# THE DISCRETE PLATEAU PROBLEM: CONVERGENCE RESULTS

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ABSTRACT. We solve the problem of finding and justifying an optimal fully discrete finite element procedure for approximating minimal, including unstable, surfaces. In a previous paper we introduced the general framework and some preliminary estimates, developed the algorithm and give the numerical results. In this paper we prove the convergence estimate.

## 1. Introduction

We recall from [DH4] that a *minimal surface* or *solution of the Plateau Problem* can be characterised in a number of different ways. For our purposes it is convenient to begin with the following formulation, which we restate more precisely later, cf. (3).

Let $D$ be the unit disc in $\Re^2$ and $\Gamma$ a smooth Jordan curve in $\Re^n$. Let $\mathcal{F}$ be the class of *harmonic* maps $u : \overline{D} \to \Re^n$ such that $u|_{\partial D} : \partial D \to \Gamma$ is monotone and satisfies a certain integral "three-point condition", cf. (1). The function $u \in \mathcal{F}$ is said to be a *minimal surface* if $u$ is stationary in $\mathcal{F}$ for the Dirichlet energy $\mathcal{D}(u) = \frac{1}{2} \int_D |\nabla u|^2$. Such maps $u$ provide an harmonic conformal parametrisation of the corresponding minimal surface.

Let $D_h$ be a quasi-uniform triangulation of $D$ with grid size controlled by $h$. Let $\mathcal{F}_h$ be the class of *discrete harmonic* maps $u_h : \overline{D}_h \to \Re^n$ for which $u_h(\phi_j) \in \Gamma$ whenever $\phi_j$ is a boundary node of $D_h$, and which satisfy an analogue of the previous integral "three-point condition". Note that we do not require "monotonicity" of $u_h|_{\partial D_h}$. The function $u_h \in \mathcal{F}_h$ is said to be a *discrete minimal surface* if $u_h$ is stationary within $\mathcal{F}_h$ for the Dirichlet energy $\mathcal{D}(u_h) = \frac{1}{2} \int_{D_h} |\nabla u_h|^2$; cf. (27) for the precise formulation.

The main result (Theorem 5.5) is that if $u$ is a nondegenerate minimal surface spanning $\Gamma$ then there exist discrete minimal surfaces $u_h$, unique in a ball of "almost" constant radius $\epsilon_0 |\log h|^{-1}$, such that $\|u - u_h\|_{H^1(D_h)} \leq ch$, where $c$ depends on $\Gamma$ and the nondegeneracy constant $\lambda$ for $u$ but is independent of $h$. Recall that nondegeneracy corresponds to the absence of zero eigenvalues for the second variation of the Dirichlet energy at $u$, and generically corresponds to the absence of branch points. The constant $c$ blows up as $\lambda \to 0$, and this is consistent with the numerical results in [DH4]. We give an outline of the proof of the main result in Section 2.3.

## 2. Formulation of the problem

**2.1. The energy functional.** We first recall some notation from [DH4] Section 3, to which we refer for further discussion. See Section 3 of the present paper for the properties of $H^{1/2}$.

Let $D$ be the open unit disc in $\Re^2$, with boundary $\partial D$. It will be convenient to let $S^1$ denote another, distinct, copy of the unit circle. Let $\Gamma$ be a Jordan curve in $\Re^n$ with regular $C^r$-parametrisation $\gamma : S^1 \to \Gamma$ where $r \geq 3$.

For $f : \partial D \to \Re^n$ we denote by $\Phi(f) : \overline{D} \to \Re^n$ its unique harmonic extension to $D$ specified by

$$\Delta \Phi(f) = 0 \quad \text{in } D, \qquad \Phi(f) = f \quad \text{on } \partial D.$$

Then $\Phi : H^{1/2}(\partial D, \Re^n) \to H^1(D, \mathbb{R}^n)$ is a bounded linear map with bounded inverse.

Harmonic maps are uniquely determined by the associated boundary maps. We will use the Hilbert space $H$ of functions defined by

$$H \quad = \quad \{\xi : \partial D \to \text{Re} \mid |\xi|_{H^{1/2}} < \infty \text{ and (1) is satisfied}\},$$

where

(1) $$\int_0^{2\pi} \xi(\phi)\, d\phi = 0, \quad \int_0^{2\pi} \xi(\phi) \cos \phi\, d\phi = 0, \quad \int_0^{2\pi} \xi(\phi) \sin \phi\, d\phi = 0.$$

The norm on $H$ is the usual norm $\| \cdot \|_{H^{1/2}}$. The corresponding affine space of maps $s : \partial D \to S^1$ such that $s(\phi) = \phi + \sigma(\phi)$ for some $\sigma \in H$ is denoted by $\mathcal{H}$. We also need the Banach space $T$ defined by $T = H \cap C^0(\partial D, \Re)$ with norm $\|\xi\|_T = \|\xi\|_{H^{1/2}} + \|\xi\|_{C^0}$. The corresponding affine space $\mathcal{T}$ is defined by $\mathcal{T} = \mathcal{H} \bigcap C^0(\partial D, S^1)$. With some abuse of standard notation, we write $\|s\| = 1 + \|\sigma\|$ for various norms on $\sigma$.

The energy functional $E$ is defined on $\mathcal{H}$ by

(2) $$E(s) = \frac{1}{2} \int_D |\nabla \Phi(\gamma \circ s)|^2 = \mathcal{D}(\Phi(\gamma \circ s)).$$

Finiteness of $E$ follows from (8). We say the harmonic function $u = \Phi(\gamma \circ s)$ is *a minimal surface spanning* $\Gamma$, or *a solution of the Plateau Problem for* $\Gamma$, if and only if $s$ is monotone and stationary for $E$, i.e.

(3) $$\langle E'(s), \xi \rangle = 0 \qquad \forall \xi \in T.$$

As discussed in [DH4], this is equivalent to other formulations of the notion of a minimal surface.

For the proof of asymptotic convergence of the numerical method we need the following regularity result, which follows from standard regularity results (cf. [DH4] Theorem 3.2) and the Implicit Function Theorem used to write $\Gamma$ locally as the graph of a $C^{k,\alpha}$ function.

**Proposition 2.1.** *If* $\gamma \in C^{k,\alpha}$ *where* $k \geq 1$ *and* $0 < \alpha < 1$, *and* $s \in \mathcal{T}$ *is monotone and stationary for* $E$, *then*

$$\|s\|_{C^{k,\alpha}} \leq c = c(\|\gamma\|_{C^{k,\alpha}}, \| |\gamma'|^{-1} \|_{L^\infty}).$$

We next recall some properties of the energy functional from [DH4] Section 3.3. Using the notation

(4) $$u = \Phi(\gamma \circ s), \quad v = \Phi(\gamma' \circ s\ \xi), \quad w = \Phi(\gamma'' \circ s\ \xi^2),$$

we get by formal computation

$$(5) \qquad E(s) \;=\; \frac{1}{2}\int_D |\nabla u|^2,$$

$$(6) \qquad \langle E'(s),\xi\rangle \;=\; \frac{d}{dt}\Big|_{t=0} E(s+t\xi) = \int_D \nabla u\nabla v,$$

$$(7) \qquad E''(s)(\xi,\xi) \;=\; \frac{d^2}{dt^2}\Big|_{t=0} E(s+t\xi) = \int_D \nabla u\nabla w + \int_D |\nabla v|^2,$$

with an analogous expression for $E''(s)(\xi,\eta)$ obtained by bilinearity in the case of distinct variations.

If $\gamma \in C^3$ then $E \in C^2(\mathcal{T},\Re)$, and the Fréchet derivatives are given by (6) and (7). The functional $E$ is *not* differentiable on $\mathcal{H}$, but if $\gamma$ and $s$ are as smooth as is necessary for the following estimates, then one has

$$(8) \qquad E(s) \;\le\; c\|\gamma\|_{C^1}^2 \|s\|_{H^{1/2}}^2,$$

$$(9) \qquad |\langle E'(s),\xi\rangle| \;\le\; c\|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi\|_{H^{1/2}},$$

$$(10) \qquad |E''(s)(\xi,\eta)| \;\le\; c\|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi\|_{H^{1/2}} \|\eta\|_{H^{1/2}}.$$

In particular, these will be used in case $s$ is stationary for $E$.

It will be important to consider the behaviour of the second derivatives of $E$ near a stationary point $s \in \mathcal{T}$. The second derivative $E''(s)$ can then be interpreted as a self-adjoint bounded map $\nabla^2 E(s)\colon H \to H$. Let

$$(11) \qquad H = H^- \oplus H^0 \oplus H^+, \qquad \xi = \xi^- + \xi^0 + \xi^+ \quad \text{if } \xi \in H,$$

be the orthogonal decomposition generated by the eigenfunctions of $\nabla^2 E(s)$ having negative, zero and positive eigenvalues respectively.

**Proposition 2.2.** *Suppose $\gamma \in C^{3,\alpha}$. Suppose $s$ is monotone and stationary for $E$. Then $H^-$ and $H^0$ are finite dimensional. If $\xi \in H^- \oplus H^0$ then $\xi \in H^{3/2}(\partial D)$ and*

$$(12) \qquad \|\xi\|_{H^{3/2}} \le \nu\|\xi\|_{H^{1/2}}$$

*with $\nu = \nu(\|\gamma\|_{C^{3,\alpha}}, \| \,|\gamma'|^{-1}\,\|_{L^\infty})$.*

*Proof.* The finite dimensionality of $H^-$ and $H^0$ follow from elliptic theory; see the proof of [St, Proposition II.5.6]. It is also shown there, with a different notation, that

$$\begin{aligned}
\|\Phi(\gamma' \circ s\; \xi)\|_{H^2(D)} &\le\; c\Big(\|\gamma\|_{C^3}, \| \,|\gamma'|^{-1}\,\|_{L^\infty}, \|s\|_{C^3}, \|\xi\|_{H^{1/2}}\Big)\\
&\le\; c\Big(\|\gamma\|_{C^{3,\alpha}}, \| \,|\gamma'|^{-1}\,\|_{L^\infty}, \|\xi\|_{H^{1/2}}\Big)
\end{aligned}$$

by Proposition 2.1. But

$$\begin{aligned}
\|\xi\|_{H^{3/2}} &\le\; \left\|\frac{\gamma' \circ s}{|\gamma' \circ s|^2}\right\|_{C^2} \|\gamma' \circ s\; \xi\|_{H^{3/2}} \quad \text{by (52)}\\
&\le\; c\Big(\|\gamma\|_{C^3}, \| \,|\gamma'|^{-1}\,\|_{L^\infty}\Big)\|\Phi(\gamma' \circ s\; \xi)\|_{H^2(D)} \quad \text{from (52) and (40).}
\end{aligned}$$

Hence

$$\|\xi\|_{H^{3/2}} \le c\Big(\|\gamma\|_{C^{3,\alpha}}, \| \,|\gamma'|^{-1}\,\|_{L^\infty}, \|\xi\|_{H^{1/2}}\Big),$$

and so the required result follows by scaling. $\qquad\square$

Higher regularity on $\gamma$ implies higher regularity on $\xi$. In particular, $\gamma \in C^{4,\alpha}$ implies $\xi \in H^{5/2}(\partial D)$; see the proof of [St, Proposition II.5.6].

If $s$ is monotone and stationary for $E$, we say $s$ is *nondegenerate* if $H^0 = \{0\}$. The corresponding minimal surface $u = \Phi(\gamma \circ s)$ is also said to be *nondegenerate*. If $s$ is nondegenerate it follows that there exists a $\lambda > 0$ such that for $\xi \in H$,

$$(13) \qquad E''(s)(\xi, \xi^+ - \xi^-) = E''(s)(\xi^+, \xi^+) - E''(s)(\xi^-, \xi^-) \geq \lambda \|\xi\|_{H^{1/2}}^2.$$

We call $\lambda$ the *nondegeneracy constant* for $s$.

2.2. **The discrete energy functional.** We recall the necessary notation from [DH4] Section 4 and prove some preliminary estimates.

Let $\mathcal{G}_h$ be a quasi-uniform triangulation of $D$ with grid size comparable to $h$. Let

$$D_h = \bigcup \{G \mid G \in \mathcal{G}_h\},$$

$$\partial D_h = \bigcup \{E_j \mid 1 \leq j \leq M\} \quad \text{where the } E_j \text{ are the boundary edges,}$$

and let

$$\mathcal{B}_h = \{\phi_1, \dots, \phi_M\} \quad \text{be the set of boundary nodes.}$$

The projection $\pi : \partial D \to \partial D_h$ is defined by

$$(14) \qquad \pi\left(e^{i((1-t)\phi_j + t\phi_{j+1})}\right) = (1-t)e^{i\phi_j} + te^{i\phi_{j+1}}$$

for $0 \leq t \leq 1$, $1 \leq j \leq M$.

