \xi_2 < j_2 h + \frac{1}{2}h\}$. Then we define $X_j = F^{(k)}\tilde{X}_j$, $B_j = F^{(k)}\tilde{B}_j$. Thus if $j_2 = 0$, $X_j \in \partial\Omega$, and if $j_2 > 0$, $X_j \in \Omega$. The domain $\Omega^{(0)}$ is subdivided without mapping.

We define functions $\varphi_k \in C_0^\infty(\Omega^{(k)})$ such that $\sum_{k=0}^K \varphi_k(x) \equiv 1$ for $x \in \bar{\Omega}$. Then we define $\omega_0^{(k)} = \varphi_k \omega_0$, $\alpha_j = \omega_0^{(k)}(X_j)|A(\tilde{X}_j)|h^2$, where $A(\tilde{X}_j)$ is the Jacobian of $F^{(k)}$ at the point $\tilde{X}_j$. Let $J^{(k)}$ be the set of all nodes X_j already defined in $\Omega^{(k)}$, and $J^{(0)}$ the set of nodes in $\Omega^{(0)}$. Denote by J_1 the union $\bigcup_{k=0}^K J^{(k)}$. We notice that the meshes overlap each other and some nodes are in the exterior of $\bar{\Omega}$. Let $J = \{j \in J_1; X_j \in \bar{\Omega}\}$. For a given time t, if for one index j the distance $\mathrm{dist}(X_j^\varepsilon(t), \partial\Omega) < \varepsilon$, then we take the symmetric point of $X_j^\varepsilon(t)$ at the exterior of Ω as follows: Let $j = (j_1, j_2.k)$; then we define

$$j' = Ej = (j_1, -j_2, k), \quad j_0 = (j_1, 0, k),$$

and let $J(t)$ be the set $\{j'\}$.

For a single mesh the centers of any two neighboring vortex blobs cannot approach each other, but that is not the case now. This is the main difference between this overlapped mesh and a single mesh. Since vortex blobs are applied instead of point vortices, we will see that, however, consistency and stability still hold.

3. CONVERGENCE

First of all let us introduce some new functions and notations to meet the needs of our convergence proof. Let u, ω, ψ be the solutions to (1)-(4) and (5)–(7) on the domain $\bar{\Omega} \cap [0, T]$. We extend ψ smoothly so that it is compactly supported in $(\bigcup_{k=0}^{K} \Omega^{(k)}) \times [0, T]$. From ψ we get extensions of u and ω by definition. Extending to zero on the whole space, we get an equation for ω and u:

$$(12) \qquad \frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = F$$

on $\mathbb{R}^2 \times [0, T]$. By (5), $F \equiv 0$ in $\bar{\Omega} \times [0, T]$. We define $X_j(t)$ and $\alpha_j(t)$ by

$$(13) \qquad \frac{dX_j}{dt} = u(X_j(t), t), \quad X_j(0) = X_j, \quad j \in J_1,$$

$$(14) \qquad \begin{aligned} \frac{d\alpha_j}{dt} &= \varphi_k(X_j) F(X_j(t), t) |A(\tilde{X}_j)| h^2, \\ \alpha_j(0) &= \omega_0^{(k)}(X_j) |A(\tilde{X}_j)| h^2, \quad j \in J_1. \end{aligned}$$

Letting $J_0 = \{j \in J_1; X_j \in \partial\Omega\}$, we define the L^p-norm of the error of characteristic curves,

$$(15) \quad \|e(t)\|_{0,p,h} = \left(\sum_{j \in J} h^2 |X_j(t) - X_j^\varepsilon(t)|^p \right)^{\frac{1}{p}} + \varepsilon^{\frac{1}{p}} \left(\sum_{j \in J_0} h |X_j(t) - X_j^\varepsilon(t)|^p \right)^{\frac{1}{p}},$$

$$(16) \qquad \|e(t)\|_{0,\infty,h} = \max_{j \in J} |X_j(t) - X_j^\varepsilon(t)|.$$

Denote $u = G(\omega)$ if u and ω is related by (6). We decompose the error of velocity into three parts:

$$u - u^\varepsilon = v_1 + v_2 + v_3,$$

where

$$v_1 = u - G(\omega(\cdot, t) * \zeta_\varepsilon),$$

$$v_2 = G\left(\omega(\cdot, t) * \zeta_\varepsilon - \sum_{j \in J_1} \alpha_j(t) \zeta_\varepsilon(\cdot - X_j(t)) \right),$$

$$v_3 = G\left(\sum_{j \in J_1} \alpha_j(t) \zeta_\varepsilon(\cdot - X_j(t)) - \sum_{j \in J \cup J(t)} \alpha_j^\varepsilon \zeta_\varepsilon(\cdot - X_j^\varepsilon(t)) \right).$$

Let us estimate the semi-norm $|\cdot|_{l,p}$ of v_i, $i = 1, 2, 3$. We will always denote by C a generic positive constant which is independent of the mesh parameter h and ε, and by $C_1, C_2, \cdots$ some particular positive constants.

