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# ON THE EQUATION $\sum_{p|N} \frac{1}{p} + \frac{1}{N} = 1$, PSEUDOPERFECT NUMBERS, AND PERFECTLY WEIGHTED GRAPHS

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Abstract. We present all solutions to the equation $\sum_{p|N} \frac{1}{p} + \frac{1}{N} = 1$ with at most eight primes, improve the bound on the nonsolvability of the Erdös-Moser equation $\sum_{j=1}^{m-1} j^n = m^n$, and discuss the computational search techniques used to generate examples of perfectly weighted graphs.

Recent study of the unit fraction equation

$$\sum_{i=1}^{k} \frac{1}{n_i} + \prod_{i=1}^{k} \frac{1}{n_i} = 1, \tag{1}$$

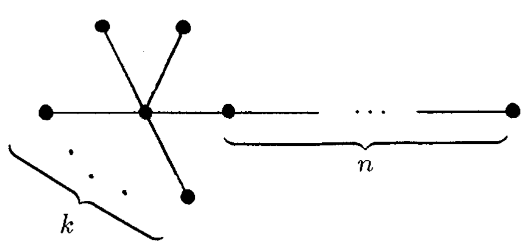
$n_1 < n_2 < \cdots < n_k$, has sparked renewed interest in the relation

$$\sum_{p|N} \frac{1}{p} + \frac{1}{N} = 1, \tag{2}$$

where the sum is taken over all distinct prime divisors of $N$. One purpose of this paper is to present all solutions of equation (2) with $k \leq 8$ primes. There is exactly one solution for each $k$ in this range, verifying conjectures of Ke and Sun [9], and Cao, Liu and Zhang [7]. In the second section, properties of solutions will be applied to the Erdös-Moser equation

$$\sum_{j=1}^{m-1} j^n = m^n. \tag{3}$$

We improve the bound on $m$ to $10^{9.3\times 10^6}$ for the conjecture that no nontrivial solution to (3) exists. Finally, we apply search techniques developed in connection with equations (1) and (2) to the topic of perfectly weighted graphs (see [4]). Specifically, for $n \geq 3$ we have found all perfectible graphs of the following form.

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## 1. Primary pseudoperfect numbers

Recall that a positive integer is called *perfect* if it is the sum of all its proper divisors, and *pseudoperfect* if it is the sum of *some* of its proper divisors ([8, p. 46]). A positive integer $N = \prod_{i=1}^{k} n_i$ with factors $n_i$ satisfying equation (1) is clearly pseudoperfect since

$$N = \sum_{i=1}^{k} \frac{N}{n_i} + 1.$$

All solutions $n_1, \ldots, n_k$ to equation (1) are known for $k \leq 7$ ([5], [3]). For $k = 8$, the list of known solutions continues to grow, with 89 solutions announced by Brenton and Bruner in 1994 ([2]). At present 112 solutions are known to the authors.

In the case where the divisors $n_i$ are precisely the distinct prime divisors of $N$, we obtain equation (2). Conversely, since equation (2) implies that $N$ is square-free, a solution to (2) is a special case of (1). We will call an integer $N = \prod_{i=1}^{k} p_i$ satisfying (2) a *primary pseudoperfect number*. Through search methods described in Section 4, all primary pseudoperfect numbers with $k \leq 8$ prime factors have been found.

**Theorem 1.** *Table 1 comprises the complete list of solutions to the equation*

$$\sum_{p|N} \frac{1}{p} + \frac{1}{N} = 1$$

*with eight or fewer primes.*

Table 1. Primary pseudoperfect numbers with $k \leq 8$ prime factors

| $k$ | $N$ | Factors |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 6 | 2,3 |
| 3 | 42 | 2,3,7 |
| 4 | 1806 | 2,3,7,43 |
| 5 | 47058 | 2,3,11,23,31 |
| 6 | 2214502422 | 2,3,11,23,31,47059 |
| 7 | 52495396602 | 2,3,11,17,101,149,3109 |
| 8 | 8490421583559688410706771261086 | 2,3,11,23,31,47059,2217342227,1729101023519 |

No solutions to equation (2) are known of length greater than 8. We do not know whether there are infinitely many solutions. As in the case of perfect numbers, no odd primary pseudoperfect number is known.