In order to have a discrete analogue $E_h$ of the functional $E$ we define the following discrete analogues of $H^1(D; \Re^n)$, $H^{1/2}(\partial D; \Re^n)$, $H$, $T$, $\mathcal{H}$ and $\mathcal{T}$:

$$(15) \quad X_h^n = \left\{u_h \in C^0(D_h; \Re^n) \mid u_h \in P_1(G) \text{ for } G \in \mathcal{G}_h\right\},$$

$$(16) \quad x_h^n = \left\{f_h \in C^0(\partial D_h; \Re^n) \mid f_h \in P_1(E_j) \text{ for } 1 \leq j \leq M\right\},$$

$$(17) \quad H_h = \{\xi_h \in C^0(\partial D; \Re) \mid \xi_h \in P_1(\pi^{-1}(E_j)) \text{ if } 1 \leq j \leq M, \ \xi_h \text{ satisfies } (1)\},$$

$$(18) \quad \mathcal{H}_h = \{s_h \in C^0(\partial D; S^1) \mid s_h(\phi) = \phi + \sigma_h(\phi) \text{ for some } \sigma_h \in H_h\}.$$

Thus $H_h \subset T \subset H$, $\mathcal{H}_h \subset \mathcal{T} \subset \mathcal{H}$, and the space of variations at any $s_h \in \mathcal{H}_h$ is naturally identified with $H_h$. Setting $n = 1$, we similarly define $X_h$ and $x_h$.

We will make frequent use of the following inverse-type estimates:

**Proposition 2.3.** *If $\xi_h \in H_h$ then*

$$(19) \qquad\qquad \|\xi_h\|_{H^1} \quad \leq \quad ch^{-1/2}\|\xi_h\|_{H^{1/2}},$$

$$(20) \qquad\qquad \|\xi_h\|_{C^0} \quad \leq \quad c|\log h|^{1/2}\|\xi_h\|_{H^{1/2}}.$$

*Proof.* The first estimate is standard. The second is shown in [DH1] Proposition 5.3. □

Suppose $f \in C^0(\partial D; \Re^n)$. One defines the "linear interpolants"

$$I_h f \in x_h^n, \quad I_h f\left((1-t)e^{i\phi_j} + te^{i\phi_{j+1}}\right) = (1-t)f(e^{i\phi_j}) + tf(e^{i\phi_{j+1}}),$$

$$I_h^{\partial D} f \in C^0(\partial D; \Re^n), \quad I_h f\left((1-t)e^{i\phi_j} + te^{i\phi_{j+1}}\right) = (1-t)f(e^{i\phi_j}) + tf(e^{i\phi_{j+1}}),$$

where $0 \le t \le 1$, $1 \le j \le M$. Here and elsewhere, $\phi_{M+1} = \phi_1$. Note the different domains of $I_h f$ and $I_h^{\partial D} f$. Note also that the image of $I_h(\gamma \circ s)$ is a polygonal approximation to $\Gamma$, and $I_h(\gamma \circ s)(\phi_j) = \gamma \circ s(\phi_j) \in \Gamma$ for $\phi_j \in \mathcal{B}_h$. Finally,

$$(21) \qquad\qquad I_h^{\partial D} f = I_h f \circ \pi.$$

Another type of approximation operator we require is a map $p_h : T\ (\mathcal{T}) \to H_h\ (\mathcal{H}_h)$. The usual interpolation operator does not preserve the normalisation conditions (1). However, if we first interpolate and then project onto $H_h$, the resulting operator still satisfies all the usual estimates. The proof of the following is essentially given in [DH1] Proposition 5.2.

**Proposition 2.4.** *There is a bounded linear operator* $p_h : T \to H_h$ *such that, in particular,*

$$(22) \qquad\qquad \|\xi - p_h \xi\|_{H^s} \le ch^{k-s} \|\xi\|_{H^k}$$

*for* $s = 0, \frac{1}{2}, 1$ *and* $k = 1, \frac{3}{2}, 2$. *Moreover,*

$$(23) \qquad \|\xi - p_h \xi\|_{C^{0,1}} \le ch \|\xi\|_{C^2}, \qquad \|p_h \xi\|_{C^{0,1}} \le c\|\xi\|_{C^{0,1}},$$

$$(24) \qquad \|\xi - p_h \xi\|_{C^0} \le ch^2 \|\xi\|_{C^2}, \qquad \|\xi - p_h \xi\|_{C^0} \le ch \|\xi\|_{C^1}.$$

*If* $s \in \mathcal{T}$ *and* $s(\phi) = \phi + \sigma(\phi)$, *then* $p_h s$ *is defined by* $p_h s(\phi) = \phi + p_h \sigma(\phi)$, $s - p_h s := \sigma - p_h \sigma$, *and hence* $p_h s$ *satisfies estimates similar to those for* $p_h \xi$.

For $f_h \in x_h$ the *discrete harmonic extension* $\Phi_h f_h \in X_h$ is defined by

$$(25) \qquad\qquad \triangle_h \Phi_h f_h = 0 \quad \text{in} \quad D_h, \qquad \Phi_h f_h = f_h \quad \text{on} \quad \partial D_h,$$

where $\triangle_h$ is the discrete Laplacian and the first equation in (25) is interpreted as $\int_{D_h} \nabla(\Phi_h f_h)\nabla \psi_h = 0$ for all $\psi_h \in X_h$ such that $\psi_h = 0$ on $\partial D_h$. If $f_h \in x_h^n$ the discrete harmonic extension $\Phi_h f_h$ is defined componentwise.

For $s_h \in \mathcal{H}_h$, the *discrete energy functional* $E_h$ is defined by

$$(26) \qquad E_h(s_h) = \frac{1}{2} \int_{D_h} |\nabla \Phi_h I_h(\gamma \circ s_h)|^2 = \mathcal{D}_h(\Phi_h I_h(\gamma \circ s_h)).$$

Note that $E_h$ is of course *not* the restriction of $E$ to $\mathcal{H}_h$. The discrete harmonic function $u_h = \Phi_h I_h(\gamma \circ s_h)$ is said to be *a discrete minimal surface spanning* $\Gamma$, or *a solution of the discrete Plateau Problem for* $\Gamma$, if and only if

$$(27) \qquad\qquad \langle E_h'(s_h), \xi_h \rangle = 0 \qquad \forall \xi_h \in H_h.$$

Note that we do not require monotonicity of $s_h$, as is the case for $s$ in (3).

The derivatives of $E_h$, cf. (5)–(7), are given by

$$(28) \qquad E_h(s_h) \quad = \quad \frac{1}{2} \int_{D_h} |\nabla u_h|^2,$$

$$(29) \qquad \langle E_h'(s_h), \xi_h \rangle \quad = \quad \int_{D_h} \nabla u_h \nabla v_h,$$

$$(30) \qquad E_h''(s_h)(\xi_h, \xi_h) \quad = \quad \int_{D_h} \nabla u_h \nabla w_h + \int_{D_h} |\nabla v_h|^2,$$

where

$$(31) \qquad u_h = \Phi_h I_h(\gamma \circ s_h), \quad v_h = \Phi_h I_h(\gamma' \circ s_h\ \xi_h), \quad w_h = \Phi_h I_h(\gamma'' \circ s_h\ \xi_h^2).$$

For $\xi_h \in H_h$ and $s$ stationary for $E$, we define the projection of the decomposition (11):

(32)
$$\xi_h^{(-)} = p_h \xi_h^-, \quad \xi_h^{(0)} = p_h \xi_h^0, \quad \xi_h^{(+)} = p_h \xi_h^+.$$

Note that $\xi_h^-$, $\xi_h^0$ and $\xi_h^+$ do not normally belong to $H_h$; in particular, the first two are smooth functions. However, if $H^- = H^0 = \{0\}$ then $\xi_h = \xi_h^+ = \xi_h^{(+)}$.

**Proposition 2.5.** *If $\xi_h \in H_h$ then*

(33)
$$\xi_h = \xi_h^{(-)} + \xi_h^{(0)} + \xi_h^{(+)}$$

*and $\xi_h^{(-)}, \xi_h^{(0)}, \xi_h^{(+)} \in H_h$. Moreover, with $\nu$ as in (12),*

(34)
$$\|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} \le ch\nu\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^- - \xi_h^{(-)}\|_{H^1} \le ch^{1/2}\nu\|\xi_h\|_{H^{1/2}},$$

(35)
$$\|\xi_h^0 - \xi_h^{(0)}\|_{H^{1/2}} \le ch\nu\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^0 - \xi_h^{(0)}\|_{H^1} \le ch^{1/2}\nu\|\xi_h\|_{H^{1/2}},$$

(36)
$$\|\xi_h^+ - \xi_h^{(+)}\|_{H^{1/2}} \le ch\nu\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^+ - \xi_h^{(+)}\|_{H^1} \le ch^{1/2}\nu\|\xi_h\|_{H^{1/2}}.$$

*Proof.* Since $p_h\xi_h = \xi_h$, (33) follows from (11). Next, from (22) and (12),

$$\|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} = \|\xi_h^- - p_h\xi_h^-\|_{H^{1/2}} \le ch\|\xi_h^-\|_{H^{3/2}} \le ch\nu\|\xi_h^-\|_{H^{1/2}}.$$

The proof of the other inequality in (34), and of (35), is similar.

For (36) just note that, from (33) and (11),

$$\xi_h^+ - \xi_h^{(+)} = -(\xi_h^- - \xi_h^{(-)}) - (\xi_h^0 - \xi_h^{(0)}),$$

and use the previous estimates. $\qquad\square$

We remark that the decomposition (33), unlike (11), is *not* an orthogonal decomposition. We also note that the powers of $h$ in (34)–(36) can be increased by $1/2$ if we assume $\|\xi\|_{H^2} \le \nu\|\xi\|_{H^{1/2}}$; this latter holds if $\gamma \in C^{4,\alpha}$, cf. the remarks after Proposition 2.2.

2.3. **Structure of the proof.** The main error estimates, Theorems 5.4 and 5.5, are proved in Section 5. To motivate our approach and the need for the various preliminary estimates in Sections 4 and 5, we outline in an informal manner the structure of the proof of Theorem 5.4. See Figure 1.

Assume $s$ is a nondegenerate stationary point for $E$ with nondegeneracy constant $\lambda$. One applies the Inverse Function Theorem to the derivative $E_h' : \mathcal{H}_h \to H_h'$ (the dual space of $H_h$) in a neighbourhood of the point $p_h s$; remember that $H_h$ is the tangent space for $\mathcal{H}_h$, i.e. the space of variations at any $t_h \in \mathcal{H}_h$. Identify $E_h''$ with the derivative $(E_h')'$ of $E_h'$; thus $(E_h')'(t_h) : H_h \to H_h'$ for $t_h \in \mathcal{H}_h$.

There are three estimates to be proved.

**A**: $\|E_h'(p_h s)\| \le c_1 h$ for $h$ sufficiently small. This follows immediately from the consistency estimate (Proposition 4.2) for the first derivative. See (109).

**B$_1$**: $\|\langle (E_h')'(p_h s), \xi_h \rangle\| \ge \frac{\lambda}{2}\|\xi_h\|$ for $h$ sufficiently small; i.e. "the slope of $E_h'$ at $p_h s$ is bounded away from 0 by $\lambda/2$". This is a nondegeneracy estimate on $E_h''(p_h s)$ and is established in Proposition 5.3, using the one-sided consistency estimate (Proposition 4.3) for the second derivative, and the regularity of members of the negative eigenspaces to compensate for the one-sided nature of that estimate. See (114).

**B$_2$**: $\|(E_h')'(p_h s) - (E_h')'(p_h s + \eta_h)\| \le \lambda/4$ for $h$ sufficiently small and $\eta_h$ sufficiently close to $p_h s$, more precisely for $\eta_h \le \epsilon_0/|\log h|$; i.e. "the slope of $E_h'$ near $p_h s$
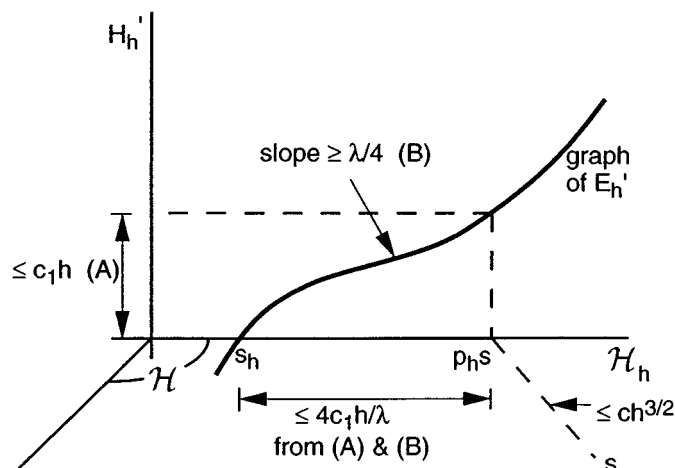
FIGURE 1

differs from that at $p_h s$ by at most $\lambda/4$". This is a consequence of Proposition 5.2. See (115). As indicated in the diagram, it follows that there exists a unique stationary point $s_h$ for $E_h$ such that $\|p_h s - s_h\|_{H^{1/2}} \leq 4c_1 h/\lambda$, and hence such that $\|s - s_h\|_{H^{1/2}} \leq ch/\lambda$.

## 3. Preliminary estimates

One can define the $H^s(\partial D)$ and $H^s(D)$ norms for any real $s$; but apart from non-negative integers $s$ we will only need the following cases.

