

ADVANCES IN ALIQUOT SEQUENCES

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ABSTRACT. In this paper we describe some advances in the knowledge of the behavior of aliquot sequences starting with a number less than 10000. For some starting values, it is shown for the first time that the sequence terminates. The current record for the maximum of a terminating sequence is located in the one starting at 4170; it converges to 1 after 869 iterations getting a maximum of 84 decimal digits at iteration 289.

For a positive integer n , let $\sigma(n)$ denote the sum of the divisors of n , and $s(n) = \sigma(n) - n$. A perfect number is a number n such that $s(n) = n$, and an amicable pair of numbers is (n, m) satisfying $s(n) = m$, $s(m) = n$. In a similar way, cycles of numbers (a_1, a_2, \dots, a_l) such that $s(a_i) = a_{i+1}$ for $1 \leq i \leq l - 1$ and $s(a_l) = a_1$ are known as aliquot cycles or sociable numbers.

An aliquot sequence is $\{s^k(n)\}_{k=0}^{\infty}$ defined by taking $s^0(n) = n$ and $s^{k+1}(n) = s(s^k(n))$. For one of these sequences, there are four possibilities: (i) it terminates at 1 (the previous term being a prime), (ii) it reaches a perfect number, (iii) it reaches an amicable pair or a cycle, (iv) it is unbounded. The Catalan-Dickson conjecture [1], [2] says that (iv) does not actually happen. But other researchers disagree with this conjecture and think that there are unbounded sequences; in fact, the alternative conjecture from Guy-Selfridge [6] (see also [3], [4], [5]) states that there are many sequences that go to infinity, perhaps almost all those that start at an even number.

The nicest way of disproving the Catalan-Dickson conjecture would be to find an n so that $s^k(n) > s^{k-1}(n)$ for every k . Although it seems likely that such n does not exist, H. W. Lenstra has proved that, for every k , there is an m so that $m < s(m) < \dots < s^k(m)$ (the proof is constructive and it can be found in [3]).

Several papers deal with the behavior of these sequences. The reader will find an extensive bibliography on this subject in [5]. But, before going on to explain our results, we will summarize the main computational advances in the study of aliquot sequences.

The smallest n for which there was ever doubt was 138, but Lehmer showed that the sequence terminates at $s^{117}(138) = 1$. Since then, the first number whose behavior is not known is 276. Many researchers have investigated the behavior of

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this sequence (for instance, G. A. Paxson, H. Cohen, D. H. Lehmer, H. J. Godwin, J. L. Selfridge, M. C. Wunderlich, T. Struppeck, R. K. Guy, A. Guy and M. Dickerman), but it is not yet known if its end exists.

But there are many other sequences whose end is in doubt. For instance, there are five main sequences starting in a number smaller than 1000 whose behavior is unknown (Lehmer five: 276, 552, 564, 660, and 966). Before the year 1980, there were fourteen main sequences starting between 1000 and 2000 (Godwin fourteen), but at the present time only twelve remain unknown (Godwin showed that the 1848 sequence terminates, as did Dickerman for the 1248 sequence). In [4], a table is included showing the status of the sequences starting with numbers less than or equal to 7044. Finally, the latest results in this field can be found in Chapters B6 and B7 in [5]; here it is stated that “we have found that those for 2580, 2850, 4488, 4830, 6792, 7752, 8862 and 9540 also terminate”.

We have been checking the aliquot sequences for numbers under 10000. We have used PARI-GP¹ and, mainly, UBASIC,² and we have run the programs on about twenty computers from the Universidad de La Rioja and the Instituto P. M. Sagasta (Logroño, Spain) for about two years, mostly during nights and weekends. The main factoring method used has been a combination of both the elliptic curve and the multiple polynomial quadratic sieve algorithms, and we have checked the primality of the factors with the APR primality test. Besides, we wanted to avoid unexpected errors so we have made a further check of the files containing the sequences. For this task, we have written a program using the MIRACL³ package. This program allows us to verify the already computed sequences easily and quickly; we have executed this program on other computers than the ones used to calculate the sequences.