Lemma 1. *If $\zeta \in L^\infty(\mathbb{R}^2)$ and*

$$(17) \qquad \int_{\mathbb{R}^2} x^\alpha \zeta(x)\, dx = 0, \qquad |\alpha| = 1,$$

then for integers $l \geq 0$

$$(18) \qquad |v_1(\cdot, t)|_{l,p,\Omega} \leq C\varepsilon^2, \qquad 1 \leq p \leq +\infty.$$

Proof. See [18] for details. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2. *Let $p > 1$, $l \geq 0$ and $\zeta \in W^{m+l-1,\infty}(\mathbb{R}^2)$ for $m \geq 1$. Then for any constant $r \in [1,2)$ and an arbitrary integer $N \geq 3$, we have*

$$(19) \qquad |v_2(\cdot, t)|_{l,p,\Omega} \leq C \left(\left(1 + \frac{h}{\varepsilon} \right)^{\frac{2}{r}} \frac{h^m}{\varepsilon^{m+l-1}} + \frac{h^N}{\varepsilon^{N+l-1}} \right).$$

Proof. Since v_2 is a linear combination with respect to k's, it would suffice to consider one k only. We define characteristic curves for all initial points, $\xi(\tau; x, t)$ by

$$(20) \qquad\qquad\qquad \frac{d\xi}{dt} = u(\xi(\tau; x, t), \tau),$$

$$(21) \qquad\qquad\qquad\qquad \xi(t; x, t) = x.$$

Then we consider

$$v_2^{(k)} = G \left((\varphi_k(\xi(0; \cdot, t))\omega(\cdot, t)) * \zeta_\varepsilon - \sum_{j \in J^{(k)}} \alpha_j(t)\zeta_\varepsilon(\cdot - X_j(t)) \right).$$

Since the mapping $x \to \xi$ defined by (20) and (21) is measure preserving, we have

$$((\varphi_k(\xi(0; \cdot, t))\omega(\cdot, t)) * \zeta_\varepsilon)(x)$$

$$= \int \zeta_\varepsilon(x - y)\varphi_k(\xi(0; y, t))\omega(y, t)\, dy$$

$$= \int \zeta_\varepsilon(x - \xi(t; \eta, 0))\varphi_k(\eta)\omega(\xi(t; \eta, 0), t)\, d\eta$$

$$= \int \zeta_\varepsilon(x - \xi(t; F^{(k)}z, 0))\varphi_k(F^{(k)}z)\omega(\xi(t; F^{(k)}z, 0), t)|A(z)|\, dz.$$

We define

$$g(x, z, t) = \partial_x^\gamma \zeta_\varepsilon(x - \xi(t; F^{(k)}z, 0))\varphi_k(F^{(k)}z)\omega(\xi(t; F^{(k)}z, 0), t)|A(z)|;$$

then by the equations (12)–(14) we have

$$\alpha_j(t) = \varphi_k(X_j)\omega(\xi(t; X_j, 0), t)|A(\tilde{X}_j)|h^2,$$

but $X_j = F^{(k)}\tilde{X}_j$, so

$$g(x, \tilde{X}_j, t) = \alpha_j(t)\partial^\gamma \zeta_\varepsilon(x - X_j(t))/h^2.$$

Consequently

$$\partial^\gamma(((\varphi_k(\xi(0; \cdot, t))\omega(\cdot, t)) * \zeta_\varepsilon)(x) - \sum_{j \in J^{(k)}} \alpha_j(t)\zeta_\varepsilon(x - X_j(t)))$$

$$= \int g(x, z, t)\, dz - \sum_{j \in J^{(k)}} g(x, \tilde{X}_j, t)h^2.$$

Let $|\gamma| = l$; then, following the same lines as the proof in [18], we get the estimation (19) for $v_2^{(k)}$, which completes the proof. $\qquad\square$

Lemma 3. *If there are constants C_1, C_2 such that $|\nabla u^\varepsilon| \leq C_1$, $h \leq C_2\varepsilon$, and if $\zeta \in W^{l,\infty}(\mathbb{R}^2)$ for $l \geq 0$, then for sufficiently small ε we have*

$$|v_3(\cdot, t)|_{l,p,\Omega} \leq \frac{C}{\varepsilon^l} \left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}} \left(\|e(t)\|_{0,p,h} + \varepsilon^{2+\frac{1}{p}}\right),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $|\nabla u| \leq C$ and $|\nabla u^\varepsilon| \leq C_1$, there is a constant $C_0 > 1$ such that if $\mathrm{dist}(X_j, \bar{\Omega}) > C_0\varepsilon$, then supp $\zeta_\varepsilon(\cdot - X_j(t))$ and supp $\zeta_\varepsilon(\cdot - X_j^\varepsilon(t))$ have no intersection with $\bar{\Omega}$. We define $J_2 = \{j; 0 < \mathrm{dist}(X_j, \bar{\Omega}) \leq C_0\varepsilon\}$; then

$$v_3 = G\left(\sum_{j \in J \cup J_2} (\alpha_j(t)\zeta_\varepsilon(\cdot - X_j(t)) - \alpha_j^\varepsilon \zeta_\varepsilon(\cdot - X_j^\varepsilon(t)))\right).$$

Let

$$v_3 = v_{3,1} + v_{3,2} + v_{3,3},$$

where

$$v_{3,1} = G\left(\sum_{j \in J} (\alpha_j(t)\zeta_\varepsilon(\cdot - X_j(t)) - \alpha_j^\varepsilon \zeta_\varepsilon(\cdot - X_j^\varepsilon(t)))\right),$$

$$v_{3,2} = G\left(\sum_{j \in J_2} (\alpha_j(t) - \alpha_j^\varepsilon)\zeta_\varepsilon(\cdot - X_j(t))\right),$$

$$v_{3,3} = G\left(\sum_{j \in J_2} \alpha_j^\varepsilon(\zeta_\varepsilon(\cdot - X_j(t)) - \zeta_\varepsilon(\cdot - X_j^\varepsilon(t)))\right).$$

We notice that $\alpha_j(t) = \alpha_j^\varepsilon = \alpha_j$ for $j \in J$; hence

$$(22) \qquad |v_{3,1}|_{l,p,\Omega} \leq \frac{C}{\varepsilon^l}\left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}} \|e(t)\|_{0,p,h}.$$

The proof of (22) is essentially the same as that in [18], except we notice that the estimate for the cardinal number, $\mathrm{card}J_x$, is multiplied by a constant K.