For $f : \partial D \to \Re$ the $H^{1/2}(\partial D)$ seminorm is defined by

$$(37) \qquad |f|^2_{H^{1/2}(\partial D)} = \int_{\partial D} \int_{\partial D} \frac{|f(\phi) - f(\overline{\phi})|^2}{|\phi - \overline{\phi}|^2} \, d\phi \, d\overline{\phi},$$

and for $u : D \to \Re$ the $H^{1/2}(D)$ seminorm is defined by

$$(38) \qquad |u|^2_{H^{1/2}(D)} = \int_D \int_D \frac{|u(x) - u(\overline{x})|^2}{|x - \overline{x}|^3} \, dx \, d\overline{x}.$$

In both cases the corresponding norm is given by

$$(39) \qquad \| \cdot \|^2_{H^{1/2}} = \| \cdot \|^2_{L^2} + | \cdot |^2_{H^{1/2}}.$$

Also,

$$|f|_{H^{3/2}(\partial D)} = |f'|_{H^{1/2}(\partial D)}, \qquad |u|_{H^{3/2}(D)} = |\nabla u|_{H^{1/2}(D)},$$
$$\| \cdot \|^2_{H^{3/2}} \quad = \quad \| \cdot \|^2_{L^2} + | \cdot |^2_{H^{3/2}}.$$

If $u \in H^{s+1/2}(D)$ for $s = 1/2, 1, 3/2$ (in fact for $s > 0$), then $u$ has a well-defined trace $f$ on $\partial D$ and

$$(40) \qquad \|f\|_{H^s(\partial D)} \leq c\|u\|_{H^{s+1/2}(D)}.$$

More precisely, the previous estimate is true for $u \in C^\infty(\overline{D})$, and the definition of trace is extended by continuity and density to the general case. Such an estimate is not true for smooth $u$ if $s = 0$, but in this case if $\triangle u \in L^2(D)$ the following

estimate holds, and then the trace operator is again defined by continuity and density to satisfy

$$(41) \qquad \|f\|_{L^2(\partial D)} \leq c \left( \|u\|_{H^{1/2}(D)} + \|\triangle u\|_{L^2(D)} \right).$$

See [LM], Chapter 1, Section 9.2, page 193 (remarks before Theorem 8.1) and page 187, Theorem 7.3. *Note* that by considering kernels it is clear that in the case $s = 1/2$ one can replace the norms in (40) by the corresponding seminorms. We will frequently use this fact without further remark.

Conversely, if $f \in H^s(\partial D)$ for $s = 0, 1/2, 1, 3/2$ (with similar results for any real $s$), then there is a unique harmonic function $\Phi(f)$ defined on $D$ with trace $f$ as before, and in particular

$$(42) \qquad \|\Phi(f)\|_{H^{s+1/2}(D)} \leq c \|f\|_{H^s(\partial D)}.$$

See [LM], Theorem 7.4, page 188. *Note* that for $s = 1/2, 1, 3/2$ the norms can be replaced by seminorms, as follows again by considering kernels; we will use this fact without further comment.

In the case of $L^2$ boundary data one also has, with $u = \Phi(f)$,

$$(43) \qquad \lim_{r \to 1^-} u(r\theta) \;=\; f(\theta) \quad \text{for a.e. } \theta \in \partial D,$$

$$(44) \qquad \left\| \sup_{0 \leq r < 1} |u(r \cdot)| \right\|_{L^2(\partial D)} \;\leq\; c \|f\|_{L^2(\partial D)}.$$

See [JK], Theorem (1.16), page 11. In particular, $u(r \cdot) \to f$ in $L^2(\partial D)$ as $r \to 1^-$. (The formulae (43) and (44) are established using the Poisson integral representation of $u$ from $f$. The function $u$ agrees with $\Phi(f)$ in (42), since it agrees for $f$ in the dense subset $C^\infty(\partial D) \subset L^2(\partial D)$ by the maximum principle, and since the Poisson integral map and $\Phi$ are both bounded as maps from $L^2(\partial D)$ into $L^2(D)$, for example.)

If $f : \partial D \to \Re$ has the Fourier series expansion

$$f(\phi) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi),$$

then one can define

$$(45) \qquad \|f\|^2_{L^2(\partial D)} \;=\; a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

$$(46) \qquad |f|^2_{H^s(\partial D)} \;=\; \sum_{n=1}^{\infty} n^{2s}(a_n^2 + b_n^2) \quad s > 0,$$

$$(47) \qquad \|f\|^2_{H^s(\partial D)} \;=\; \|f\|^2_{L^2(\partial D)} + |f|^2_{H^s(\partial D)}.$$

These norms and seminorms are equivalent to the usual definitions in case $s \in \mathbb{N}$, and the previous definitions in case $s = 1/2$ or $3/2$. The harmonic extension of $f$ is given by

$$(\Phi f)(r, \phi) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\phi + b_n \sin n\phi).$$

**Proposition 3.1.** *Suppose* $f, g : \partial D \to \Re$. *Then*

$$(48) \qquad\qquad |fg|_{H^{1/2}} \;\leq\; \|f\|_{C^0}|g|_{H^{1/2}} + |f|_{H^{1/2}}\|g\|_{C^0},$$

$$(49) \qquad\qquad \|fg\|_{H^{1/2}} \;\leq\; c\|f\|_{C^{0,1}}\|g\|_{H^{1/2}},$$

$$(50) \qquad\qquad |fg|_{H^1} \;\leq\; \|f\|_{C^0}|g|_{H^1} + |f|_{H^1}\|g\|_{C^0},$$

$$(51) \qquad\qquad \|fg\|_{H^1} \;\leq\; c\|f\|_{C^{0,1}}\|g\|_{H^1},$$

$$(52) \qquad\qquad \|fg\|_{H^{3/2}} \;\leq\; c\|f\|_{C^2}\|g\|_{H^{3/2}}.$$

*Proof.* The first two inequalities follow from (37). The next two inequalities are standard. The last inequality follows from (49). □

The next proposition will be applied in case $g$ is (a component of) $\gamma$, $\gamma'$ or $\gamma''$, and $s$ is either smooth or piecewise linear and continuous. In particular, $g$ will be at least $C^1$ but $s$ may be only $C^{0,1}$ in some cases.

**Proposition 3.2.** *Suppose* $s = id + \sigma : \partial D \to S^1$, $g : S^1 \to \Re$. *Then*

$$(53) \qquad\qquad \|g \circ s\|_{C^0} \;\leq\; \|g\|_{C^0} \leq \|g\|_{C^0}\|s\|_{C^0},$$

$$(54) \qquad\qquad \|g \circ s\|_{C^{0,1}} \;\leq\; c\|g\|_{C^1}\|s\|_{C^{0,1}},$$

$$(55) \qquad\qquad \|g \circ s\|_{C^1} \;\leq\; c\|g\|_{C^1}\|s\|_{C^1},$$

$$(56) \qquad\qquad \|g \circ s\|_{C^2} \;\leq\; c\|g\|_{C^2}\|s\|_{C^2}^2,$$

$$(57) \qquad\qquad \|g \circ s\|_{L^2} \;\leq\; c\|g\|_{C^0} \leq c\|g\|_{C^0}\|s\|_{L^2},$$

$$(58) \qquad\qquad \|g \circ s\|_{H^{1/2}} \;\leq\; c\|g\|_{C^1}\|s\|_{H^{1/2}},$$

$$(59) \qquad\qquad \|g \circ s\|_{H^1} \;\leq\; c\|g\|_{C^1}\|s\|_{H^1}.$$

*Proof.* Recall that $\|s\|_{C^0} = 1 + \|\sigma\|_{C^0} \geq 1$. Then (53) is immediate, and similarly for (57). Also (54) is immediate. For (55), (56) and (59), note that

$$(g \circ s)' = g' \circ s \; s', \quad (g \circ s)'' = g'' \circ s \; (s')^2 + g' \circ s \; s''.$$

Inequality (58) follows from (37). □

The following proposition will typically be applied in case $g$ is $\gamma$, $\gamma'$ or $\gamma''$ and in particular is $C^1$; and either $s_1 = s_0$ and $s_2 = p_h s_0$, or $s_1 = p_h s_0$ and $s_2 = p_h s_0 + \eta_h$ for an arbitrary $\eta_h \in H_h$. Note that in (61) only the $C^0$ norm of $s_1 - s_2$ is required, and in case $s_1 - s_2 \in H_h$ this will be estimated by the inverse estimate (20).

**Proposition 3.3.** *Suppose* $s_i = id + \sigma_i : \partial D \to S^1$ *for* $i = 1, 2$ *and* $g : S^1 \to \Re$. *Then*

$$(60) \quad \|g \circ s_1 - g \circ s_2\|_{L^2} \;\leq\; c|g|_{C^1}\|s_1 - s_2\|_{L^2},$$

$$(61) \quad |g \circ s_1 - g \circ s_2|_{H^{1/2}} \;\leq\; c\|g\|_{C^2}(\|s_1\|_{C^{0,1}} + \|s_1 - s_2\|_{C^0})\|s_1 - s_2\|_{H^{1/2}},$$

$$(62) \quad |g \circ s_1 - g \circ s_2|_{H^1} \;\leq\; c\|g\|_{C^2}\|s_1\|_{C^{0,1}}\|s_1 - s_2\|_{H^1},$$

$$(63) \quad |g \circ s_1 - g \circ s_2|_{C^{0,1}} \;\leq\; c\|g\|_{C^2}\|s_1\|_{C^{0,1}}\|s_1 - s_2\|_{C^{0,1}}.$$

*Proof.* The proof of (60) is immediate.
For (61) write

$$\eta(\phi) = s_2(\phi) - s_1(\phi).$$

Then

$$\left|\Big(g(s_1(\phi)) - g(s_1(\phi) + \eta(\phi))\Big) - \Big(g(s_1(\overline{\phi})) - g(s_1(\overline{\phi}) + \eta(\overline{\phi}))\Big)\right|$$

$$= \left|-\int_0^1 g'(s_1(\phi) + t\eta(\phi))dt\ \eta(\phi) + \int_0^1 g'(s_1(\overline{\phi}) + t\eta(\overline{\phi}))dt\ \eta(\overline{\phi})\right|$$

$$\leq \int_0^1 |g'(s_1(\phi) + t\eta(\phi))|dt\ |\eta(\phi) - \eta(\overline{\phi})|$$

$$+ \int_0^1 \left|g'(s_1(\phi) + t\eta(\phi)) - g'(s_1(\overline{\phi}) + t\eta(\overline{\phi}))\right| dt\ |\eta(\overline{\phi})|$$

$$\leq \|g\|_{C^1}|\eta(\phi) - \eta(\overline{\phi})| + \|g\|_{C^2}\Big(|s_1(\phi) - s_1(\overline{\phi})| + |\eta(\phi) - \eta(\overline{\phi})|\Big)|\eta(\overline{\phi})|.$$

It now follows from (37) that

$$|g \circ s_1 - g \circ (s_1 + \eta)|_{H^{1/2}}$$
$$\leq \|g\|_{C^1}|\eta|_{H^{1/2}} + \|g\|_{C^2}(c\|s_1\|_{C^{0,1}}\|\eta\|_{L^2} + |\eta|_{H^{1/2}}\|\eta\|_{C^0})$$
$$\leq c\|g\|_{C^2}(\|s_1\|_{C^{0,1}} + \|\eta\|_{C^0})\|\eta\|_{H^{1/2}},$$

recalling that $\|s_1\|_{C^{0,1}} \geq 1$. This establishes (61).

For (62) we compute

$$\|(g \circ s_1 - g \circ s_2)'\|_{L^2}$$
$$= \|g' \circ s_1\ s_1' - g' \circ s_2\ s_2'\|_{L^2}$$
$$\leq \|g' \circ s_1 - g' \circ s_2\|_{L^2}\|s_1'\|_{L^\infty} + \|g' \circ s_2\|_{C^0}\|s_1' - s_2'\|_{L^2}$$
$$\leq c|g|_{C^2}\|s_1 - s_2\|_{L^2}|s_1|_{C^{0,1}} + |g|_{C^1}|s_1 - s_2|_{H^1} \quad \text{from (60)}$$
$$\leq c\|g\|_{C^2}\|s_1\|_{C^{0,1}}\|s_1 - s_2\|_{H^1}.$$

For (63) we similarly estimate $\|(g \circ s_1 - g \circ s_2)'\|_{L^\infty}$.  $\qquad\square$

The next proposition will be used repeatedly in the consistency and non-degeneracy estimates of Sections 4 and 5, particularly in case $s = 1$. In Corollary 3.5 we see that the $H^1(D_h)$ seminorm of the discrete harmonic extension of a *discrete* function $f$ can be estimated by the $H^1(D)$ seminorm of the smooth harmonic extension of the function. This is not true for general smooth $f$. The corollary is due to Bramble, Pasciak and Schatz [BPS].