If we allow prime *powers* among the divisors, we have two additional solutions.

| $k$ | $N$ | Factors |
|---|---|---|
| 7 | 144508961850 | 2,3,11,25,29,1097,2753 |
| 8 | 20882840055109264384350 | 2,3,11,25,29,1097,2753,144508961851 |

These, together with the 8 solutions of Table 1, constitute the list of known solutions in primes $p_i$ to the equation

$$\sum_{i=1}^{k} \frac{1}{p_i^{\alpha_i}} + \frac{1}{\prod_{i=1}^{k} p_i^{\alpha_i}} = 1. \tag{4}$$

There is also independent interest in the companion equation

$$\sum_{p|N} \frac{1}{p} - \frac{1}{N} = 1. \tag{5}$$

Reference [1] discusses the history of this equation and presents the eleven solutions that were known as of 1996. Recently, two new solutions have been found:

$$\begin{aligned} &191066718142050798455575991633850 6\\ &\quad = 2 * 3 * 7 * 43 * 1831 * 138683 * 2861051 * 1456230512169437 \end{aligned}$$

by M. Hogan and C. Mangilin, and

42000179497077470620387115096706566324041957537516306092287644161425572 11-
58209843254519032347481 8
=2∗3∗11∗23∗31∗47059∗2217342227∗1729101023519
∗8491659218261819498490029296021∗58254480569119734123541298976556403

by R. Girgensohn (both unpublished).

## 2. The Erdös-Moser equation

More than four decades ago Paul Erdös conjectured that no solution exists to the equation

$$1^n + 2^n + \cdots + (m-1)^n = m^n$$

except the trivial solution $1^1 + 2^1 = 3^1$. Although the conjecture remains unproven (see [8, p. 153–154]), in 1953 Leo Moser [11] verified that no solution exists for $m < 10^{10^6}$. This bound has recently been used by Pieter Moree [10] to obtain similar results for the equation $\sum_{j=1}^{m-1} j^n = am^n$. Moser's proof proceeds by using elementary number theoretic considerations to show that if $(m, n)$ is a solution, then the following expressions involving the prime divisors of $m-1$ and of $2m \pm 1$ must be integers:

$$\sum_{p|(m-1)} \frac{1}{p} + \frac{1}{m-1} = t_1, \tag{$\alpha$}$$

$$\sum_{p|(2m-1)} \frac{1}{p} + \frac{2}{2m-1} = t_2, \tag{$\beta$}$$

$$\sum_{p|(2m+1)} \frac{1}{p} + \frac{4}{2m+1} = t_3. \tag{$\gamma$}$$

Furthermore, if $m$ is odd, then $m \equiv 3 \mod 8$ and

$$\sum_{p|\frac{(m+1)}{2}} \frac{1}{p} + \frac{1}{\frac{(m+1)}{2}} = t_4. \tag{$\delta$}$$

No solution to any of these is known for $t_i > 1$. For $t_1, t_4 = 1$, equations $(\alpha)$ and $(\delta)$ imply that $m-1$ and $\frac{m+1}{2}$ are a pair of primary pseudoperfect integers. No nontrivial solution is known to either $(\beta)$ or $(\gamma)$. All of this comprises strong support for the conjecture that no solution to (3) exists.

Considering first the case $m \equiv \pm 1 \mod 6$, Moser notes that except for the primes 2 and 3, no prime can divide any two of $m \pm 1$, $2m \pm 1$. Therefore, the prime divisors of the square-free integer $M = \frac{(4m^4-5m^2+1)}{12} = \frac{1}{12}(m-1)(m+1)(2m-1)(2m+1)$ satisfy

$$\begin{aligned}(6)\qquad &\sum_{p|M} \frac{1}{p} + \frac{1}{m-1} + \frac{2}{m+1} + \frac{2}{2m-1} + \frac{4}{2m+1} \\ &= t_1 + t_2 + t_3 + t_4 - \frac{1}{2} - \frac{1}{3} \geq 3\frac{1}{6}.\end{aligned}$$

In the remaining cases $m \equiv 3 \mod 6$ and $m$ even, similar analysis applied to $M' = \frac{1}{4}(m-1)(m+1)(2m-1)(2m+1)$ and to $M'' = (m-1)(2m-1)(2m+1)$, respectively, lead to similar inequalities, which are greatly more restrictive than (6), since in these cases the small primes 3, respectively 2, do not appear in the sum. The bound $m > 10^{10^6}$ then follows from estimates on the rate of growth of $\sum \frac{1}{p}$ taken over all primes.

In 1953 "computation" was the unwanted stepchild of "pure" mathematics, in part because adequate computational tools were lacking. Moser himself is (justly) proud of having achieved the startlingly immense bound $10^{10^6}$ by techniques of analytic number theory "without laborious computations" ([11, p. 84]).