We turn now to estimating $v_{3,2}$. Since $F \equiv 0$ in $\bar{\Omega} \times [0, T]$ and F is smooth, we have from (14) that

$$\alpha_j(t) = \omega_0^{(k)}(X_j)|A(\tilde{X}_j)|h^2 + O(\varepsilon)h^2.$$

Also,

$$\alpha_j^\varepsilon = \alpha_{Ej}^\varepsilon = \omega_0^{(k)}(X_{Ej})|A(\tilde{X}_{Ej})|h^2.$$

Because $j \in J_2$, dist $(\tilde{X}_j, \tilde{X}_{Ej}) \leq C\varepsilon$; hence

$$|\alpha_j(t) - \alpha_j^\varepsilon| \leq C\varepsilon h^2.$$

For a fixed point x we define $J_x = \{j \in J_2; x \in \mathrm{supp}\,\zeta_\varepsilon(\cdot - X_j(t))\}$. Let $B_j(t)$ be the image of B_j under the mapping (20), (21); then the diameter of $B_j(t)$ is bounded

by $C_3 h$. We construct a disk with center x and radius $\varepsilon + C_3 h$; then, if $j \in J_x$, $B_j(t)$ is contained in the disk. Since the mapping (20), (21) is measure preserving, we have

$$\text{card } J_x \leq \frac{\pi (\varepsilon + Ch)^2 K}{h^2} \leq C \left(1 + \frac{\varepsilon}{h} \right)^2 .$$

Let

$$g = \sum_{j \in J_2} (\alpha_j(t) - \alpha_j^\varepsilon) \zeta_\varepsilon (x - X_j(t));$$

then by [18], Lemma 5.1,

$$|g|_{l-1,p} \leq C \left(1 + \frac{\varepsilon}{h} \right)^{\frac{2}{q}} \left(\sum_{j \in J_2} \|(\alpha_j(t) - \alpha_j^\varepsilon(t)) \partial^\gamma \zeta_\varepsilon (\cdot - X_j(t))\|_{0,p}^p \right)^{\frac{1}{p}},$$

where $l \geq 1, |\gamma| = l - 1$. We consider the set

$$D = \{x; 0 < \text{dist} (x, \bar{\Omega}) \leq C_0 \varepsilon + C_3 h\};$$

then area $D \leq C(C_0 \varepsilon + C_3 h)$, and it is easy to see that

$$\text{card } J_2 \leq \frac{C(C_0 \varepsilon + C_3 h)}{h^2} \leq C \left(\frac{1}{h} + \frac{\varepsilon}{h^2} \right) .$$

Besides, we have

$$\|\partial^\gamma \zeta_\varepsilon\|_{0,p} \leq \frac{C}{\varepsilon^{l-1+\frac{2}{q}}};$$

therefore

$$|g|_{l-1,p} \leq C \left(1 + \frac{\varepsilon}{h} \right)^{\frac{2}{q}} \left(\frac{1}{h} + \frac{\varepsilon}{h^2} \right)^{\frac{1}{p}} \varepsilon h^2 \frac{1}{\varepsilon^{l-1+\frac{2}{q}}}$$

$$= \frac{C}{\varepsilon^l} \varepsilon^{2+\frac{1}{p}} \left(1 + \frac{h}{\varepsilon} \right)^{1+\frac{1}{q}} .$$

It remains to consider the case of $l = 0$. Let $g^{(k)} = \varphi_k g$. We construct local coordinates (n, τ) on $\Omega^{(k)}$ and let $n = 0$ correspond to $\partial \Omega$. Let $\Omega_d = \{(n, \tau); 0 < n < d, |\tau| < C\}$. We estimate $|g^{(k)}|_{-1,p,\Omega_d}$. Let $\tilde{g} = \int_d^n g \, dn$. For $\varphi \in C_0^\infty(\Omega_d)$ we have

$$\int_{\Omega_d} \varphi g \, dn \, d\tau = \int d\tau \int \varphi \frac{\partial \tilde{g}}{\partial n} \, dn = - \int d\tau \int \tilde{g} \frac{\partial \varphi}{\partial n} \, dn$$

$$\leq \|\tilde{g}\|_{0,p} \left\| \frac{\partial \varphi}{\partial n} \right\|_{0,q} .$$

We have

$$\|g\|_{0,\infty} \leq \sup |(\alpha_j(t) - \alpha_j^\varepsilon) \zeta_\varepsilon (x - X_j(t))| \text{card } J_x \leq C\varepsilon \left(1 + \frac{h}{\varepsilon} \right)^2 .$$

The support of g is contained in $\{(n, \tau); 0 \leq n \leq C\varepsilon\}$; hence

$$\|\tilde{g}\|_{0,p} \leq \varepsilon^{2+\frac{1}{p}} \left(1 + \frac{h}{\varepsilon} \right)^2 ,$$

and consequently

$$\|g\|_{-1,p} \leq C\varepsilon^{2+\frac{1}{p}} \left(1 + \frac{h}{\varepsilon}\right)^2.$$

By the L^p-norm estimate [2], [3] we get

(23) $|v_{3,2}|_{l,p,\Omega} \leq \dfrac{C}{\varepsilon^l}\varepsilon^{2+\frac{1}{p}}, \quad l \geq 0.$