**Proposition 3.4.** *If* $f \in H^s(\partial D, \Re^n)$ *where* $s = 1, 3/2$, *then*

$$(64) \qquad |\Phi(f) - \Phi_h I_h(f)|_{H^1(D_h)} \leq ch^{s-1/2}|f|_{H^s(\partial D)},$$

$$(65) \qquad |\Phi_h I_h(f)|_{H^1(D_h)} \leq |f|_{H^{1/2}(\partial D)} + ch^{s-1/2}|f|_{H^s(\partial D)}.$$

*Proof.* It is sufficient to take $n = 1$. Let

$$u = \Phi(f), \quad u_h = \Phi_h I_h(f).$$

Note that $I_h f$ is well-defined, as $f \in C^0(\partial D_h)$. Since $u_h = I_h u$ on $\partial D_h$, it follows in the usual way from the weak form of Laplace's equation and (25) that

$$(66) \qquad 0 = \int_{D_h} (\nabla u_h - \nabla u)(\nabla u_h - \nabla I_h u),$$

and so

$$(67) \qquad \int_{D_h} |\nabla u - \nabla u_h|^2 = \int_{D_h} (\nabla u - \nabla u_h)(\nabla u - \nabla I_h u).$$

Hence

$$
\begin{aligned}
|u - u_h|_{H^1(D_h)} &\leq |u - I_h u|_{H^1(D_h)} \\
&\leq ch^{s-1/2}|u|_{H^{s+1/2}(D_h)} \leq ch^{s-1/2}|u|_{H^{s+1/2}(D)} \\
&\leq ch^{s-1/2}|u|_{H^s(\partial D)},
\end{aligned}
$$

from a standard interpolation result and (42). This gives (64), and (65) then follows from (42). $\square$

**Corollary 3.5.** *Suppose $f_h \in H_h$. Then*

$$
|\Phi_h I_h(f_h)|_{H^1(D_h)} \leq c|\Phi(f_h)|_{H^1(D)} \tag{68}
$$

*Proof.* This follows from (65) with $s = 1$, the inverse estimate (19), and (40). $\square$

Proposition 3.7 will allow us to estimate various quantities involving a harmonic function and the discrete disc $D_h$, in terms of the trace of the harmonic function on $D$. But first we need an elementary lemma.

**Lemma 3.6.** *Suppose $u \in H^1(D)$, and $\pi:\partial D \to \partial D_h$ is as in (14). Then*

$$
\|u - u \circ \pi\|_{L^2(\partial D)} \leq ch|u|_{H^1(D \setminus D_h)} \tag{69}
$$

*Proof.* Let $L_\theta$ be the straight line segment joining $\theta \in \partial D$ to $\pi(\theta) \in \partial D_h$. Then

$$
|u(\theta) - u \circ \pi(\theta)|^2 \leq \left(\int_{L_\theta} |\nabla u|\right)^2 \leq ch^2 \int_{L_\theta} |\nabla u|^2.
$$

Each $z \in D \setminus D_h$ can be written uniquely as $(\theta, y) \in \partial D \times \Re^+$, where $z \in L_\theta$ and $y$ is the distance of $z$ from $\theta \in \partial D$. The corresponding map $(\theta, y) \mapsto z$ has Jacobian $J$ satisfying $1 - ch^2 \leq J \leq 1$. Hence

$$
\int_{\partial D} |u - u \circ \pi|^2 \leq ch^2 \int_{\partial D} d\theta \int_{L_\theta} |\nabla u|^2 \leq ch^2 \int_{D \setminus D_h} |\nabla u|^2.
$$

$\square$

In the following proposition and elsewhere, $\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ denote normal and tangential derivatives on the relevant curve.

**Proposition 3.7.** *Suppose $u$ is harmonic in $D$ with trace $u|_{\partial D} \in L^2(\partial D)$ $H^1(\partial D)$ as appropriate. Then*

$$
\|u\|_{L^2(D \setminus D_h)} \leq ch\|u\|_{L^2(\partial D)}, \tag{70}
$$

$$
\|\nabla u\|_{L^2(D \setminus D_h)} \leq ch|u|_{H^1(\partial D)}, \tag{71}
$$

$$
\|u - u \circ \pi\|_{L^2(\partial D)} \leq ch^2|u|_{H^1(\partial D)}, \tag{72}
$$

$$
\left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial D_h)} \leq c|u|_{H^1(\partial D)}. \tag{73}
$$

*Proof.* Let $D(\rho)$ be the disc of radius $\rho$. From (44)

$$
\|u\|_{L^2(\partial D(\rho))} \leq c\|u\|_{L^2(\partial D)}.
$$

Integrating the square of this inequality with respect to $\rho$ from $1 - ch^2$ to $1$ now establishes (70).

Since $\nabla u$ is also harmonic, it follows that

$$
\|\nabla u\|_{L^2(D \setminus D_h)} \leq ch\|\nabla u\|_{L^2(\partial D)}.
$$

But

$$(74) \qquad \int_{\partial D} \left| \frac{\partial u}{\partial \nu} \right|^2 = \int_{\partial D} \left| \frac{\partial u}{\partial \tau} \right|^2,$$

from a Fourier series expansion, and so (71) follows.

Inequality (72) follows from (71) and the previous lemma.

For (73) we have that if $\triangle u = 0$ and $u|_{\partial D} \in H^1(\partial D)$, then $\triangle \nabla u = 0$ and $\nabla u \in H^{1/2}(D)$ by (42). Hence $\nabla u|_{\partial D}$ is well-defined, and $\nabla u|_{\partial D} \in L^2(\partial D)$ from (41). From (44) applied to $\nabla u$,

$$\|\nabla u\|_{L^2(\partial D_h)} \leq c \|\nabla u\|_{L^2(\partial D)}$$

and in particular, using (74),

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_h)} \leq c |u|_{H^1(\partial D)}.$$

This establishes (73).                                                     □

## 4. Consistency estimates for the energy

In this section we compare $E$ and its derivatives at $s$ with $E_h$ and its derivatives at $p_h s$. Apart from their intrinsic significance, these estimates will be needed in the next section to establish the main convergence results.

*Remarks on the Proofs.* In the proof of each of the three propositions in this section, Proposition 3.4 is used to estimate the difference between a harmonic function and the corresponding discrete harmonic function, and also to estimate various discrete harmonic "error" terms.

Interpolation results in terms of the $H^2$ *piecewise* seminorm, together with inverse estimates, are used to estimate the $L^2$ norm of the quantities

$$\gamma \circ p_h s - I_h^{\partial D}(\gamma \circ p_h s),$$
$$\gamma' \circ p_h s \, \xi_h - I_h^{\partial D}(\gamma' \circ p_h s \, \xi_h),$$

and

$$\gamma'' \circ p_h s \, \xi_h^2 - I_h^{\partial D}(\gamma'' \circ p_h s \, \xi_h)^2.$$

Since piecewise second derivatives of $p_h s$ and $\xi_h$ vanish, this enables us to gain an extra power of $h$ over what one might at first expect.

In each of the three proofs the term $I_2$ is estimated with an integration by parts. In Proposition 4.1 this improves the order of convergence from $O(h)$ to $O(h^2)$. In the other two propositions, one could not otherwise expect any order of convergence; see also the remark preceding Proposition 4.3 concerning $\|\nabla(v - v_h)\|_{L^2(D_h)}$.

**Proposition 4.1.** *Let $s \in \mathcal{H} \cap C^2$, $\gamma \in C^2$. Then*

$$|E_h(p_h s) - E(s)| \leq c h^2 \|\gamma\|_{C^2}^2 \|s\|_{C^2}^4.$$

*Proof.* Let

$$(75) \qquad u = \Phi(\gamma \circ s), \quad u_h = \Phi_h I_h(\gamma \circ p_h s).$$

Then

$$E_h(p_h s) - E(s) = \frac{1}{2} \int_{D_h} |\nabla u_h|^2 - \frac{1}{2} \int_D |\nabla u|^2$$

$$= \frac{1}{2} \int_{D_h} |\nabla (u - u_h)|^2 - \int_{D_h} \nabla u \nabla (u - u_h) - \frac{1}{2} \int_{D \setminus D_h} |\nabla u|^2$$

$$(76) \qquad = I_1 + I_2 + I_3.$$

We will estimate these terms separately.

For $I_1$,

$$(2I_1)^{1/2} = |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ p_h s)|_{H^1(D_h)}$$
$$\leq |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ s)|_{H^1(D_h)} + |\Phi_h I_h(\gamma \circ s - \gamma \circ p_h s)|_{H^1(D_h)}$$
$$\leq ch|\gamma \circ s|_{H^{3/2}} + |\gamma \circ s - \gamma \circ p_h s|_{H^{1/2}} + ch^{1/2}|\gamma \circ s - \gamma \circ p_h s|_{H^1}$$
$$\text{from (64) with } s = 3/2 \text{ and (65) with } s = 1.$$

But

$$h|\gamma \circ s|_{H^{3/2}} \leq ch|\gamma \circ s|_{C^2} \leq ch\|\gamma\|_{C^2}\|s\|_{C^2}^2 \quad \text{from (56)};$$

also

$$|\gamma \circ s - \gamma \circ p_h s|_{H^{1/2}} \leq c\|\gamma\|_{C^2}(\|s\|_{C^{0,1}} + \|p_h s\|_{C^{0,1}})\|s - p_h s\|_{H^{1/2}} \text{ by (61)}$$
$$\leq ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|s\|_{H^2} \quad \text{by (23) and (22)}$$
$$\leq ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^2}^2,$$

and

$$h^{1/2}|\gamma \circ s - \gamma \circ p_h s|_{H^1} \leq ch^{1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|s - p_h s\|_{H^1} \quad \text{from (62)}$$
$$\leq ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^2}^2 \quad \text{from (22)}.$$

Hence

$$(77) \qquad\qquad I_1 \leq ch^2\|\gamma\|_{C^2}^2\|s\|_{C^2}^4.$$

For $I_2$ one has

$$|I_2| = \left| \int_{\partial D_h} \frac{\partial u}{\partial \nu}(u - u_h) \right|$$

$$\leq \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_h)} \|u - u_h\|_{L^2(\partial D_h)}$$

$$\leq c|u|_{H^1(\partial D)}\|u - u_h\|_{L^2(\partial D_h)} \quad \text{by (73)}$$

$$\leq c\|\gamma\|_{C^1}\|s\|_{C^1}\|u \circ \pi - u_h \circ \pi\|_{L^2(\partial D)} \quad \text{by (59) and (14)}$$

$$\leq c\|\gamma\|_{C^1}\|s\|_{C^1} \left( \|u \circ \pi - u\|_{L^2(\partial D)} + \|u - u_h \circ \pi\|_{L^2(\partial D)} \right).$$

But

$$\|u \circ \pi - u\|_{L^2(\partial D)} \leq ch^2|u|_{H^1(\partial D)} = ch^2|\gamma \circ s|_{H^1} \leq ch^2\|\gamma\|_{C^1}\|s\|_{C^1}$$

by (72) and (59). Also

$$\|u - u_h \circ \pi\|_{L^2(\partial D)} = \|\gamma \circ s - I_h^{\partial D}(\gamma \circ p_h s)\|_{L^2} \quad \text{by (75) and (21)}$$
$$\leq \|\gamma \circ s - \gamma \circ p_h s\|_{L^2} + \|\gamma \circ p_h s - I_h^{\partial D}(\gamma \circ p_h s)\|_{L^2}$$
$$\leq \|\gamma\|_{C^1}\|s - p_h s\|_{L^2} + ch^2|\gamma \circ p_h s|_{H^2},$$

by (60) and a standard interpolation estimate, where the $H^2$ semi-norm is to be understood in a piecewise sense. On each arc segment, $(\gamma \circ p_h s)'' = \gamma'' \circ p_h s \, ((p_h s)')^2$ since $(p_h s)'' = 0$, so we can continue the estimate by

$$\leq c\|\gamma\|_{C^1} h^2 \|s\|_{C^2} + ch^2 \|\gamma\|_{C^2} \|s\|_{C^1}^2 \leq ch^2 \|\gamma\|_{C^2} \|s\|_{C^2}^2,$$

using (24) and (23). Hence

$$(78) \qquad\qquad |I_2| \leq ch^2 \|\gamma\|_{C^2}^2 \|s\|_{C^2}^3.$$

Finally,

$$|I_3| \; = \; \frac{1}{2}\|\nabla u\|_{L^2(D \setminus D_h)}^2 \; \leq \; ch^2 |u|_{H^1(\partial D)}^2 \quad \text{by (71)}$$
$$(79) \qquad\qquad = \; ch^2 |\gamma \circ s|_{H^1}^2 \; \leq \; ch^2 \|\gamma\|_{C^1}^2 \|s\|_{C^1}^2.$$

The proposition now follows from (76), (77), (78) and (79). $\qquad\qquad\square$

The next proposition shows that $E_h'(p_h s)$ is an $O(h)$ approximation to $E'(s)|_{H_h}$ in the $H_h'$ norm for $\gamma$ and $s$ sufficiently smooth. The actual power of $h$ is important in the proof of Theorem 5.4.