Times change: now we can actually calculate these large numbers that previously could only be roughly estimated. All calculations reported in this paper were done on a network of 20 Sun Sparc stations over the course of about 10 months.

**Theorem 2.** *Let $(m, n)$ be a solution to the Erdös-Moser equation (3), with $n > 1$. Then $m > 1.485 \times 10^{9321155}$.*

*Proof.* As above, the critical case is $m \equiv \pm 1 \mod 6$. In this case put $M = \frac{4m^4-5m^2+1}{12}$. We claim that $M$ has at least 4990906 prime factors. For if not, then

$$\sum_{p|M} \frac{1}{p} \leq \sum_{i=1}^{4990905} \frac{1}{p_i},$$

where $p_i$ is the $i$th prime. But by direct computation

$$\sum_{i=1}^{4990905} \frac{1}{p_i} = 3.1666666588101728584 \cdots < 3\frac{1}{6} - 10^{-9}.$$

Since, by Moser's bound $m > 10^{10^6}$, this is less than $3\frac{1}{6} - \frac{1}{m-1} - \frac{2}{m+1} - \frac{2}{2m-1} - \frac{4}{2m+1}$, contradicting (6). Thus $M \geq \prod_{i=1}^{4990906} p_i$. Again, direct computation gives

$$\sum_{i=1}^{4990906} \log p_i = 8.5851010694053365252 \cdots * 10^7.$$

Solving the resulting inequality

$$\frac{m^4}{3} > M > e^{8.5851010694053365252 * 10^7}$$

gives the required bound $m > 1.485 * 10^{9321155}$. The cases $m \equiv 0 \mod 3$ and $m$ even are similar. $\square$

*Remark.* While this bound appears to be the best available by the method pioneered by Moser, the authors hope that new insights will eventually make it possible to reach the more natural benchmark $10^{10^7}$.

## 3. Perfectly weighted graphs

The concept of a perfectly weighted graph was introduced by Brenton and Drucker in [4] in connection with the problem of classifying isolated singular points of algebraic surfaces by properties of the local fundamental group.

**Definitions.** Let $G$ be a tree (a connected graph with no circuits) on $n$ vertices $v_1, \ldots, v_n$, with an integer weight $w_i > 1$ assigned to each vertex $v_i$. Then the weighted graph $G = G(w_1, \ldots, w_n)$ is called *perfectly weighted* if the corresponding matrix

$$M_G = \begin{bmatrix} w_1 & 0 & \ldots & 0 \\ 0 & w_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & w_n \end{bmatrix} - [\text{the adjacency matrix of } G]$$

is positive definite with determinant 1. An unweighted tree $G$ is *perfectible* if there exist weights $w_i$ for its vertices such that the resulting weighted graph $G(w_1, \ldots, w_n)$ is perfectly weighted.

An isolated singular point $x$ of an $m$-dimensional complex analytic variety $X$ is called *homologically trivial* if $x$ admits a neighborhood $U$ in $X$ which is homeomorphic to the cone on a homology $(2m-1)$-sphere. That is, $H_i(\partial U, Z) = 0$ for $0 < i < 2m - 1$.

The main theorem from [4] gives the following relation between perfectly weighted graphs and homologically trivial singularities in complex dimension 2.

**Lemma 3.1.** *Let $X$ be a complex surface with a singularity at the point $x \in X$ and with no other singular points. Let $\widetilde{X}$ be the minimal nonsingular model of $X$ and let $\rho : \widetilde{X} \to X$ be the minimal resolution of singularities. Denote by $C$ the exceptional curve $\rho^{-1}(x)$, and write $C = \bigcup_{i=1}^{n} C_i$ with each $C_i$ irreducible. Suppose that the resolution $\rho$ is normal and that each component $C_i$ is rational. Then $x \in X$ is homologically trivial if and only if the dual intersection graph of $\rho$ is a perfectly weighted tree.*