Finally let us estimate $v_{3,3}$. At the time $t = 0$ we have

$$X_j(0) = X_j = F^{(k)}\tilde{X}_j$$

and

$$\begin{aligned} X_j^\varepsilon(0) &= 2X_{j_0}^\varepsilon(0) - X_{Ej}^\varepsilon(0) \\ &= 2X_{j_0} - X_{Ej} = 2F^{(k)}\tilde{X}_{j_0} - F^{(k)}\tilde{X}_{Ej}. \end{aligned}$$

Applying Taylor's expansion about the point $\tilde{X}_{Ej}$ we get

$$F^{(k)}\tilde{X}_j = F^{(k)}\tilde{X}_{Ej} + DF^{(k)}(\tilde{X}_j - \tilde{X}_{Ej}) + O(\varepsilon^2),$$

$$F^{(k)}\tilde{X}_{j_0} = F^{(k)}\tilde{X}_{Ej} + DF^{(k)}(\tilde{X}_{j_0} - \tilde{X}_{Ej}) + O(\varepsilon^2).$$

But we have

$$\tilde{X}_j - \tilde{X}_{Ej} = 2(\tilde{X}_{j_0} - \tilde{X}_{Ej}),$$

and hence

$$\begin{aligned} X_j(0) - X_j^\varepsilon(0) &= (X_j - X_{Ej}) - 2(X_{j_0} - X_{Ej}) \\ &= (F^{(k)}\tilde{X}_j - F^{(k)}\tilde{X}_{Ej}) - 2(F^{(k)}\tilde{X}_{j_0} - F^{(k)}\tilde{X}_{Ej}) \end{aligned}$$

(24) $= O(\varepsilon^2).$

Applying Taylor's expansion to the function u, we get

$$u(X_{j_0}(t), t) = u(X_{Ej}(t), t) + Du(X_{j_0}(t) - X_{Ej}(t)) + O(\varepsilon^2),$$

$$u(X_j(t), t) = u(X_{Ej}(t), t) + Du(X_j(t) - X_{Ej}(t)) + O(\varepsilon^2).$$

Then by the equation (13) we get

$$\frac{d(X_{j_0}(t) - X_{Ej}(t))}{dt} = Du(X_{j_0}(t) - X_{Ej}(t)) + O(\varepsilon^2),$$

$$\frac{d(X_j(t) - X_{Ej}(t))}{dt} = Du(X_j(t) - X_{Ej}(t)) + O(\varepsilon^2).$$

Let $f(t) = (X_j(t) - X_{Ej}(t)) - 2(X_{j_0}(t) - X_{Ej}(t))$; then

$$\frac{df}{dt} = Du \cdot f + O(\varepsilon^2).$$

By (24) $f(0) = O(\varepsilon^2)$; hence $f(t) = O(\varepsilon^2)$. Also, we have

$$X_j^\varepsilon(t) = 2X_{j_0}^\varepsilon(t) - X_{Ej}^\varepsilon(t);$$

therefore

$$X_j(t) - X_j^\varepsilon(t) = 2(X_{j_0}(t) - X_{j_0}^\varepsilon(t)) - (X_{Ej}(t) - X_{Ej}^\varepsilon(t)) + O(\varepsilon^2).$$

Consequently

$$-v_{3,3} = G\left(\sum_{j\in J_2} \alpha_j^\varepsilon \int_0^1 D\zeta_\varepsilon(x - X_j(t) + \theta(X_j(t) - X_j^\varepsilon(t)))\, d\theta(X_j(t) - X_j^\varepsilon(t))\right)$$

$$= w_1 + w_2 + w_3,$$

where

$$w_1 = 2G\bigg(\sum_{j\in J_2} \alpha_j^\varepsilon \int_0^1 D\zeta_\varepsilon(x - X_j(t)$$

$$+ \theta(X_j(t) - X_j^\varepsilon(t)))\, d\theta(X_{j_0}(t) - X_{j_0}^\varepsilon(t))\bigg),$$

$$-w_2 = G\bigg(\sum_{j\in J_2} \alpha_j^\varepsilon \int_0^1 D\zeta_\varepsilon(x - X_j(t)$$

$$+ \theta(X_j(t) - X_j^\varepsilon(t)))\, d\theta(X_{Ej}(t) - X_{Ej}^\varepsilon(t))\bigg),$$

$$w_3 = G\left(O(\varepsilon^2) \sum_{j\in J_2} \alpha_j^\varepsilon \int_0^1 D\zeta_\varepsilon(x - X_j(t) + \theta(X_j(t) - X_j^\varepsilon(t)))\, d\theta\right).$$

The estimate for w_2 is the same as $v_{3,1}$. Thus we have

$$(25) \qquad |w_2|_{l,p,\Omega} \le \frac{C}{\varepsilon^l}\left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}} \|e(t)\|_{0,p,h}.$$

The estimate for w_3 is the same as $v_{3,2}$, except that

$$\mathrm{card} J_x \le C\left(1 + \frac{\varepsilon}{h} + \frac{\|e(t)\|_{0,\infty,h}}{h}\right)^2;$$

therefore

$$(26) \qquad |w_3|_{l,p,\Omega} \le \frac{C}{\varepsilon^l}\varepsilon^{2+\frac{1}{p}}\left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^2.$$

It remains to estimate w_1. By analogy to (25) we get

$$|w_1|_{l,p,\Omega} \le \frac{C}{\varepsilon^l}\left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}}\left(\sum_{j\in J_2} h^2|X_{j_0}(t) - X_{j_0}^\varepsilon(t)|^p\right)^{\frac{1}{p}}.$$