**Proposition 4.2.** *Let $s \in \mathcal{H} \cap C^2$, $\gamma \in C^3$. Then for any $\xi_h \in H_h$*

$$|\langle E'(s), \xi_h \rangle - \langle E_h'(p_h s), \xi_h \rangle| \leq ch\|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}.$$

*Proof.* Define $u$ and $u_h$ as in (75), and for $\xi_h \in H_h$ define

$$(80) \qquad\qquad v = \Phi(\gamma' \circ s \, \xi_h), \quad v_h = \Phi_h I_h(\gamma' \circ p_h s \, \xi_h).$$

Then from (29),

$$\langle E'(s), \xi_h \rangle - \langle E_h'(p_h s), \xi_h \rangle$$
$$= \int_D \nabla u \nabla v - \int_{D_h} \nabla u_h \nabla v_h$$
$$= \int_{D_h} (\nabla u \nabla v - \nabla u_h \nabla v_h) + \int_{D \setminus D_h} \nabla u \nabla v$$
$$= \int_{D_h} \nabla(u - u_h)\nabla v_h + \int_{D_h} \nabla u \nabla(v - v_h) + \int_{D \setminus D_h} \nabla u \nabla v$$
$$(81) \qquad = \; I_1 + I_2 + I_3.$$

Now

$$|I_1| \; \leq \; \|\nabla(u - u_h)\|_{L^2(D_h)} \|\nabla v_h\|_{L^2(D_h)}$$
$$\leq \; ch\|\gamma\|_{C^2} \|s\|_{C^2}^2 \|\nabla v_h\|_{L^2(D_h)} \quad \text{from (77)}.$$

Also

$$\|\nabla v_h\|_{L^2(D_h)} \; = \; |\Phi_h I_h(\gamma' \circ p_h s \, \xi_h)|_{H^1(D_h)}$$
$$\leq \; |\gamma' \circ p_h s \, \xi_h|_{H^{1/2}} + ch^{1/2}|\gamma' \circ p_h s \, \xi_h|_{H^1} \quad \text{by (65)}.$$

The first term is estimated by

$$|\gamma' \circ p_h s \, \xi_h|_{H^{1/2}} \; \leq \; c\|\gamma' \circ p_h s\|_{C^{0,1}} \|\xi_h\|_{H^{1/2}} \quad \text{from (49)}$$
$$\leq \; c\|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}} \quad \text{using (23)};$$

the second by

$$
\begin{aligned}
|\gamma' \circ p_h s\ \xi_h|_{H^1} &\leq\ c\|\gamma' \circ p_h s\|_{C^{0,1}}\|\xi_h\|_{H^1} \quad \text{by (51)} \\
&\leq\ c\|\gamma\|_{C^2}\|p_h s\|_{C^{0,1}}\|\xi_h\|_{H^1} \quad \text{by (54)} \\
&\leq\ ch^{-1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}} \quad \text{by (23) and (19).}
\end{aligned}
$$

(82)

So altogether,

(83) $$|I_1| \leq ch\|\gamma\|_{C^2}^2\|s\|_{C^1}^3\|\xi_h\|_{H^{1/2}}.$$

For $I_2$, as for $I_2$ in the proof of the previous proposition,

$$
\begin{aligned}
|I_2| &=\ \left|\int_{\partial D_h} \frac{\partial u}{\partial \nu}(v - v_h)\right| \\
&\leq\ c\|\gamma\|_{C^1}\|s\|_{C^1}\left(\|v \circ \pi - v\|_{L^2(\partial D)} + \|v - v_h \circ \pi\|_{L^2(\partial D)}\right).
\end{aligned}
$$

From (72) and the argument for (82),

$$
\begin{aligned}
\|v \circ \pi - v\|_{L^2(\partial D)} &\leq\ ch^2|v|_{H^1(\partial D)}\ =\ ch^2|\gamma' \circ s\ \xi_h|_{H^1} \\
&\leq\ ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\|v - v_h \circ \pi\|_{L^2(\partial D)} \\
&=\ \|\gamma' \circ s\ \xi_h - I_h^{\partial D}(\gamma' \circ p_h s\ \xi_h)\|_{L^2} \\
&\leq\ \|(\gamma' \circ s - \gamma' \circ p_h s)\ \xi_h\|_{L^2} + \|\gamma' \circ p_h s\ \xi_h - I_h^{\partial D}(\gamma' \circ p_h s\ \xi_h)\|_{L^2} \\
&\leq\ \|\gamma\|_{C^2}\|s - p_h s\|_{C^0}\|\xi_h\|_{L^2} + ch^2|\gamma' \circ p_h s\ \xi_h|_{H^2} \\
&\qquad \text{where the } H^2 \text{ semi-norm is understood in the piecewise sense} \\
&\leq\ ch^2\|\gamma\|_{C^2}\|s\|_{C^2}\|\xi_h\|_{L^2} \quad \text{from (24)} \\
&\qquad + ch^2\left(\|\gamma\|_{C^3}\|s\|_{C^1}^2\|\xi_h\|_{L^2} + \|\gamma\|_{C^2}\|s\|_{C^1}|\xi_h|_{H^1}\right) \quad \text{since on each arc} \\
&\qquad \text{segment, } (\gamma' \circ p_h s\ \xi_h)'' = \gamma''' \circ p_h s((p_h s)')^2\ \xi_h + 2\gamma'' \circ p_h s(p_h s)'\ \xi_h' \\
&\leq\ ch^2\|\gamma\|_{C^3}\|s\|_{C^2}^2\|\xi_h\|_{H^1} \\
&\leq\ ch^{3/2}\|\gamma\|_{C^3}\|s\|_{C^2}^2\|\xi_h\|_{H^{1/2}} \quad \text{by (19).}
\end{aligned}
$$

Altogether,

(84) $$|I_2| \leq ch^{3/2}\|\gamma\|_{C^3}^2\|s\|_{C^2}^3\|\xi_h\|_{H^{1/2}}.$$

Finally,

$$
\begin{aligned}
|I_3| &\leq\ \|\nabla u\|_{L^2(D\setminus D_h)}\|\nabla v\|_{L^2(D\setminus D_h)} \\
&\leq\ ch\|\gamma\|_{C^1}\|s\|_{C^1}\|\nabla v\|_{L^2(D\setminus D_h)} \quad \text{by (79)} \\
&\leq\ ch^2\|\gamma\|_{C^1}\|s\|_{C^1}|v|_{H^1(\partial D)} \quad \text{by (71)} \\
&\leq\ ch^2\|\gamma\|_{C^1}\|s\|_{C^1}|\gamma' \circ s\ \xi_h|_{H^1} \\
&\leq\ ch^2\|\gamma\|_{C^1}\|s\|_{C^1}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^1} \quad \text{by (51) and (55)} \\
&\leq\ ch^{3/2}\|\gamma\|_{C^2}^2\|s\|_{C^1}^2\|\xi_h\|_{H^{1/2}} \quad \text{by (19).}
\end{aligned}
$$

(85)

The proposition follows from (81), (83), (84) and (85). $\qquad\square$

*Remark.* The following proposition, and the related Proposition 5.3, are essential for the proof of the main theorem. Assuming $\gamma$ and $s$ are sufficiently smooth, we see from (87) that if $E''(s)$ is positive definite (for example, if $s$ is a strictly stable

local minimum for $E$) then so is $E_h''(p_h s)$ for $h$ sufficiently small, with the same "ellipticity" constant up to $O(h^{1/2}|\log h|^{1/2})$.

One might hope to estimate the positive term

$$(86) \qquad \|\nabla(v - v_h)\|_{L^2(D_h)}^2 = \|\Phi(\gamma' \circ s\ \xi_h) - \Phi_h I_h(\gamma' \circ p_h s\ \xi_h)\|_{H^1(D_h)}^2$$

in (87) by $ch^\alpha \|\xi_h\|_{H^{1/2}}^2$ for some $\alpha > 0$. But we cannot expect this, as we see by considering the "model" estimates:

$$\|\Phi(\xi_h) - \Phi_h I_h(\xi_h)\|_{H^1(D_h)} \le ch^{1/2}\|\xi_h\|_{H^1} \le c\|\xi_h\|_{H^{1/2}}$$

from (64) and (19) — these estimates cannot be improved.

Nonetheless, in Proposition 5.3 we *do* estimate (86) in case $\xi_h = p_h \xi$, where $\xi$ is in the negative eigenspace of $E''(s)$. The point is that $\xi$ is then smooth by elliptic regularity theory (although $\xi_h$ is of course not smooth), and in fact we show in (106) that in this case

$$\|\nabla(v - v_h)\|_{L^2(D_h)}^2 \le ch\|\xi_h\|_{H^{1/2}}^2.$$

As a consequence we deduce in Proposition 5.3 that if $E''(s)$ is nondegenerate then so is $E_h''(p_h s)$ for $h$ sufficiently small, with the same nondegeneracy constant up to $O(h^{1/2}|\log h|^{1/2})$.

**Proposition 4.3.** *Let $s \in C^2 \cap \mathcal{H}$ and $\gamma \in C^3$. Then for any $\xi_h \in H_h$*

$$(87) \qquad E_h''(p_h s)(\xi_h, \xi_h) - E''(s)(\xi_h, \xi_h) = \|\nabla(v - v_h)\|_{L^2(D_h)}^2 + R$$

*where*

$$v = \Phi(\gamma' \circ s\ \xi_h), \quad v_h = \Phi_h I_h(\gamma' \circ p_h s\ \xi_h),$$

*and*

$$|R| \le ch^{1/2}|\log h|^{1/2}\|\gamma\|_{C^3}^2\|s\|_{C^2}^3\|\xi_h\|_{H^{1/2}}^2.$$

*Proof.* Let $u$ and $u_h$ be as in (75). Let

$$(88) \qquad w = \Phi(\gamma'' \circ s\ \xi_h^2), \quad w_h = \Phi_h I_h(\gamma'' \circ p_h s\ \xi_h^2).$$

From (30),

$$\Big(E''(s) - E_h''(p_h s)\Big)(\xi_h, \xi_h)$$

$$= \int_D |\nabla v|^2 + \int_D \nabla u \nabla w - \int_{D_h} |\nabla v_h|^2 - \int_{D_h} \nabla u_h \nabla w_h$$

$$= -\int_{D_h} |\nabla(v - v_h)|^2 + 2\int_{D_h} \nabla v \nabla(v - v_h) + \int_{D_h} \nabla(u - u_h)\nabla w_h$$

$$+ \int_{D_h} \nabla(w - w_h)\nabla u + \int_{D\setminus D_h} |\nabla v|^2 + \int_{D\setminus D_h} \nabla u \nabla w$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

We estimate the terms other than $I_1$.

For $I_2$ we have

$$|I_2| = 2\left|\int_{D_h} \nabla v \nabla(v - v_h)\right| = 2\left|\int_{\partial D_h} \frac{\partial v}{\partial \nu}\ (v - v_h)\right|$$

$$\le 2\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\partial D_h)} \|v - v_h\|_{L^2(\partial D_h)}.$$

The first factor is estimated with the use of (73) and arguing as for (82):

$$\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\partial D_h)} \leq c|v|_{H^1(\partial D)} = c|\gamma' \circ s\,\xi_h|_{H^1} \leq ch^{-1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}.$$

The second factor was estimated in the estimate for $I_2$ in the proof of Proposition 4.2:

$$\|v - v_h\|_{L^2(\partial D_h)} \leq ch^{3/2}\|\gamma\|_{C^3}\|s\|_{C^2}^2\|\xi_h\|_{H^{1/2}}.$$

Altogether for $I_2$,

(89) $$|I_2| \leq ch\|\gamma\|_{C^3}^2\|s\|_{C^2}^3\|\xi_h\|_{H^{1/2}}^2.$$

For $I_3$,

$$|I_3| = \left|\int_{D_h} \nabla(u - u_h)\nabla w_h\right| \leq \|\nabla(u - u_h)\|_{L^2(D_h)}\|\nabla w_h\|_{L^2(D_h)}$$

$$\leq ch\|\gamma\|_{C^2}\|s\|_{C^2}^2\|\nabla w_h\|_{L^2(D_h)} \quad \text{from (77).}$$

From (65) we get

$$\|\nabla w_h\|_{L^2(D_h)} = |\Phi_h I_h(\gamma'' \circ p_h s\,\xi_h^2)|_{H^1(D_h)}$$

$$\leq |\gamma'' \circ p_h s\,\xi_h^2|_{H^{1/2}} + ch^{1/2}|\gamma'' \circ p_h s\,\xi_h^2|_{H^1}.$$

But

$$|\gamma'' \circ p_h s\,\xi_h^2|_{H^{1/2}} \leq c\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h^2\|_{H^{1/2}} \quad \text{from (49) and (54)}$$

$$\leq c\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{C^0}\|\xi_h\|_{H^{1/2}} \quad \text{from (48)}$$

$$\leq c|\log h|^{1/2}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2 \quad \text{from (20).}$$

Similarly,

$$|\gamma'' \circ p_h s\,\xi_h^2|_{H^1} \leq c\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h^2\|_{H^1} \quad \text{using (51)}$$

$$\leq c\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{C^0}\|\xi_h\|_{H^1} \quad \text{from (50)}$$

(90) $$\leq ch^{-1/2}|\log h|^{1/2}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2$$

from (20) and (19). Hence

$$\|\nabla w_h\|_{L^2(D_h)} \leq c|\log h|^{1/2}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2,$$

and so

(91) $$|I_3| \leq ch|\log h|^{1/2}\|\gamma\|_{C^3}^2\|s\|_{C^2}^3\|\xi_h\|_{H^{1/2}}^2.$$

For $I_4$ we have

$$I_4 = \int_{D_h} \nabla(w - w_h)\nabla u = \int_{D_h} (w - w_h)\frac{\partial u}{\partial \nu},$$

and so, as for $I_2$ in the proof of Proposition 4.1,

(92) $$|I_4| \leq c\|\gamma\|_{C^1}\|s\|_{C^1}\left(\|w \circ \pi - w\|_{L^2(\partial D)} + \|w - w_h \circ \pi\|_{L^2(\partial D)}\right).$$