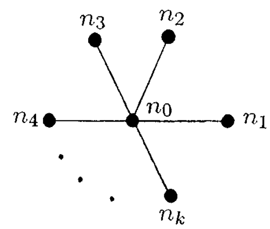
Here the *dual intersection graph* (call it $G_\rho$) is the graph on vertices $v_1, \ldots, v_n$, where $v_i$ meets $v_j$ if and only if $C_i$ meets $C_j$ in $\widetilde{X}$ and with weight $w_i$ on $v_i$ equal to the negative of the Chern class of the normal bundle of the embedding of $C_i$ in $\widetilde{X}$. The essential element of the proof is the following presentation, due to Mumford [12], of the local fundamental group. Under the hypotheses of Lemma 3.1, if $G_\rho$ is a tree, then the first homotopy group of a tubular neighborhood $T = \rho^{-1}(\partial U)$ of the exceptional curve $C$ in $\widetilde{X}$ is given by generators $x_1, \ldots, x_n$ with relations

$$\prod_{j=1}^{n} x_j^{C_i \cdot C_j} = 1$$

for all $i$, and $x_i x_j = x_j x_i$ if $C_i$ meets $C_j$, where $C_i \cdot C_j$ is the intersection number (the negative of the $i,j$th entry of $G_\rho$). Since the intersection matrix $[C_i \cdot C_j]$ is always negative definite in the complex case, the corresponding first homology group

$H_1(T, Z) = \pi_1(T)/(xyx^{-1}y^{-1} = 1)$ is a finite group of order $D = (-1)^n \det[C_i \cdot C_j] = \det[M_{G_\rho}]$. Thus, if $x \in X$ is homologically trivial, then $\pi_1(T)$ is a *perfect group* (generated by commutators), and the converse follows from Poincaré duality.

In the special case in which $G = G_\rho$ is the weighted star




direct computation shows that

$$D = \det[M_G] = \prod_{i=0}^{k} n_i - \sum_{i=1}^{k} \prod_{j \neq i} n_j.$$

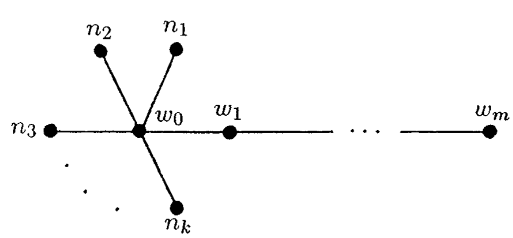
This is equivalent to

$$\sum_{i=1}^{k} \frac{1}{n_i} + \frac{D}{\prod_{i=1}^{k} n_i} = n_0,$$

which exhibits the connection between this topic and our equations (1) and (2). Explicitly (allowing $n_0 = 1$), the group generated by $x_0, \ldots, x_n$ with relations $x_0 = \prod_{i=1}^{k} x_i = x_i^{n_i}$ for all $i$ is perfect if and only if the integer $N = \prod_{i=1}^{k} n_i$ is pseudoperfect with factors $n_i$ satisfying equation (1). We find it interesting that although the terms "perfect number" and "perfect group" were coined independently, the results of this paper reveal a relation between these apparently disparate topics.

From the point of view of number theory, perhaps the most interesting graphs are the so-called "weighted flowers", which are weighted graphs of the form

$$J_{k,m} = J_{k,m}(n_1, \ldots, n_k; w_0, \ldots, w_m)$$




**Lemma 3.2.** *For the weighted graph $G = J_{k,m}(n_1, \ldots, n_k;\ w_0, \ldots, w_m)$ pictured above, we have*

$$\sum_{i=1}^{k} \frac{1}{n_i} + \frac{D}{Q \prod_{i=1}^{k} n_i} = \frac{P}{Q}, \tag{7}$$

*where $\frac{P}{Q}$ has the continued fraction expansion*

$$[w_0, w_1, \ldots, w_m] = w_0 - \cfrac{1}{w_1 - \cfrac{1}{w_2 - \ddots - \cfrac{1}{w_m}}}$$

*and where $D = \det[M_G]$.*

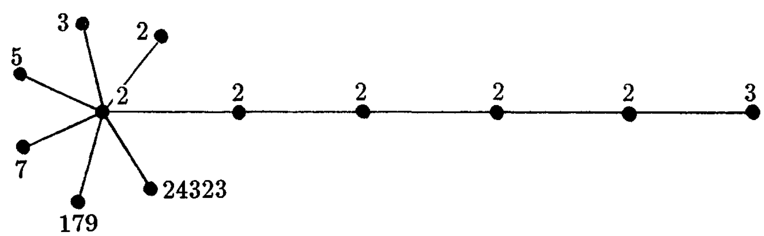
A simple proof follows by induction on $m$ (cf. [4, Lemma 4.3]).

In view of Lemma 3.2, we can use computational techniques similar to those employed in finding solutions of equation (1) to find perfect weights for graphs of this type. A perfectible graph is called *minimal* if it contains no proper perfectible subgraphs. Table 2 presents the complete list of minimal perfectible flowers $J_{k,m}$ with $m > 2$. Since a graph containing a perfectible subgraph is itself perfectible, we have the following result.