We notice that $j_0 = (j_1, 0, k)$ and $j = (j_1, j_2, k)$; hence, for fixed j_0, $\mathrm{card}\,\{j\} = \mathrm{card}\,\{j_2\} \le C(1 + \frac{\varepsilon}{h})$. Then

$$\sum_{j\in J_2} h^2|X_{j_0}(t) - X_{j_0}^\varepsilon(t)|^p \le C\left(1 + \frac{\varepsilon}{h}\right) h \sum_{j_0\in J_0} h|X_{j_0}(t) - X_{j_0}^\varepsilon(t)|^p.$$

By the definition (15) we have

$$(27) \qquad |w_1|_{l,p,\Omega} \le \frac{C}{\varepsilon^l}\left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}} \|e(t)\|_{0,p,h}.$$

The combination of (22), (23), (25)–(27) gives the desired estimate. $\qquad\square$

We now estimate $\|e(t)\|_{0,p,h}$. By (8) and (13) we have

$$X_j(t) - X_j^\varepsilon(t) = \int_0^t (u(X_j(s), s) - u^\varepsilon(X_j^\varepsilon(s), s))\, ds = I_j^{(1)} + I_j^{(2)},$$

where

$$I_j^{(1)} = \int_0^t (u - u^\varepsilon)(X_j^\varepsilon(s), s)\, ds,$$

$$I_j^{(2)} = \int_0^t (u^\varepsilon(X_j(s), s) - u^\varepsilon(X_j^\varepsilon(s), s))\, ds.$$

Lemma 4. *We assume that the moment condition (17) holds. If $|\nabla u^\varepsilon| \le C_1$, $\zeta \in W^{m+1,\infty}(\mathbb{R}^2)$, $m \ge 2$, $p > 2$, and*

$$(28) \qquad\qquad\qquad C_4^{-1}\varepsilon^a \le h \le C_4 \varepsilon^{\frac{m+1}{m}},$$

where $\frac{p}{2} > a \ge \frac{m+1}{m}$, then we have

$$(29) \qquad\qquad\qquad \|e(t)\|_{0,p,h} \le C\varepsilon^2$$

for $t \in [0, T]$, provided ε is sufficiently small.

Proof. We start by estimating the first term of (15). By Lemmas 1, 2, 3 we have

$$|u - u^\varepsilon|_{l,p,\Omega} \le C \Bigg\{ \varepsilon^2 + \left(1 + \frac{h}{\varepsilon}\right)^{\frac{2}{r}} \frac{h^m}{\varepsilon^{m+l-1}} + \frac{h^N}{\varepsilon^{N+l-1}}$$

$$(30) \qquad\qquad + \frac{1}{\varepsilon^l}\left(1 + \frac{\|e(t)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}} \left(\|e(t)\|_{0,p,h} + \varepsilon^{2+\frac{1}{p}}\right) \Bigg\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, N is large enough, and $l = 0, 1, 2$. By [13]

$$\left(\sum_{j \in J} h^2 |(u - u^\varepsilon)(X_j(t), t)|^p\right)^{\frac{1}{p}} \le CK^{\frac{1}{p}}(\|u - u^\varepsilon\|_{0,p,\Omega} + h|u - u^\varepsilon|_{1,p,\Omega}).$$

Due to (30) and the assumption we obtain

$$\left(\sum_{j \in J} h^2 |I_j^{(1)}|^p\right)^{\frac{1}{p}}$$

$$(31) \qquad \le C \int_0^t \left\{ \varepsilon^2 + \left(1 + \frac{\|e(s)\|_{0,\infty,h}}{\varepsilon}\right)^{\frac{2}{q}} \left(\|e(s)\|_{0,p,h} + \varepsilon^{2+\frac{1}{p}}\right) \right\}\, ds.$$

By the assumption we have

$$\left(\sum_{j \in J} h^2 |I_j^{(2)}|^p\right)^{\frac{1}{p}} \le C \int_0^t \left(\sum_{j \in J} h^2 |X_j(s) - X_j^\varepsilon(s)|^p\right)^{\frac{1}{p}}\, ds$$

$$(32) \qquad\qquad\qquad\qquad \le C \int_0^t \|e(s)\|_{0,p,h}\, ds.$$

We now estimate the second term of (15). Analogously to (32),

$$(33) \qquad \varepsilon^{\frac{1}{p}} \left(\sum_{j \in J_0} h |I_j^{(2)}|^p \right)^{\frac{1}{p}} \leq C \int_0^t \|e(s)\|_{0,p,h} \, ds.$$

Let

$$f(\eta) = (u - u^\varepsilon)(\xi(t; F^{(k)}\eta, 0), t);$$

then by [13] we have analogously

$$\left(\sum_{j \in J_0} h |f(\tilde{X}_j)|^p \right)^{\frac{1}{p}} \leq C K^{\frac{1}{p}} (\|f\|_{0,p,\mathbb{R}} + h |f|_{1,p,\mathbb{R}}).$$

We set $\eta = \varepsilon \eta'$ and apply the trace theorem [1] to get

$$\|f(\varepsilon \cdot)\|_{0,p,\mathbb{R}} \leq C \|f(\varepsilon \cdot)\|_{1,p,\mathbb{R}^2},$$

$$|f(\varepsilon \cdot)|_{1,p,\mathbb{R}} \leq C(|f(\varepsilon \cdot)|_{1,p,\mathbb{R}^2} + |f(\varepsilon \cdot)|_{2,p,\mathbb{R}^2}).$$

Therefore

$$\|f\|_{0,p,\mathbb{R}} \leq \varepsilon^{-\frac{1}{p}} C(\|f\|_{0,p,\mathbb{R}^2} + \varepsilon |f|_{1,p,\mathbb{R}^2}),$$

$$|f|_{1,p,\mathbb{R}} \leq \varepsilon^{-\frac{1}{p}} C(|f|_{1,p,\mathbb{R}^2} + \varepsilon |f|_{2,p,\mathbb{R}^2}).$$