But

$$\|w \circ \pi - w\|_{L^2(\partial D)} \leq ch^2|w|_{H^1(\partial D)} \quad \text{by (72)}$$

$$\leq ch^2|\gamma'' \circ s\,\xi_h^2|_{H^1}$$

$$\leq ch^{3/2}|\log h|^{1/2}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2 \quad \text{as for (90).}$$

Also

$$
\begin{aligned}
\|w &- w_h \circ \pi\|_{L^2(\partial D)} \\
&= \quad \|\gamma'' \circ s\, \xi_h^2 - I_h^{\partial D}(\gamma'' \circ p_h s\, \xi_h^2)\|_{L^2} \quad \text{from (88) and (21)} \\
&\leq \quad \|\gamma'' \circ s\, \xi_h^2 - \gamma'' \circ p_h s\, \xi_h^2\|_{L^2} + \|\gamma'' \circ p_h s\, \xi_h^2 - I_h^{\partial D}(\gamma'' \circ p_h s\, \xi_h^2)\|_{L^2} \\
&\leq \quad \|\gamma'' \circ s - \gamma'' \circ p_h s\|_{C^0}\|\xi_h\|_{L^4}^2 + ch|\gamma'' \circ p_h s\, \xi_h^2|_{H^1} \\
&\leq \quad c\|\gamma\|_{C^3}\|s - p_h s\|_{C^0}\|\xi_h\|_{H^{1/2}}^2 + ch^{1/2}|\log h|^{1/2}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2 \\
&\qquad \text{using a Sobolev embedding theorem and (90)} \\
&\leq \quad ch^{1/2}|\log h|^{1/2}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2 \quad \text{from (24)}.
\end{aligned}
$$

Hence

$$
(93) \qquad |I_4| \leq ch^{1/2}|\log h|^{1/2}\|\gamma\|_{C^3}^2\|s\|_{C^1}^2\|\xi_h\|_{H^{1/2}}^2.
$$

From the final part of the argument for $I_3$ in the proof of the previous proposition,

$$
(94) \qquad I_5 = \|\nabla v\|_{L^2(D\setminus D_h)}^2 \leq ch\|\gamma\|_{C^2}^2\|s\|_{C^1}^2\|\xi_h\|_{H^{1/2}}^2.
$$

Finally,

$$
\begin{aligned}
|I_6| &\leq \quad \|\nabla u\|_{L^2(D\setminus D_h)}\|\nabla w\|_{L^2(D\setminus D_h)} \\
&\leq \quad ch^2|u|_{H^1(\partial D)}|w|_{H^1(\partial D)} \quad \text{by (71)} \\
&= \quad ch^2|\gamma \circ s|_{H^1}|\gamma'' \circ s\, \xi_h^2|_{H^1} \\
&\leq \quad ch^2\|\gamma\|_{C^1}\|s\|_{C^1}|\gamma'' \circ s\, \xi_h^2|_{H^1} \quad \text{by (59)} \\
(95) \qquad &\leq \quad ch^{3/2}|\log h|^{1/2}\|\gamma\|_{C^3}^2\|s\|_{C^1}^2\|\xi_h\|_{H^{1/2}}^2 \quad \text{as for (90)}.
\end{aligned}
$$

Now we collect the estimates (89), (91), (93), (94) and (95), and prove the proposition. □

## 5. THE MAIN ERROR ESTIMATES

We will apply the following quantitative version of the Inverse Function Theorem with $\mathcal{X} = \mathcal{H}_h$ and $Y = H_h'$ in the proof of Theorem 5.4.

**Lemma 5.1.** *Let $\mathcal{X}$ be an affine Banach space with Banach space $X$ as tangent space, and let $Y$ be a Banach space. Suppose $x_0 \in \mathcal{X}$ and $f \in C^1(\mathcal{X}, Y)$. Assume there are positive constants $\alpha$, $\beta$, $\delta$ and $\epsilon$ such that*

$$
(96) \qquad \|f(x_0)\|_Y \quad \leq \quad \delta,
$$

$$
(97) \qquad \|f'(x_0)^{-1}\|_{L(Y,X)} \quad \leq \quad \alpha^{-1},
$$

$$
(98) \qquad \|f'(x) - f'(x_0)\|_{L(X,Y)} \quad \leq \quad \beta \quad \textit{for all } x \in \overline{B}_\epsilon(x_0),
$$

*where*

$$
(99) \qquad \beta < \alpha, \quad \delta \leq (\alpha - \beta)\epsilon.
$$

*Then there exists a unique $x_* \in \overline{B}_\epsilon(x_0)$ such that $f(x_*) = 0$.*

*Proof.* This follows from the proof of the Inverse Function Theorem in [Be, pp. 113–114]. The modifications necessary since $\mathcal{X}$ is an *affine* space are trivial. □

The next proposition establishes that for any $\beta > 0$,

$$
\|E_h''(p_h s) - E_h''(p_h s + \eta_h)\|_{(H_h \times H_h)'} \leq \beta
$$

provided $|\log h|\,\|\eta_h\|_{H^{1/2}}$ is sufficiently small (depending on $\beta$ but independent of $h$). This will be used to establish the appropriate version of (98) in the proof of Proposition 5.4.

*Remark on the Proof.* The precise form of estimate (61), and the inverse estimate (20), are used to control $|g \circ s_h - g \circ (s_h + \eta_h)|_{H^{1/2}}$ for $g = \gamma,\ \gamma'$ and $\gamma''$.

**Proposition 5.2.** *Let $s \in \mathcal{H} \cap C^1$ and $\gamma \in C^4$. Then for any $\eta_h, \xi_h \in H_h$*

$$\left|\Big(E_h''(p_h s) - E_h''(p_h s + \eta_h)\Big)(\xi_h, \xi_h)\right|$$
$$\leq\ c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)^2 |\log h|\,\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^4}^2\|s\|_{C^1}^2\|\xi_h\|_{H^{1/2}}^2.$$

*Proof.* For simplicity we write $s_h$ for $p_h s$.
  From (30),

$$\Big(E_h''(s_h) - E_h''(s_h + \eta_h)\Big)(\xi_h, \xi_h)$$
$$=\ \int_{D_h} |\nabla v_h|^2 + \int_{D_h} \nabla u_h \nabla w_h - \int_{D_h} |\nabla \overline{v}_h|^2 - \int_{D_h} \nabla \overline{u}_h \nabla \overline{w}_h,$$

with

$$
\begin{array}{llll}
u_h &=& \Phi_h I_h(\gamma \circ s_h), & \overline{u}_h &=& \Phi_h I_h(\gamma \circ (s_h + \eta_h)), \\
v_h &=& \Phi_h I_h(\gamma' \circ s_h\ \xi_h), & \overline{v}_h &=& \Phi_h I_h(\gamma' \circ (s_h + \eta_h)\ \xi_h), \\
w_h &=& \Phi_h I_h(\gamma'' \circ s_h\ \xi_h^2), & \overline{w}_h &=& \Phi_h I_h(\gamma'' \circ (s_h + \eta_h)\ \xi_h^2).
\end{array}
$$

So

$$\left|\Big(E_h''(s_h) - E_h''(s_h + \eta_h)\Big)(\xi_h, \xi_h)\right|$$
$$\leq\ \int_{D_h} |\nabla(v_h - \overline{v}_h)|\,(|\nabla v_h| + |\nabla \overline{v}_h|) + \int_{D_h} |\nabla(u_h - \overline{u}_h)|\,|\nabla w_h|$$
$$+ \int_{D_h} |\nabla \overline{u}_h|\,|\nabla(w_h - \overline{w}_h)|$$
$$\leq\ \|\nabla(u_h - \overline{u}_h)\|_{L^2(D_h)}\|\nabla w_h\|_{L^2(D_h)}$$
$$+ \|\nabla(v_h - \overline{v}_h)\|_{L^2(D_h)}\big(\|\nabla v_h\|_{L^2(D_h)} + \|\nabla \overline{v}_h\|_{L^2(D_h)}\big)$$
$$+ \|\nabla(w_h - \overline{w}_h)\|_{L^2(D_h)}\|\nabla \overline{u}_h\|_{L^2(D_h)}$$
$$(100) \qquad =\ A_1 B_1 + A_2(B_{21} + B_{22}) + A_3 B_3.$$

Now we estimate these terms separately.
  By (65)

$$A_1\ =\ \left|\Phi_h I_h\Big(\gamma \circ s_h - \gamma \circ (s_h + \eta_h)\Big)\right|_{H^1(D_h)}$$
$$\leq\ |\gamma \circ s_h - \gamma \circ (s_h + \eta_h)|_{H^{1/2}} + ch^{1/2}|\gamma \circ s_h - \gamma \circ (s_h + \eta_h)|_{H^1}$$
$$\leq\ c\|\gamma\|_{C^2}\big(\|s_h\|_{C^{0,1}} + \|\eta_h\|_{C^0}\big)\|\eta_h\|_{H^{1/2}} + ch^{1/2}\|\gamma\|_{C^2}\|s_h\|_{C^{0,1}}\|\eta_h\|_{H^1}$$
$$\qquad \text{by (61) and (62)}$$
$$\leq\ c\|\gamma\|_{C^2}\|s\|_{C^1}\|\eta_h\|_{H^{1/2}}\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)$$
$$+ c\|\gamma\|_{C^2}\|s\|_{C^1}\|\eta_h\|_{H^{1/2}} \quad \text{by (20), (19) and (23)}$$
$$\leq\ c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^2}\|s\|_{C^1}.$$

Again from (65)

$$
\begin{aligned}
A_2 &= \left| \Phi_h I_h\Big(\gamma' \circ s_h\ \xi_h - \gamma' \circ (s_h + \eta_h)\ \xi_h\Big) \right|_{H^1(D_h)} \\
&\leq \ \left|(\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h))\ \xi_h\right|_{H^{1/2}} \\
&\quad + ch^{1/2}\left|(\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h))\ \xi_h\right|_{H^1} \\
&\leq \ \left|\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)\right|_{H^{1/2}}\|\xi_h\|_{C^0} \\
&\quad + \|\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)\|_{C^0}|\xi_h|_{H^{1/2}} \quad \text{by (48)} \\
&\quad + ch^{1/2}\|\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)\|_{C^0}|\xi_h|_{H^1} \\
&\quad + ch^{1/2}|\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)|_{H^1}\|\xi_h\|_{C^0} \quad \text{by (50)} \\
&\leq \ c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{C^0} \\
&\quad + c\|\gamma\|_{C^2}\|\eta_h\|_{C^0}|\xi_h|_{H^{1/2}} + ch^{1/2}\|\gamma\|_{C^2}\|\eta_h\|_{C^0}|\xi_h|_{H^1} \quad \text{using the argu-} \\
&\qquad \text{ment for } A_1 \text{ on the first and last terms in the preceding inequality} \\
&\leq \ c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)|\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}},
\end{aligned}
$$

again using the inverse estimates (20) and (19) on $\eta_h$ and $\xi_h$.

In the same way we proceed for $A_3$ to first obtain the analogue of the next-to-last inequality above:

$$
\begin{aligned}
A_3 &= \left| \Phi_h I_h(\gamma'' \circ s_h\ \xi_h^2 - \gamma'' \circ (s_h + \eta_h)\xi_h^2) \right|_{H^1(D_h)} \\
&\leq \ c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)\|\eta_h\|^{1/2}\|\gamma\|_{C^4}\|s\|_{C^1}\|\xi_h^2\|_{C^0} \\
&\quad + c\|\gamma\|_{C^3}\|\eta_h\|_{C^0}|\xi_h^2|_{H^{1/2}} + ch^{1/2}\|\gamma\|_{C^3}\|\eta_h\|_{C^0}|\xi_h^2|_{H^1} \\
&\leq \ c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)|\log h|\,\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^4}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2,
\end{aligned}
$$

from the inverse estimates (19) and (20) applied to $\eta_h$ and $\xi_h$, and since

$$
\|\xi_h^2\|_{H^{1/2}} \leq 2\|\xi_h\|_{C^0}\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^2\|_{H^1} \leq 2\|\xi_h\|_{C^0}\|\xi_h\|_{H^1}.
$$

For $B_3$,

$$
\begin{aligned}
B_3 &= \ |\Phi_h I_h(\gamma \circ (s_h + \eta_h))|_{H^1(D_h)} \\
&\leq \ |\gamma \circ (s_h + \eta_h)|_{H^{1/2}} + ch^{1/2}|\gamma \circ (s_h + \eta_h)|_{H^1} \quad \text{by (65)} \\
&\leq \ \|\gamma\|_{C^1}(\|s_h\|_{H^{1/2}} + \|\eta_h\|_{H^{1/2}}) + ch^{1/2}\|\gamma\|_{C^1}(\|s_h\|_{H^1} + \|\eta_h\|_{H^1}) \\
&\leq \ c\|\gamma\|_{C^1}(\|s_h\|_{H^{1/2}} + \|\eta_h\|_{H^{1/2}}) \quad \text{by (19)} \\
&\leq \ c(1 + \|\eta_h\|_{H^{1/2}})\|\gamma\|_{C^1}\|s\|_{C^1}.
\end{aligned}
$$