**Theorem 3.** *Let $G = J_{k,m}$ be a flower with $m > 2$. Then $G$ is perfectible if and only if $G$ contains one of the five graphs in Table 2.*

Verifying that each of these weighted graphs is perfectly weighted is a direct application of Lemma 3.2. The proof of Theorem 3 consists of verifying that each graph is minimal and that the list is complete. This was accomplished by exhaustive searches, as discussed in Section 4.

The graphs in Table 2 are pictured with one set of perfect weights. The perfect weights may not be unique. For instance,




is another set of perfect weights on $J_{6,5}$.

The graphs $J_{3,28}$, $J_{4,6}$, and $J_{6,5}$ were discusssed in [4]. $J_{10,3}$ was also introduced in [4], but at that time it was not known whether it was minimal or not. $J_{7,4}$ was derived only in 1995 after an earlier discovery by K. Conway (unpublished) of a set of perfect weights for $J_{8,4}$ shown below.

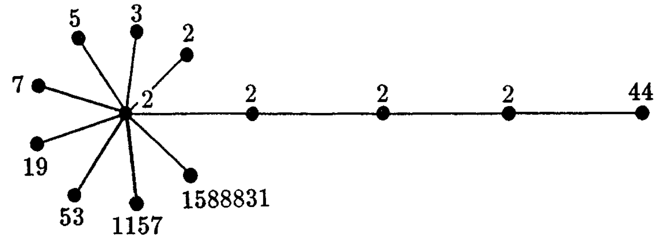
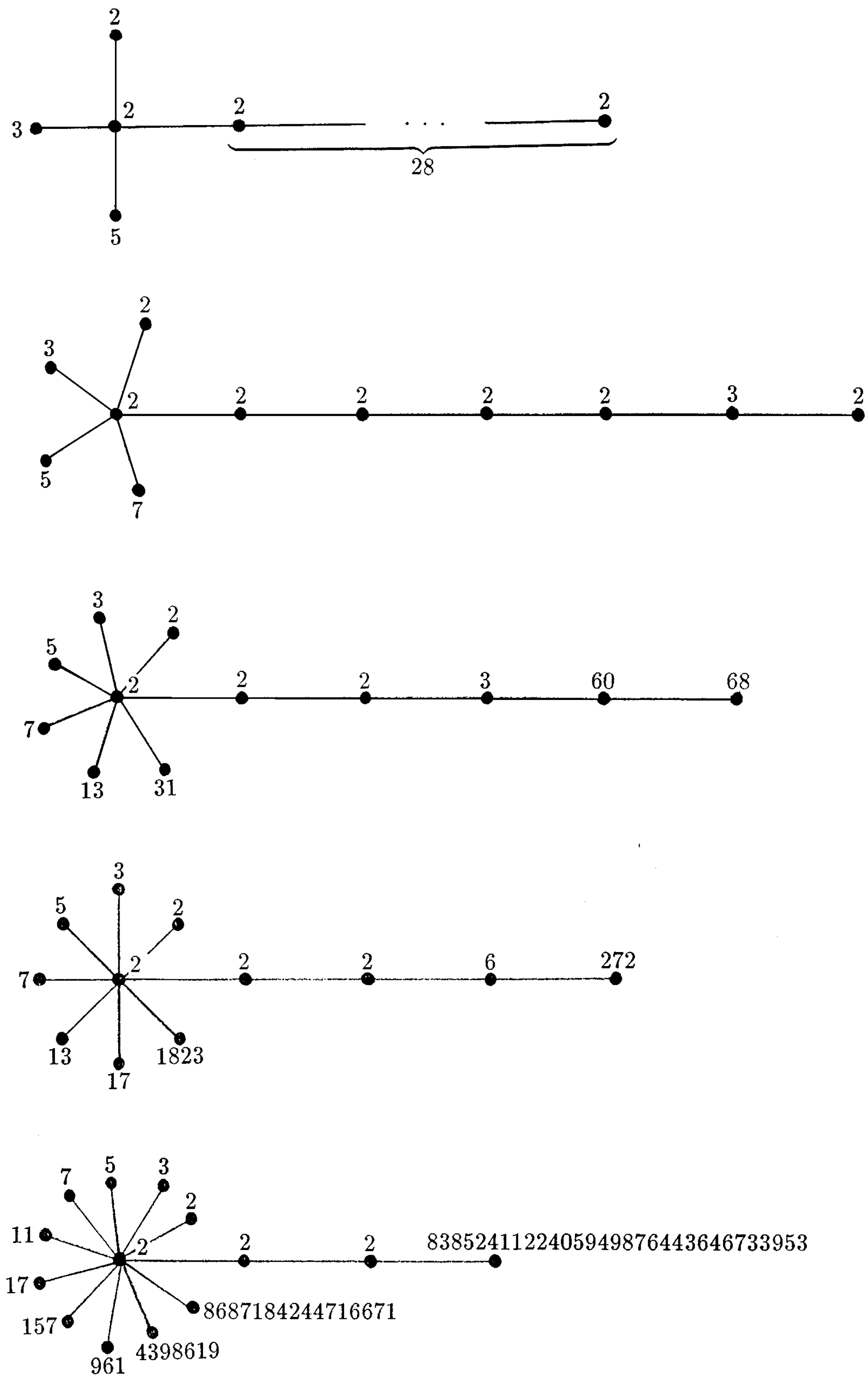