Consequently

$$\left(\sum_{j \in J_0} h |f(\tilde{X}_j)|^p \right)^{\frac{1}{p}} \leq \varepsilon^{-\frac{1}{p}} C(\|f\|_{0,p,\mathbb{R}^2} + \varepsilon |f|_{1,p,\mathbb{R}^2} + \varepsilon h |f|_{2,p,\mathbb{R}^2}).$$

Therefore

$$\varepsilon^{\frac{1}{p}} \left(\sum_{j \in J_0} h |I_j^{(1)}|^p \right)^{\frac{1}{p}} \leq C \int_0^t \varepsilon^{\frac{1}{p}} \left(\sum_{j \in J_0} h |f(\tilde{X}_j)|^p \right)^{\frac{1}{p}} ds$$

$$\leq C \int_0^t (\|f\|_{0,p,\mathbb{R}^2} + \varepsilon |f|_{1,p,\mathbb{R}^2} + \varepsilon h |f|_{2,p,\mathbb{R}^2}) \, ds.$$

Since $(F^{(k)})^{-1}$ is smooth, we get

$$\varepsilon^{\frac{1}{p}} \left(\sum_{j \in J_0} h |I_j^{(1)}|^p \right)^{\frac{1}{p}}$$

$$\leq C \int_0^t (\|u - u^\varepsilon\|_{0,p,\mathbb{R}^2} + \varepsilon |u - u^\varepsilon|_{1,p,\mathbb{R}^2} + \varepsilon h |u - u^\varepsilon|_{2,p,\mathbb{R}^2}) \, ds.$$

It is analogous to (31) that

$$\varepsilon^{\frac{1}{p}} \left(\sum_{j \in J_0} h |I_j^{(1)}|^p \right)^{\frac{1}{p}}$$

$$(34) \qquad \leq C \int_0^t \left\{ \varepsilon^2 + \left(1 + \frac{\|e(s)\|_{0,\infty,h}}{\varepsilon} \right)^{\frac{2}{q}} \left(\|e(s)\|_{0,p,h} + \varepsilon^{2+\frac{1}{p}} \right) \right\} ds.$$

The combination of (31)–(34) gives

$$\|e(t)\|_{0,p,h}$$

$$(35) \qquad \leq C \int_0^t \left\{ \varepsilon^2 + \left(1 + \frac{\|e(s)\|_{0,\infty,h}}{\varepsilon} \right)^{\frac{2}{q}} \left(\|e(s)\|_{0,p,h} + \varepsilon^{2+\frac{1}{p}} \right) \right\} ds.$$

We claim that

$$(36) \qquad \qquad \|e(t)\|_{0,p.h} \leq C_5 \varepsilon^2,$$

where C_5 is a constant to be determined. Since $\|e(0)\|_{0,p,h} = 0$ and $\|e(t)\|_{0,p,h}$ is continuous, (36) holds on an interval $[0, T_*]$. By (28)

$$\|e(t)\|_{0,\infty,h} \leq h^{-\frac{2}{p}} \|e(t)\|_{0,p,h}$$

$$\leq C_4^{\frac{2}{p}} \varepsilon^{-\frac{2a}{p}} \|e(t)\|_{0,p,h} \leq C_4^{\frac{2}{p}} C_5 \varepsilon^{2 - \frac{2a}{p}}.$$

By the assumption $p > 2a$ we have on $[0, T_*]$ that

$$\left(1 + \frac{1}{\varepsilon} \|e(t)\|_{0,\infty,h} \right)^{\frac{2}{q}} \leq C_6,$$

where

$$(37) \qquad \qquad C_6 \geq \left(1 + C_4^{\frac{2}{p}} C_5 \varepsilon^{2 - \frac{2a}{p}} \right)^{\frac{2}{q}}.$$

(35) is reduced to

$$\|e(t)\|_{0,p,h} \leq C_7 \int_0^t \left\{ \varepsilon^2 + \|e(s)\|_{0,p,h} \right\} ds.$$

Applying Gronwall's inequality, we get

$$(38) \qquad \qquad \|e(t)\|_{0,p,h} \leq C_7 t e^{C_7 t} \varepsilon^2.$$

First we take a constant $C_6 > 1$, then we set $C_5 = C_7 T e^{C_7 T}$, then we take ε_0 sufficiently small that (37) holds for $\varepsilon \leq \varepsilon_0$. If $T_* < T$, then by (38) we have

$$\|e(t)\|_{0,p,h} < C_5 \varepsilon^2,$$

which means the interval $[0, T_*]$ can be extended. Thus $T = T_*$. □

Corollary 1. *Under the conditions of Lemma 4, (29) holds for all $\varepsilon > 0$, and the condition $a < \frac{p}{2}$ is not necessary.*

In fact we can take p large enough so that $p > 2a$, and then by Lemma 4 we have (29). For $r \in [1, p)$, we can apply the Hölder inequality to get

$$\|e(t)\|_{0,r,h} \leq C \|e(t)\|_{0,p,h},$$

because Ω is bounded. Besides, if $\varepsilon > \varepsilon_0$, we have

$$\|e(t)\|_{0,p,h} \leq C\varepsilon_0^{-2}\varepsilon^2,$$

since $\|e(t)\|_{0,p,h}$ is bounded.

Finally, let us prove the convergence theorem.

Theorem 1. *If the moment condition* (17) *holds,* $\zeta \in W^{m+1,\infty}(\mathbb{R}^2)$ *for* $m \geq 2$, $p \in [1,\infty)$, *and if the condition* (28) *holds for* $a \geq \frac{m+1}{m}$, *then*

(39) $$|\nabla u^\varepsilon| \leq C, \qquad x \in \bar{\Omega},$$

(40) $$\|u - u^\varepsilon\|_{0,p} + \|e(t)\|_{0,p,h} \leq C\varepsilon^2,$$

where $t \in [0,T]$.