For $B_{22}$,

$$
\begin{aligned}
B_{22} &= |\Phi_h I_h(\gamma' \circ (s_h + \eta_h)\, \xi_h)|_{H^1(D_h)} \\
&\leq |\gamma' \circ (s_h + \eta_h)\, \xi_h|_{H^{1/2}} + ch^{1/2}|\gamma' \circ (s_h + \eta_h)\, \xi_h|_{H^1} \quad \text{by (65)} \\
&\leq \|\gamma' \circ (s_h + \eta_h)\|_{C^0}|\xi_h|_{H^{1/2}} + |\gamma' \circ (s_h + \eta_h)|_{H^{1/2}}\|\xi_h\|_{C^0} \\
&\quad + ch^{1/2}\Big(\|\gamma' \circ (s_h + \eta_h)\|_{C^0}|\xi_h|_{H^1} + |\gamma' \circ (s_h + \eta_h)|_{H^1}\|\xi_h\|_{C^0}\Big) \\
&\quad \text{by (48)} \\
&\leq c\|\gamma\|_{C^1}|\xi_h|_{H^{1/2}} + c\|\gamma\|_{C^2}|s_h + \eta_h|_{H^{1/2}}\|\xi_h\|_{C^0} \\
&\quad + ch^{1/2}\Big(\|\gamma\|_{C^1}|\xi_h|_{H^1} + \|\gamma\|_{C^2}\|s_h + \eta_h\|_{H^1}\|\xi_h\|_{C^0}\Big) \\
&\leq c\|\gamma\|_{C^2}(\|s_h + \eta_h\|_{H^{1/2}})|\log h|^{1/2}\|\xi_h\|_{H^{1/2}} \quad \text{by (19) and (20)} \\
&\leq c(1 + \|\eta_h\|_{H^{1/2}})|\log h|^{1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}.
\end{aligned}
$$

For $B_{21}$ we have the same estimate with $\eta_h$ set equal to 0:

$$
B_{21} \leq c|\log h|^{1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}.
$$

Finally,

$$
\begin{aligned}
B_1 &= |\Phi_h I_h(\gamma'' \circ s_h\, \xi_h^2)|_{H^1(D_h)} \\
&\leq |\gamma'' \circ s_h\, \xi_h^2|_{H^{1/2}} + ch^{1/2}|\gamma'' \circ s_h\, \xi_h^2|_{H^1} \quad \text{by (65)} \\
&\leq \|\gamma'' \circ s_h\|_{C^0}|\xi_h^2|_{H^{1/2}} + |\gamma'' \circ s_h|_{H^{1/2}}\|\xi_h^2\|_{C^0} \\
&\quad + ch^{1/2}\Big(\|\gamma'' \circ s_h\|_{C^0}|\xi_h^2|_{H^1} + |\gamma'' \circ s_h|_{H^1}\|\xi_h^2\|_{C^0}\Big) \\
&\leq c\|\gamma\|_{C^2}\|\xi_h\|_{C^0}\|\xi_h\|_{H^{1/2}} + c\|\gamma\|_{C^3}\|s_h\|_{H^{1/2}}\|\xi_h\|_{C^0}^2 \\
&\quad + ch^{1/2}\Big(\|\gamma\|_{C^2}\|\xi_h\|_{C^0}\|\xi_h\|_{H^1} + \|\gamma\|_{C^3}\|s_h\|_{H^1}\|\xi_h\|_{C^0}^2\Big) \\
&\leq c|\log h|\,\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2.
\end{aligned}
$$

Substituting the previous estimates in (100), we get

$$
\begin{aligned}
&\left|\Big(E_h''(s_h) - E_h''(s_h + \eta_h)\Big)(\xi_h, \xi_h)\right| \\
&\leq c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^2}\|s\|_{C^1} \\
&\qquad \times |\log h|\,\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2 \\
&\quad + c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)|\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^3}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}} \\
&\qquad \times \Big(1 + \|\eta_h\|_{H^{1/2}}\Big)|\log h|^{1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}} \\
&\quad + c\Big(1 + \|\eta_h\|_{H^{1/2}}\Big)\|\gamma\|_{C^1}\|s\|_{C^1} \\
&\qquad \times \Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)|\log h|\,\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^4}\|s\|_{C^1}\|\xi_h\|_{H^{1/2}}^2 \\
&\leq c\Big(1 + |\log h|^{1/2}\|\eta_h\|_{H^{1/2}}\Big)^2|\log h|\,\|\eta_h\|_{H^{1/2}}\|\gamma\|_{C^4}^2\|s\|_{C^1}^2\|\xi_h\|_{H^{1/2}}^2.
\end{aligned}
$$

$\square$

It follows from the next result that $E_h''(p_h s)$ is nondegenerate with nondegeneracy constant arbitrarily close to $\lambda$, provided $h$ is sufficiently small. This will be used to establish the appropriate version of (97) in the proof of Theorem 5.4.

*Remark on the Proof.* (See also the Remark preceding Proposition 4.3). Nondegeneracy for "interpolants" of functions from the positive and negative spaces for $E''(s)$ must be treated separately—see the separate treatment of $I_1$ and $I_2$ below. In the first case, nondegeneracy follows more or less directly from Proposition 4.3. In the second case, the term $\|\nabla(v - v_h)\|^2_{L^2(D_h)}$, where $v = \Phi\left(\gamma' \circ s\, \xi_h^{(-)}\right)$ and $v_h = \Phi_h I_h\left(\gamma' \circ s_h\, \xi_h^{(-)}\right)$, must be estimated. The main points are to apply the regularity results (34) and Proposition 2.2 to members of the negative space.

**Proposition 5.3.** *Let* $s \in \mathcal{H} \cap C^2$ *and* $\gamma \in C^3$. *Suppose* $E''(s)$ *is nondegenerate with nondegeneracy constant* $\lambda$, *as in* (13). *Let* $\nu$ *be as in* (12). *Then*

$$E''_h(p_h s)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) \geq \left(\lambda - ch^{1/2}|\log h|^{1/2}\|\gamma\|^2_{C^3}\nu^2\|s\|^4_{C^2}\right)\|\xi_h\|^2_{H^{1/2}}$$

*for every* $\xi_h \in H_h$.

*Proof.* We again denote $p_h s$ by $s_h$ and split

$$E''_h(s_h)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) = E''_h(s_h)(\xi_h^{(+)}, \xi_h^{(+)}) - E''_h(s_h)(\xi_h^{(-)}, \xi_h^{(-)})$$

$$(101) \qquad\qquad\qquad = I_1 + I_2.$$

It follows from (87), after discarding a positive term, that

$$\begin{aligned}
I_1 &\geq E''(s)(\xi_h^{(+)}, \xi_h^{(+)}) - ch^{1/2}|\log h|^{1/2}\|\gamma\|^2_{C^3}\|s\|^3_{C^2}\|\xi_h^{(+)}\|^2_{H^{1/2}} \\
&= E''(s)(\xi_h^+, \xi_h^+) + E''(s)\left((\xi_h^{(+)}, \xi_h^{(+)}) - (\xi_h^+, \xi_h^+)\right) \\
&\quad - ch^{1/2}|\log h|^{1/2}\|\gamma\|^2_{C^3}\|s\|^3_{C^2}\|\xi_h^{(+)}\|^2_{H^{1/2}}.
\end{aligned}$$

Using the notation of (13), there exists $\lambda^+ > 0$ such that

$$E''(s)(\xi_h^+, \xi_h^+) \geq \lambda^+ \|\xi_h^+\|^2_{H^{1/2}}.$$

Also

$$\begin{aligned}
&\left|E''(s)\left((\xi_h^{(+)}, \xi_h^{(+)}) - (\xi_h^+, \xi_h^+)\right)\right| \\
&= E''(s)\left(\xi_h^{(+)} - \xi_h^+, \xi_h^{(+)} + \xi_h^+\right) \\
&\leq c\|\gamma\|^2_{C^2}\|s\|^2_{C^1}\|\xi_h^{(+)} - \xi_h^+\|_{H^{1/2}}\|\xi_h^{(+)} + \xi_h^+\|_{H^{1/2}} \quad \text{by (10)} \\
&\leq ch\|\gamma\|^2_{C^2}\nu^2\|s\|^2_{C^1}\|\xi_h\|^2_{H^{1/2}} \quad \text{by (36)}.
\end{aligned}$$

Using (36) again, $I_1$ can then be estimated from below by

$$(102) \qquad I_1 \geq \lambda^+\|\xi_h^+\|^2_{H^{1/2}} - ch^{1/2}|\log h|^{1/2}\|\gamma\|^2_{C^3}\nu^2\|s\|^3_{C^2}\|\xi_h\|^2_{H^{1/2}}.$$

With similar arguments, but now keeping all terms in (87) and using the notation of (13), we get

$$\begin{aligned}
I_2 &\geq \lambda^-\|\xi_h^-\|^2_{H^{1/2}} - ch^{1/2}|\log h|^{1/2}\|\gamma\|^2_{C^3}\nu^2\|s\|^3_{C^2}\|\xi_h\|^2_{H^{1/2}} \\
(103) &\quad - \|\nabla(v - v_h)\|^2_{L^2(D_h)},
\end{aligned}$$

where, here,

$$(104) \qquad\qquad v = \Phi\left(\gamma' \circ s\, \xi_h^{(-)}\right), \quad v_h = \Phi_h I_h\left(\gamma' \circ s_h\, \xi_h^{(-)}\right).$$

From (101), (102), (103), and using (13),

$$E_h''(s_h)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) \geq \left(\lambda - ch^{1/2}|\log h|^{1/2}\|\gamma\|_{C^3}^2 \nu^2 \|s\|_{C^2}^3\right)\|\xi_h\|_{H^{1/2}}^2$$

(105)
$$-\|\nabla(v - v_h)\|_{L^2(D_h)}^2.$$

It remains to estimate the last term. For this it is important to notice that $\xi_h^-$ is a smooth function. We have

$$\|\nabla(v - v_h)\|_{L^2(D_h)}$$
$$= |\Phi(\gamma' \circ s\, \xi_h^{(-)}) - \Phi_h I_h(\gamma' \circ s_h\, \xi_h^{(-)})|_{H^1(D_h)}$$
$$\leq |\Phi(\gamma' \circ s\, (\xi_h^{(-)} - \xi_h^-))|_{H^1(D_h)} + |\Phi_h I_h(\gamma' \circ s_h\, (\xi_h^{(-)} - \xi_h^-))|_{H^1(D_h)}$$
$$+ |\Phi(\gamma' \circ s\, \xi_h^-) - \Phi_h I_h(\gamma' \circ s\, \xi_h^-)|_{H^1(D_h)}$$
$$+ |\Phi_h I_h((\gamma' \circ s - \gamma' \circ s_h)\, \xi_h^-)|_{H^1(D_h)}$$
$$= I_3 + I_4 + I_5 + I_6.$$

But

$$I_3 \leq |\gamma' \circ s\, (\xi_h^{(-)} - \xi_h^-)|_{H^{1/2}}$$
$$\leq \|\gamma' \circ s\|_{C^1}\|\xi_h^{(-)} - \xi_h^-\|_{H^{1/2}}$$
$$\leq ch\|\gamma\|_{C^2}\|s\|_{C^1}\nu\|\xi_h\|_{H^{1/2}} \quad \text{by (34)}.$$

And

$$I_4 \leq |\gamma' \circ s_h\, (\xi_h^- - \xi_h^{(-)})|_{H^{1/2}} + ch^{1/2}|\gamma' \circ s_h\, (\xi_h^- - \xi_h^{(-)})|_{H^1} \quad \text{by (65)}$$
$$\leq c\|\gamma' \circ s_h\|_{C^1}\left(\|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} + h^{1/2}\|\xi_h^- - \xi_h^{(-)}\|_{H^1}\right)$$
$$\leq ch\|\gamma\|_{C^2}\|s\|_{C^1}\nu\|\xi_h\|_{H^{1/2}} \quad \text{from (34)}.$$

Also

$$I_5 \leq ch^{1/2}|\gamma' \circ s\, \xi_h^-|_{H^1} \quad \text{by (64)}$$
$$\leq ch^{1/2}\|\gamma' \circ s\|_{C^1}|\xi_h^-|_{H^1}$$
$$\leq ch^{1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\nu\|\xi_h\|_{H^{1/2}} \quad \text{by (12)}.$$

Finally,

$$I_6 \leq c|(\gamma' \circ s - \gamma' \circ s_h)\, \xi_h^-|_{H^{1/2}} + ch^{1/2}|(\gamma' \circ s - \gamma' \circ s_h)\, \xi_h^-|_{H^1} \quad \text{by (65)}$$
$$\leq c\|\gamma' \circ s - \gamma' \circ s_h\|_{C^{0,1}}\left(\|\xi_h^-\|_{H^{1/2}} + h^{1/2}\|\xi_h^-\|_{H^1}\right) \quad \text{by (49) and (51)}$$
$$\leq c\|\gamma\|_{C^3}\|s\|_{C^1}\|s - s_h\|_{C^{0,1}}\nu\|\xi_h\|_{H^{1/2}} \quad \text{by (63) and (12)}$$
$$\leq ch\|\gamma\|_{C^3}\|s\|_{C^2}^2\nu\|\xi_h\|_{H^{1/2}} \quad \text{from (23)}.$$

If we now put together the estimates for $I_3$, $I_4$, $I_5$ and $I_6$, we get

(106)
$$\|\nabla(v - v_h)\|_{L^2(D_h)} \leq ch^{1/2}\|\gamma\|_{C^3}\nu\|s\|_{C^2}^2\|\xi_h\|_{H^{1/2}}.$$

This together with (105) proves the proposition. $\qquad\square$

The next result gives the main error estimates for discrete maps $s_h$ which are stationary for $E_h$. As remarked at the beginning of this section, the proof uses Lemma 5.1. The necessary estimates were established in Propositions 4.2, 5.2 and 5.3.