TABLE 2. Minimal perfectible flowers



## 4. Search techniques

The computational results reported thus far stem from finding solutions to the equation

$$\sum_{i=1}^{k} \frac{1}{n_i} + \frac{1}{Q\prod_{i=1}^{k} n_i} = \frac{P}{Q}. \tag{8}$$

Our main computational tool is Lemma 4.1, which gives a criterion for extending a solution of equation (7) to a solution of equation (8) by the adjunction of two more terms. Lemmas 4.1 and 4.2 generalize results of [5, Proposition 12 and Lemma 17].

**Lemma 4.1.** *Given a positive integer $P$ and relatively prime positive integers $n_1, \ldots, n_k, Q$, write*

$$\sum_{i=1}^{k} \frac{1}{n_i} + \frac{D}{Q\prod_{i=1}^{k} n_i} = \frac{P}{Q},$$

*where $D = (\frac{P}{Q} - \sum_{i=1}^{k} \frac{1}{n_i})Q\prod_{i=1}^{k} n_i$. Let $F$ be a factor of $Y = Q^2\prod_{i=1}^{k} n_i^2 + D$ and write $Y = FG$. Suppose that $F$ (and hence also $G$) is congruent to $-Q\prod_{i=1}^{k} n_i$* mod $D$ *and put*

$$n_{k+1} = \frac{Q\prod_{i=1}^{k} n_i + F}{D} \quad \text{and} \quad n_{k+2} = \frac{Q\prod_{i=1}^{k} n_i + G}{D}.$$

*Then the integers $n_1, \ldots, n_{k+2}$ satisfy the equation*

$$\sum_{i=1}^{k+2} \frac{1}{n_i} + \frac{1}{Q\prod_{i=1}^{k+2} n_i} = \frac{P}{Q}.$$

*Proof.*

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{1}{n_i} + \frac{1}{n_{k+1}} + \frac{1}{n_{k+2}} + \frac{1}{Q\prod_{i=1}^{k} n_i(n_{k+1})(n_{k+2})} \\
&\quad = \frac{P}{Q} - \frac{D}{Q\prod_{i=1}^{k} n_i} + \frac{D}{Q\prod_{i=1}^{k} n_i + F} + \frac{D}{Q\prod_{i=1}^{k} n_i + G} \\
&\qquad + \frac{D^2}{Q\prod_{i=1}^{k} n_i(Q\prod_{i=1}^{k} n_i + F)(Q\prod_{i=1}^{k} n_i + G)} \\
&\quad = \frac{P}{Q} + \frac{D(Y - FG)}{Q\prod_{i=1}^{k} n_i(Q\prod_{i=1}^{k} n_i + F)(Q\prod_{i=1}^{k} n_i + G)} = \frac{P}{Q}. \qquad \square
\end{aligned}$$

In addition, given a partial solution $n_1, n_2, \ldots, n_i$, we know the bounds on a search for $n_{i+1}$.

**Lemma 4.2.** *Let $n_1 < n_2 < \cdots < n_k, k > 2$, satisfy equation* (8). *Then for each index $i \le k-2$, we have*

$$\left(\frac{P}{Q} - \sum_{j=1}^{i} \frac{1}{n_j}\right)^{-1} < n_{i+1} < (k-i)\left(\frac{P}{Q} - \sum_{j=1}^{i} \frac{1}{n_j}\right)^{-1}.$$

*Proof.*

$$\frac{1}{n_{i+1}} = \frac{P}{Q} - \sum_{j \neq i+1} \frac{1}{n_j} - \frac{1}{Q \prod_{j=1}^{k} n_j} < \frac{P}{Q} - \sum_{j=1}^{i} \frac{1}{n_j},$$

so $n_{i+1} > (\frac{P}{Q} - \sum_{j=1}^{i} \frac{1}{n_j})^{-1}$ as required.