Proof. We take $p > 2$ and notice that

$$\|(u - u^\varepsilon)(\cdot, 0)\|_{2,p,\Omega} = \|(v_1 + v_2 + v_{3,2} + w_3)(\cdot, 0)\|_{2,p,\Omega} \leq C,$$

which gives

$$\|(u - u^\varepsilon)(\cdot, 0)\|_{1,\infty,\Omega} \leq C_8.$$

Enlarge C_8 if necessary so that

(41) $$\|(v_1 + v_2)(\cdot, t)\|_{1,\infty,\Omega} \leq C_8.$$

Let

$$C_1 = \max_{(x,t)\in\bar{\Omega}\times[0,T]} |\nabla u| + C_8 + 1;$$

then $|\nabla u^\varepsilon| \leq C_1$ on a certain interval $[0, T_*]$. By Lemma 4 and Corollary 1

$$\|e(t)\|_{0,p,h} \leq C\varepsilon^2.$$

By Lemma 3

$$|v_3|_{l,p,\Omega} \leq C\varepsilon^{2-l}.$$

We take a constant s such that $2/p < s < 1$; then, by the interpolation inequality [1],

$$\|v_3\|_{1,\infty,\Omega} \leq C\|v_3\|_{1,p,\Omega}^{1-s}\|v_3\|_{2,p,\Omega}^{s} \leq C\varepsilon^{1-s}.$$

Taking ε small enough, we have

$$\|v_3\|_{1,\infty,\Omega} < 1.$$

By (41), $|\nabla u^\varepsilon| < C_1$. Using continuous extension we get (39) for $t \in [0,T]$ and small ε. Then (40) with large p, small ε follows from Lemmas 1, 2, 3, 4. The restriction for p, ε can be loosened via the argument of Corollary 1. $\qquad\square$

4. Numerical example

Let Ω be a unit disk. We consider the initial-boundary value problem,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = f,$$

$$\nabla \cdot u = 0,$$

$$u \cdot n|_{\partial\Omega} = 0,$$

$$u|_{t=0} = u_0.$$

We take the stream function,

$$\psi = (1 - |x|^2)(x_1 + x_2)e^{-t},$$

then we get the velocity u and vorticity ω by direct calculation, and then we get the initial condition for the vorticity,

$$\omega|_{t=0} = 8(x_1 + x_2).$$

Upon substituting u and ω into the equation

$$\frac{\partial\omega}{\partial t} + u \cdot \nabla\omega = F,$$

we get

$$F = -8(x_1 + x_2)e^{-t} + 16(x_1^2 - x_2^2)e^{-2t}.$$

We will compare the approximate solutions to this exact solution.

The first scheme (A) is a simple one without boundary treatment. We subdivide $\mathbb{R}^2$ into squares with side length h and pick those whose centers lie in Ω. The vortex blob function is taken as

$$\zeta(x) = \begin{cases} \frac{7}{2\pi}(1 - 10|x|^3 + 15|x|^4 - 6|x|^5), & |x| < 1, \\ 0, & |x| \geq 1; \end{cases}$$

hence

$$\int_{\mathbb{R}^2} \zeta(x)\,dx = 1.$$

The function ζ satisfies the condition (17). Since it is axisymmetric and compactly supported, the solution of $-\triangle g = \zeta_\varepsilon(x)$ on $\mathbb{R}^2$ can be expressed explicitly:

$$g(x) = \begin{cases} -\frac{7}{2\pi}\log|x|\left(\frac{1}{2}\frac{|x|^2}{\varepsilon^2} - 2\frac{|x|^5}{\varepsilon^5} + \frac{5}{2}\frac{|x|^6}{\varepsilon^6} - \frac{6}{7}\frac{|x|^7}{\varepsilon^7}\right), & |x| < \varepsilon, \\ -\frac{1}{2\pi}\log|x|, & |x| \geq \varepsilon, \end{cases}$$

which simplifies our computation. The boundary value problem (10), (11) is solved by the Fourier method. First we employ a linear combination of g's to get a particular solution to the nonhomogeneous equation, then solve the Laplace equation with nonhomogeneous boundary conditions. Let the vorticity ω^ε be expressed in terms of (9); then the equations for X_j^ε and α_j^ε are (8) and

$$\frac{d\alpha_j^\varepsilon}{dt} = F(X_j^\varepsilon(t), t)h^2.$$

We use an explicit Euler scheme to solve these ordinary differential equations. To prevent any vortex blob from moving across the boundary, we use the following modification for those vortex blobs neighboring the boundary: Let (r, θ) be the
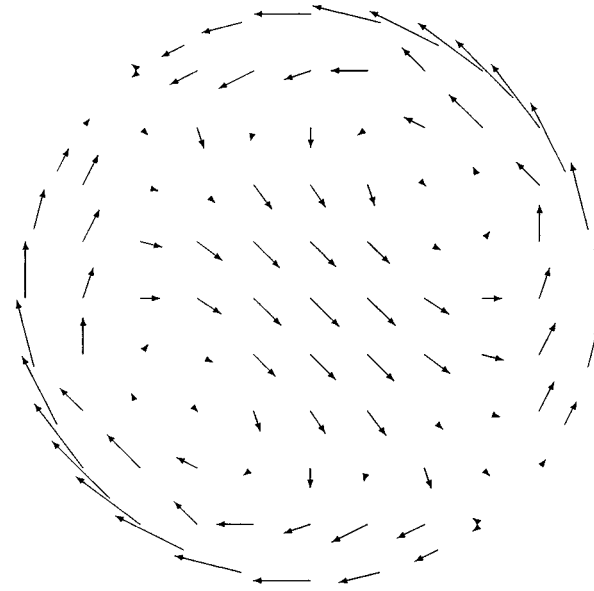
FIGURE 1