**Theorem 5.4.** *Assume $\gamma \in C^4$. Let $s$ be a monotone nondegenerate stationary point for $E$, with nondegeneracy constant $\lambda$ as in (13).*

*Then there exist positive constants $h_0$ and $c_0$ depending on $\|\gamma\|_{C^4}$ and $\| |\gamma'|^{-1} \|_{L^\infty}$, and $\lambda$ in the case of $h_0$, such that if $0 < h \leq h_0$ then there exists $s_h \in \mathcal{H}_h$ which is stationary for $E_h$ and satisfies*

$$\tag{107} \|s - s_h\|_{H^{1/2}} \leq c_0 \lambda^{-1} h.$$

*Moreover, there exists $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \| |\gamma'|^{-1} \|_{L^\infty}, \lambda) > 0$ such that $s_h$ is the unique stationary point for $E_h$ satisfying*

$$\tag{108} \|s - s_h\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}.$$

*Proof.* We will apply Lemma 5.1 with $\mathcal{X} = \mathcal{H}_h$, $X = H_h$, $Y = H_h'$ (the dual space of $H_h$), $f = E_h'$ and $x_0 = p_h s$.

Note that

$$E_h' : \mathcal{H}_h \to H_h'.$$

From Propositions 4.2 and 2.1, since $E'(s) = 0$,

$$\tag{109} \|E_h'(p_h s)\|_{H_h'} \leq c_1 h$$

where $c_1 = c_1(\|\gamma\|_{C^3}, \| |\gamma'|^{-1} \|_{L^\infty})$.

The derivative $(E_h')'$ of $E_h'$ is a map

$$(E_h')' : \mathcal{H}_h \to L(H_h, H_h')$$

and is naturally identified in the usual way with $E_h''$ via

$$\tag{110} \langle \langle (E_h')'(t_h), \xi_h \rangle, \eta_h \rangle = E_h''(t_h)(\xi_h, \eta_h)$$

for all $t_h \in \mathcal{H}_h$ and $\xi_h, \eta_h \in \mathcal{H}_h$. From Propositions 5.3, 2.1 and 2.2, since $h^{1/2} |\log h|^{1/2} \to 0$ as $h \to 0$,

$$\tag{111} E_h''(p_h s)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) \geq \frac{3\lambda}{4} \|\xi_h\|_{H^{1/2}}^2$$

for all $\xi_h \in H_h$, provided $0 < h \leq h_0 = h_0(\|\gamma\|_{C^4}, \| |\gamma'|^{-1} \|_{L^\infty}, \lambda)$. But

$$\tag{112}
\begin{aligned}
\|\xi_h^{(+)} - \xi_h^{(-)}\|_{H^{1/2}} &\leq \|\xi_h^+ - \xi_h^-\|_{H^{1/2}} + \|\xi_h^+ - \xi_h^{(+)}\|_{H^{1/2}} + \|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} \\
&\leq (1 + ch\nu) \|\xi_h\|_{H^{1/2}}
\end{aligned}$$

from (36) and (34), and since $\|\xi_h^+ - \xi_h^-\|_{H^{1/2}} = \|\xi_h\|_{H^{1/2}}$. Hence from (110), (111) and (112),

$$\tag{113} \langle (E_h')'(p_h s), \xi_h \rangle(\xi_h^{(+)} - \xi_h^{(-)}) \geq \frac{\lambda}{2} \|\xi_h\|_{H^{1/2}} \|\xi_h^{(+)} - \xi_h^{(-)}\|_{H^{1/2}}$$

for all $\xi \in H_h$, provided $0 < h \leq h_0$ for a new $h_0$ with the same dependencies as before. Thus for each $\xi_h \in H_h$ with $\|\xi_h\|_{H^{1/2}} = 1$, the map $\langle (E_h')'(p_h s), \xi_h \rangle$ $(\in H_h')$ has norm $\geq \lambda/2$. It follows, since $H_h$ and $H_h'$ have equal finite dimension, that $(E_h')'(p_h s)$ is invertible and

$$\tag{114} \left\| ((E_h')'(p_h s))^{-1} \right\|_{L(H_h', H_h)} \leq \left( \frac{\lambda}{2} \right)^{-1}.$$

Next note that from (110) and Proposition 5.2, by the symmetry of the second derivative,

$$\|(E_h')'(p_h s) - (E_h')'(p_h s + \eta_h)\|_{L(H_h', H_h)}$$
$$\leq \quad c_2 \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}}\right)^2 |\log h| \, \|\eta_h\|_{H^{1/2}}$$

for all $\eta_h \in H_h$, where $c_2 = c_2(\|\gamma\|_{C^4}, \| \, |\gamma'|^{-1} \, \|_{L^\infty})$ from Proposition 2.1.

We can now choose $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \| \, |\gamma'|^{-1} \, \|_{L^\infty}, \lambda)$ so that $|\log h| \, \|\eta_h\|_{H^{1/2}} < \epsilon_0$ implies

$$(115) \qquad \|(E_h')'(p_h s) - (E_h')'(p_h s + \eta_h)\|_{L(H_h', H_h)} \leq \frac{\lambda}{4}.$$

By further restricting $h_0$, again with the same dependencies, we can ensure that, see (109),

$$(116) \qquad 0 < h \leq h_0 \quad \text{implies} \quad c_1 h \leq \frac{\lambda}{4} \epsilon_0 \, |\log h|^{-1}.$$

From (109), (114), (115), (116) and Lemma 5.1 with $\alpha = \lambda/2$, $\beta = \lambda/4$, $\delta = c_1 h$ and $\epsilon = \epsilon_0 |\log h|^{-1}$, it follows that for $0 < h \leq h_0$ there is a unique $s_h \in \mathcal{H}_h$ which is stationary for $E_h$ and such that

$$(117) \qquad \|s_h - p_h s\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}.$$

Next apply Lemma 5.1 with $\alpha = \lambda/2$, $\beta = \lambda/4$, $\delta = c_1 h$ and $\epsilon = 4c_1 h/\lambda$. Note that $\delta = (\alpha - \beta)\epsilon \ (= c_1 h)$. Moreover, if $0 < h \leq h_0$ then $\epsilon \leq \epsilon_0 |\log h|^{-1}$ from (116). It follows that the unique stationary $s_h$ as in (117) satisfies

$$(118) \qquad \|s_h - p_h s\|_{H^{1/2}} \leq \frac{c_0}{\lambda} h,$$

where $c_0 = 4c_1$.

Since

$$\|s - p_h s\|_{H^{1/2}} \leq ch^{3/2} \|s\|_{H^2} \leq ch^{3/2} \|s\|_{C^2} \leq ch^{3/2} \|\gamma\|_{C^3},$$

we may replace $p_h s$ by $s$ in (117) and (118), after further restricting $h_0$, $\epsilon_0$ and $c_0$ if necessary. $\qquad \square$

*Remark.* Suppose $u = \Phi(\gamma \circ s) : D \to \Re^n$ is harmonic. Then

$$\begin{aligned} \|u\|_{H^1(D \setminus D_h)} &\leq & ch \|u\|_{H^1(\partial D)} &\quad \text{from (70) and (71)} \\ &\leq & ch \|\gamma\|_{C^1} \|s\|_{C^1} &\quad \text{from (57) and (59)} \\ &\leq & ch, \end{aligned}$$

where $c = c(\|\gamma\|_{C^2})$. Thus the contribution to $\|u\|_{H^1(D)}$ from the boundary strip $D \setminus D_h$ is $O(h)$. This is consistent with the order of approximation in the following theorem.

**Theorem 5.5.** *Assume $\gamma \in C^4$. Let $u$ be a nondegenerate minimal surface spanning $\Gamma$ with nondegeneracy constant $\lambda$ as in (13).*

*Then there exist positive constants $h_0$ and $c_0$ depending on $\|\gamma\|_{C^4}$ and $\| \, |\gamma'|^{-1} \, \|_{L^\infty}$, and $\lambda$ in the case of $h_0$, such that if $0 < h \leq h_0$, then there is a discrete minimal surface $u_h$ satisfying*

$$(119) \qquad \|u - u_h\|_{H^1(D_h)} \leq c_0 \lambda^{-1} h.$$

*Moreover, there exists $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \||\gamma'|^{-1}\|_{L^\infty}, \lambda) > 0$ such that if $u = \Phi(\gamma \circ s)$ and $u_h = \Phi_h I_h(\gamma \circ s_h)$, then $u_h$ is the unique discrete minimal surface satisfying*

$$(120) \qquad \|s - s_h\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}.$$

*Proof.* Let

$$u = \Phi(\gamma \circ s), \quad u_h = \Phi_h I_h(\gamma \circ s_h),$$

where $s$ and $s_h$ are as in Proposition 5.4. We begin with $h_0$, $\epsilon_0$ and $c_0$ as in Proposition 5.4.

Now

$$
\begin{aligned}
|u - u_h&|_{H^1(D_h)} \\
&= |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ s_h)|_{H^1(D_h)} \\
&\leq |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ p_h s)|_{H^1(D_h)} + |\Phi_h I_h(\gamma \circ p_h s - \gamma \circ s_h)|_{H^1(D_h)} \\
(121) \quad &= A + B.
\end{aligned}
$$

But

$$(122) \qquad A \leq ch\|\gamma\|_{C^2}\|s\|_{C^2}^2,$$

by the estimate (77) for $I_1$ in the proof of Proposition 4.1.

Also,

$$B \leq c\left(1 + |\log h|^{1/2}\|s_h - p_h s\|_{H^{1/2}}\right)\|s_h - p_h s\|_{H^{1/2}}\|\gamma\|_{C^2}\|s\|_{C^1}$$

by the estimate for $A_1$ in the proof of Proposition 5.2, with $s_h$ and $\eta_h$ there replaced by $p_h s$ and $s_h - p_h s$, respectively. Hence

$$(123) \qquad B \leq \frac{c_0}{\lambda}h \qquad \text{from (118)}$$

provided $0 < h \leq h_0$, for a new $h_0$ and $c_0$ with the same dependencies.

It follows from (121), (122) and (123), possibly again with new $c_0$ and $h_0$ with the same dependencies as before, that

$$(124) \qquad |u - u_h|_{H^1(D_h)} \leq \frac{c_0}{\lambda}h$$

if $0 < h \leq h_0$.

Now

$$(125) \qquad \|u - u_h\|_{H^1(D_h)} \leq c\left(|u - u_h|_{H^1(D_h)} + \|u - u_h\|_{L^2(\partial D_h)}\right),$$

as follows easily by integrating along rays ($c$ depends only on $n$). It is routine to estimate $\|u - u_h\|_{L^2(\partial D_h)}$. Let

$$\tilde{u}_h = \Phi_h I_h(\gamma \circ p_h s).$$

Then

$$
\begin{aligned}
\|u - u_h\|_{L^2(\partial D_h)} &\leq \|u - \tilde{u}_h\|_{L^2(\partial D_h)} + \|\tilde{u}_h - u_h\|_{L^2(\partial D_h)} \\
&\leq ch^2\|\gamma\|_{C^2}\|s\|_{C^2}^2 + \|\tilde{u}_h - u_h\|_{L^2(\partial D_h)} \\
(126) \quad &= ch^2\|\gamma\|_{C^2}\|s\|_{C^2}^2 + D
\end{aligned}
$$

from the argument used to estimate $I_2$ in the proof of Proposition 4.1, with $u_h$ there replaced by $\tilde{u}_h$. Moreover,

$$
\begin{aligned}
D &= \|I_h(\gamma \circ p_h s - \gamma \circ s_h)\|_{L^2(\partial D_h)} \\
&\leq \|I_h^{\partial D}(\gamma \circ p_h s - \gamma \circ s_h)\|_{L^2(\partial D)} \quad \text{by (21) as } |\nabla \pi| \leq 1 \\
&\leq \|\gamma \circ p_h s - \gamma \circ s_h\|_{L^2} + ch\|\gamma \circ p_h s - \gamma \circ s_h\|_{H^1} \\
&\leq c\|\gamma\|_{C^1}\|p_h s - s_h\|_{L^2} + ch\|\gamma\|_{C^2}\|s\|_{C^1}\|p_h s - s_h\|_{H^1} \\
&\qquad \text{by (60), (62) and (23)} \\
&\leq c\|\gamma\|_{C^2}\|s\|_{C^1}\|p_h s - s_h\|_{L^2} \quad \text{by a standard inverse estimate} \\
(127) \qquad &\leq \frac{c_0}{\lambda}h \quad \text{from (118)}
\end{aligned}
$$

with a new $c_0$ but with the same dependencies, for $0 < h \leq h_0$. From (124), (125), (126) and (127) it follows that if $0 < h \leq h_0$ then

$$\|u - u_h\|_{H^1(D_h)} \leq \frac{c_0}{\lambda}h,$$

where $c_0 = c_0(\|\gamma\|_{C^3}, \| |\gamma'|^{-1} \|_{L^\infty})$ and $h_0 = h_0(\|\gamma\|_{C^3}, \| |\gamma'|^{-1} \|_{L^\infty}, \lambda)$. $\qquad \square$

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