On the other hand, since $n_1 < n_2 < \cdots < n_k$, we have

$$\begin{aligned}(k-i)\frac{1}{n_{i+1}} &\geq \frac{1}{n_{i+1}} + \frac{1}{n_{i+2}} + \left(\frac{1}{n_{i+1}} - \frac{1}{n_{i+2}}\right) + \sum_{j=i+3}^{k} \frac{1}{n_j} \\ &= \sum_{j=i+1}^{k} \frac{1}{n_j} + \frac{n_{i+2} - n_{i+1}}{n_{i+1} n_{i+2}} \\ &> \sum_{j=i+1}^{k} \frac{1}{n_j} + \frac{1}{Q \prod_{j=1}^{k} n_j} = \frac{P}{Q} - \sum_{j=1}^{i} \frac{1}{n_j},\end{aligned}$$

and thus $n_{i+1} < (k-i)(\frac{P}{Q} - \sum_{j=1}^{i} \frac{1}{n_j})^{-1}$ as claimed. □

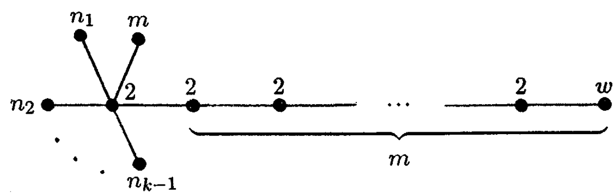
To implement these ideas in a search program for fixed $\frac{P}{Q}$ and $k$, we use Lemma 4.2 to determine all possibilities for $n_1, \ldots, n_{k-2}$, then we determine $n_{k-1}$ and $n_k$ by the technique of Lemma 4.1. The advantage of this method over simply searching for all possiblities for $n_{k-1}$ (as many as $10^{13}$ choices for $k = 8$ and $\frac{P}{Q} = 1$) is that we can reduce computation time by making use of advanced factoring techniques. In contrast, a complete tree search for $n_{k-1}$, using the bounds of Lemma 4.2, is equivalent to factoring the large integer $Y$ of Lemma 4.1 by trial division up to the square root. Computation time could be further reduced by incorporating the required congruence relations $F, G \equiv -Q \prod_{i=1}^{k} n_i \bmod D$ into the factoring methods and by taking advantage of the special form $Y = (Q \prod n_i)^2 + D$ for a known small number $D$. This program has proven to be the most useful tool for finding solutions to equation (8) and its special cases, equations (1) and (2). It yields both *nonsporadic solutions* (solutions of length $k$ resulting from extending known solutions of length $k-1$ or $k-2$) and *sporadic solutions* (solutions not generated from such solutions of smaller length).

These searches have produced the following results. With respect to equation (1), 68 nonsporadic and 44 sporadic solutions have been discovered for $k = 8$. The 68 nonsporadic solutions are easy to find and were discussed in [3]. The 44 sporadic solutions include all except those in the string 2,3,7,43. The search is also complete with respect to solutions in *prime* integers $n_i$, giving a proof of the completeness of the list of primary pseudoperfect numbers in Table 1.

Similar computational searches give results about particular perfectible graphs. $J_{8,5}$, for instance, admits at least 21 sets of perfect weights. Sixteen of these sets result from extending perfect weights on the minimal perfectible graph $J_{6,5}$, four of them result from extending $J_{7,4}$, and one from extending $J_{8,4}$. These are the nonsporadic solutions, and there are possibly sporadic solutions for perfect weights on $J_{8,5}$ which have not yet been explored.

A special case of Lemma 3.2 reveals further interesting properties of the graphs $J_{k,m}$ and a tighter relation between the topics of perfectible graphs and pseudoperfect numbers.

**Proposition 4.1.** *The weighted graph*




*is perfectly weighted if and only if* $n_1, \ldots, n_k$ *satisfy equation* (1), *where* $n_k = m^2(w-1)+m$.