position of a particle and (u_r, u_θ) be its velocity in polar coordinates. Since u_r vanishes on the boundary, we solve

$$\frac{d\bar{r}}{dt} = u_r \frac{1 - \bar{r}}{1 - r},$$
$$\bar{r}|_{t=0} = r,$$

which gives

$$\bar{r} = 1 - (1 - r)e^{-u_r t/(1-r)}.$$

Consequently after a time step Δt we take the new coordinate of this particle as $(1 - (1 - r)e^{-u_r \Delta t/(1-r)}, \theta + u_\theta \Delta t/r)$.

The second scheme (B) is the one introduced in [15] and [17]. We extend the domain Ω a little and evaluate the motion of all vortex blobs even if their centers are not in Ω. The velocity of these blobs is obtained by extrapolation. Since we don't care if the centers are in Ω, the modification for the blobs near the boundary mentioned above is not necessary.

The third scheme (C) is the one developed in this paper. We create overlapped meshes. We consider the contribution of some blobs outside the domain, but extrapolation is not employed. We use the partition of unity here, so we define a compactly supported function

$$\varphi(x) = \begin{cases} 1, & |x| < 1 - 2\delta, \\ \frac{2}{\delta^3}(|x| + \frac{5}{2}\delta - 1)(1 - \delta - |x|)^2, & 1 - 2\delta \le |x| < 1 - \delta, \\ 0, & |x| > 1 - \delta. \end{cases}$$

The flow is shown in Figure 1. The meshes of the schemes (A),(B), and (C) are shown in Figures 2, 3, and 4, respectively. For the sake of clearness we take large parameters $h = 0.25$, $\varepsilon = 0.5$ and $\delta = 0.2$, and only part of the squares are drawn.
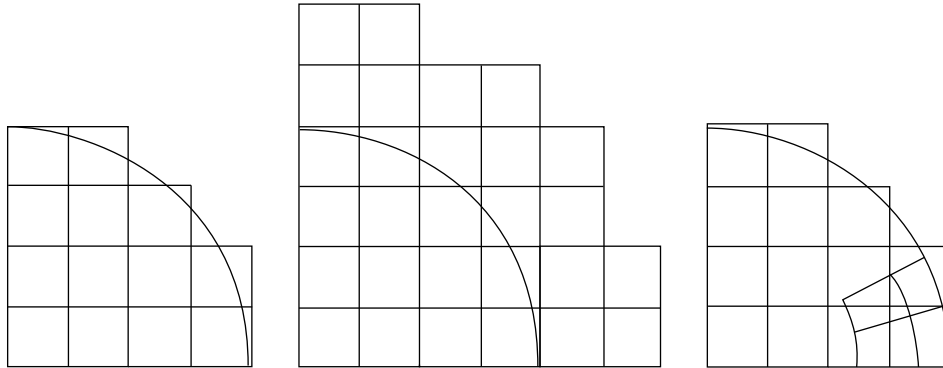
FIGURE   FIGURE   FIGURE

TABLE 1

t	(A)	(B)	(C)
0	0.475	0.139	0.204
0.1	0.486	0.152	0.182
0.2	0.433	0.161	0.167
0.3	0.367	0.162	0.154
0.4	0.373	0.155	0.142
0.5	0.340	0.156	0.153
0.6	0.304	0.152	0.169
0.7	0.252	0.152	0.210
0.8	0.218	0.144	0.221
0.9	0.183	0.139	0.254
1.0	0.182	0.129	0.232
1.1	0.162	0.132	0.218
1.2	0.186	0.117	0.210
1.3	0.176	0.115	0.178
1.4	0.162	0.105	0.175
1.5	0.156	0.100	0.155
1.6	0.142	0.0955	0.125
1.7	0.121	0.0901	0.112
1.8	0.118	0.0857	0.110
1.9	0.109	0.0828	0.112
2.0	0.0976	0.0814	0.111

In the real computation we take $h = 0.1$, $\varepsilon = 0.3$, $\delta = 0.1$ and $\Delta t = 0.01$. In the Fourier method we take 100 integral points and we truncate the Fourier series by taking 20 terms. For each scheme we evaluate 200 steps. The errors are shown in Table 1; they are defined as

$$\text{error} = \max_j \max\{|(u_1^\varepsilon - u_1)(X_j^\varepsilon(t), t)|, |(u_2^\varepsilon - u_2)(X_j^\varepsilon(t), t)|\}.$$

Let us notice that $|u_1|_{\max} = |u_2|_{\max} = 1 + \sqrt{2}$.

From the results we may conclude that:

1. To see the consistency we examine the errors at $t = 0$, because they are just the truncation errors. The best one is (B) and the worst is (A).

2. All schemes seem stable. Even more, they are asymptotically stable. After a period of oscillation the errors decrease monotonically. We think this is because these schemes are self-adaptive. In particular, the oscillation of the scheme (B) is the smallest.

3. It seems the scheme (B) is the best, but the extrapolation process in it depends on the smoothness of the solution, and it is time-consuming. We use an IBM 486 PC computer. The CPU time for them are 30, 90, and 40 minutes respectively.

4. For the scheme (A) we cannot prove anything theoretically: consistency, stability, or convergence. For all these schemes we don't know how to study the asymptotic stability.

Acknowledgment

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