For any sequence, we have stopped the calculations when both of the following happen: (i) it has a driver (see [6]), which implies that it is in a stable increasing situation, without immediate hope of finding its end; (ii) it has reached a term big enough (75 digits) and whose factorization is not easily found. To increase the number of digits of a sequence may require a lot of iterates, and we are dealing with more than 80 sequences. This involves a very large amount of computer time. For instance, we spend about three quarters of an hour to factor a number with 60 decimal digits in a Pentium 100 computer with the multiple polynomial quadratic sieve algorithm; similarly, one hour and a half for one with 65 digits, 5 hours for one with 70 digits, 18 hours for one with 75 digits, one day and a half for one with 80 digits, 4 days for one with 85 digits, and two weeks for one with 90 digits. So, finally, we stopped computing on February 13, 1997.

In our study, we have achieved several advances over the results of the aforementioned authors. Some of our conclusions follow:

- We have increased the number of iterations in each sequence. In all the cases, the last term reached has at least 75 decimal digits.
- In connection to the most pursued sequence, the one beginning with 276, we have reached 913 iterates without finishing. We have stopped at a term that has 90 decimal digits. Also, we have made a special effort with Lehmer and Godwin sequences.

¹C. Batut, D. Bernardi, H. Cohen and M. Olivier, available by anonymous ftp from megrez.ceremab.u-bordeaux.fr in `/pub/pari`.

²Yuji Kida, available by anonymous ftp from rkmath.rikkyo.ac.jp in `/ubibm`.

³Michael Scott, available by anonymous ftp from ftp.compapp.dcu.ie in `/pub/crypto`.

- The sequence starting at 8262 converges to 1 after 773 iterations, 317 being the previous prime. Its maximum is

$$\begin{aligned} s^{444}(8262) &= 1058123957450778949935095186502241023325216959345026 \\ &= 2 \cdot 223 \cdot 8831 \cdot 1256153653 \cdot 7873472183 \cdot 27163313185808805688909099, \end{aligned}$$

a number of 52 decimal digits.

- The sequence starting at 7080 converges to 1 after 1264 iterations, 37 being the previous prime. Its maximum is

$$\begin{aligned} s^{539}(7080) &= 3356676040772699493933846784218306089052226182418412008219112234936 \\ &= 2^3 \cdot 419 \cdot 34110465457 \cdot 1418417145283289 \cdot 20697305061593509314923291434129565941, \end{aligned}$$

a number of 67 decimal digits.

- The sequence starting at 3556 converges to 1 after 2058 iterations, 59 being the previous prime. Its maximum is

$$\begin{aligned} s^{551}(3556) &= 47637284184582867729108156155573671704082265291532247964505425538142202028 \\ &= 2^2 \cdot 503 \cdot 24359 \cdot 7455102267576203135197 \cdot 100819403542569743 \cdot 12931884982259497162000565321, \end{aligned}$$

a number of 75 decimal digits.

- The sequence starting at 4170 converges to 1 after 869 iterations, 79 being the previous prime. Its maximum is

$$\begin{aligned} s^{289}(4170) &= 329561080342477212747203692863366213833838703158858822327064032192093690321488891836 \\ &= 2^2 \cdot 41 \cdot 97 \cdot 20374357 \cdot 1559593537 \cdot 651966073954976081342107597832287652091395156174990523498331163, \end{aligned}$$

a number of 84 decimal digits. It is a new record for the maximum of a terminating sequence (the previous record was the 1248 sequence, which terminates after reaching a maximum of 58 digits). We found it on December 22, 1996.

- Among the sequences that finish in a cycle, the one that reaches the largest maximum is the one starting at 7422. For this sequence, we have $s^{332}(7422) = 2924$ and $s^{333}(7422) = 2620$, (2620, 2924) being an amicable pair. The sequence maximum is $s^{156}(7422) = 1269528720481893811316 = 2^2 \cdot 13 \cdot 109 \cdot 131 \cdot 110323 \cdot 15497988349$. (The same happens with the 7434 and 9894 sequences, which coincide with the sequence corresponding to 7422 after a few iterations.)

Finally, in Table 1 we summarize our work with aliquot sequences starting with $n < 10000$. In this table we show the main sequences whose status is yet unknown (by *main* we mean that all the other sequences with unknown end merge with any sequence in the table; in this case, we only show in the table the sequence that starts in the smaller number). We also include the number of decimal digits of the last term $s^k(n)$ reached for each sequence and the driver in that stage. This table is similar to the one in [4], but now the last term computed for every item is between 10^{20} and 10^{40} times bigger (also, the last number analyzed in [4] is 7044). It is important to note that, for every sequence in the table, we have given up computing iterations at a number containing not only a guide, but also a driver. As a consequence, no sequences are expected to start decreasing after only a few iterations.

The complete computations used to work out this table, and the last term reached for any sequence, can be obtained from the authors upon request. Also, they are available by anonymous ftp from navalsaz.unirioja.es in the directory `/pub/aliquot`. Equally, we have created a web page about aliquot sequences: <http://www.unirioja.es/dptos/dmc/jvarona/aliquot.html>. This provides a good starting point for future work.

TABLE 1. Aliquot sequences whose end is in doubt

n	276	552	564	660	966	1074	1134	1464	1476
k	913	719	2230	333	358	1419	2132	1703	997
digits	90	91	91	90	91	85	88	88	87
driver	$2 \cdot 3$	$2^2 \cdot 7$	$2^3 \cdot 3$	$2^5 \cdot 3 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 7$	$2^6 \cdot 127$	$2^4 \cdot 31$	$2^3 \cdot 3 \cdot 5$
n	1488	1512	1560	1578	1632	1734	1920	1992	2232
k	580	1531	1303	726	653	1262	1872	949	339
digits	84	85	89	82	82	89	82	88	89
driver	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^5 \cdot 3 \cdot 7$	$2^2 \cdot 7$	$2 \cdot 3$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3$
n	2340	2360	2484	2514	2664	2712	2982	3270	3366
k	410	895	699	2794	660	800	723	356	988
digits	78	78	78	80	79	82	81	79	79
driver	$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 7$	$2^4 \cdot 31$	$2^4 \cdot 31$	$2^3 \cdot 3$	$2^3 \cdot 3$
n	3408	3432	3564	3630	3678	3774	3876	3906	4116
k	551	882	730	1030	1070	1138	662	638	557
digits	83	85	89	80	82	85	77	83	77
driver	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3$	$2^4 \cdot 31$	$2^2 \cdot 7$	$2^3 \cdot 3$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$
n	4224	4290	4350	4380	4788	4800	4842	5148	5208
k	459	839	1062	817	2095	1070	365	1359	1668
digits	78	79	80	78	82	82	79	80	83
driver	$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$2^4 \cdot 31$	$2^2 \cdot 7$	$2 \cdot 3$	$2^3 \cdot 3 \cdot 5$
n	5250	5352	5400	5448	5736	5748	5778	6160	6396
k	1284	646	2077	1154	871	875	680	1557	1190
digits	78	83	76	86	81	85	78	81	77
driver	$2 \cdot 3$	$2^2 \cdot 7$	$2 \cdot 3$	$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^4 \cdot 31$	$2^3 \cdot 3$	$2^3 \cdot 3 \cdot 5$
n	6552	6680	6822	6832	6984	7044	7392	7560	7890
k	845	533	717	599	1568	1061	463	728	788
digits	82	80	75	75	75	88	83	79	75
driver	$2^3 \cdot 3$	$2^3 \cdot 3$	$2 \cdot 3$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3 \cdot 5$	$2^4 \cdot 31$	$2^2 \cdot 7$
n	7920	8040	8154	8184	8288	8352	8760	8844	8904
k	866	2172	515	1122	618	1185	2091	1104	834
digits	80	83	75	79	79	81	75	82	80
driver	$2^6 \cdot 127$	$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 7$	$2^2 \cdot 7$	$2 \cdot 3$	$2^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$2^4 \cdot 31$	$2^2 \cdot 7$
n	9120	9282	9336	9378	9436	9462	9480	9588	9684
k	523	373	507	492	316	404	746	1168	590
digits	89	77	81	75	75	83	77	78	84
driver	$2^3 \cdot 3$	$2^4 \cdot 31$	$2^6 \cdot 127$	$2^2 \cdot 7$	$2^3 \cdot 3$	$2^3 \cdot 3 \cdot 5$	$2 \cdot 3$	$2 \cdot 3$	$2^5 \cdot 3 \cdot 7$
n	9708	9852							
k	558	467							
digits	83	76							
driver	$2^4 \cdot 31$	$2^2 \cdot 7$							

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