*Proof.* Direct computation verifies that the continued fraction

$$[2,2,\ldots,2,w] = \frac{(m+1)w-m}{mw-(m-1)} = 1 + \frac{1}{m} - \frac{1}{n_k}$$

for $n_k$ as above. Thus, by Lemma 3.2 we have

$$\sum_{i=1}^{k-1} \frac{1}{n_i} + \frac{1}{m} + \frac{D}{(mw-(m-1))m\prod_{i=1}^{k-1} n_i} = 1 + \frac{1}{m} - \frac{1}{n_k},$$

or

$$\sum_{i=1}^{k} \frac{1}{n_i} + \frac{D}{\prod_{i=1}^{k} n_i} = 1,$$

where $D$ is the determinant of the weighted graph. Hence $D = 1$ if and only if $n_1, \ldots, n_k$ satisfy equation (1). $\square$

To apply this result we need only find solutions $n_1, \ldots, n_k$ to equation (1) in which one of the $n_i$'s happens to be congruent to $m \bmod m^2$ for some $m$ (but $n_i \neq m$ to ensure that $w = 1 + (n_i - m)/m^2$ is greater than 1). For $k \leq 8$ we found 24 distinct solution sets $n_1, \ldots, n_k$ which contain an $n_i$ with this special property for some integer $m$. Three of these sets have two different $n_i$'s with this property, and two have an $n_i$ which satisfies this congruence for two different $m$'s. This gives a total of 29 examples of perfectly weighted graphs of the type $J_{k,m}$ with $k \leq 8$ and with weights as pictured in Proposition 4.1. They are presented in Table 3.

For $m < 5$ there are no solutions of this type for $k < 10$. The most challenging case in this range was $k = 9$ and $m = 3$. In this instance we found that there are only 5 solutions to equation (1) with no $n_i = 3$:

$$2, 5, 7, 9, 31, 73, 13327, 63582361, 110273083859;$$
$$2, 5, 7, 9, 37, 61, 383, 3226871, 2344136699;$$
$$2, 5, 7, 11, 17, 149, 1431, 64911433, 1169526576259;$$
$$2, 5, 7, 11, 17, 157, 961, 4398619, 8687184244716671;$$
$$2, 5, 7, 11, 17, 167, 1257, 1919, 9373.$$

This leads immediately to the result that $J_{9,3}$ is not perfectible. First, it is easy to reduce the general case of perfect weights for $J_{9,3}$ to those pictured in Proposition 4.1. Then we check that none of the $n_i$'s appearing in the five solutions above is congruent to 3 mod 9. In a similar manner, other graphs of type $J_{k,m}$ can be shown not to be perfectible, resulting in a proof of Theorem 3.

TABLE 3. Special perfect weights on $J_{k,m}, k \leq 8$

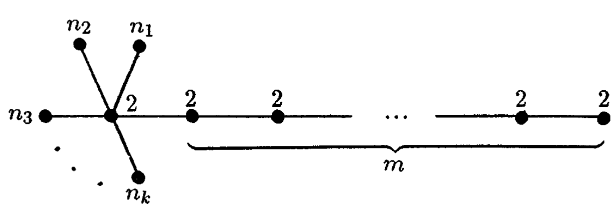
| $n_1, n_2, \ldots, n_{k-1}$ | $m$ | $w$ |
|---|---|---|
| 2,3,7,179,24323 | 5 | 3 |
| 2,3,7,55,179,24323 | 67 | 2240437 |
| 2,3,7,179,24323,10057317271 | 5 | 3 |
| 2,3,11,23,31,211031 | 71 | 13 |
| 2,3,11,23,31,12017087 | 7 | 965 |
| 2,3,7,43,1807,3263443,1348117392613837537 19 | 5 | 426002311687 |
| 2,3,7,43,1807,3263479,243811701792623 | 5 | 11527311163 |
| 2,3,7,43,1823,193667,637617223459 | 5 | 1250940688133154818523 |
| 2,3,7,43,1823,193667,637617223459 | 31 | 32542681793266254385 |
| 2,3,7,43,1831,132347,231679879 | 17 | 4142701692187 |
| 2,3,7,47,395,277442411,1701723083 | 361 | 7 |
| 2,3,7,47,403,19403,15435516179 | 5 | 3387914913502507 |
| 2,3,7.55,179,24323,10114963067949757017 1 | 67 | 2240437 |
| 2,3,7,55,179,24323,513449911932648503 | 37 | 7346471 |
| 2,3,7,179,24323,10057317271,101149630679497570171 | 5 | 3 |
| 2,3,7,179,24323,10057317287,5949978284730273323 | 5 | 3 |
| 2,3,7,179,24323,10057317311,2467064172726591731 | 5 | 3 |
| 2,3,7,179,24323,10057317467,513449911932648503 | 5 | 3 |
| 2,3,7,179,24323,10057317967,145121431390804003 | 5 | 3 |
| 2,3,7,179,24323,10057320619,30202945461748519 | 5 | 3 |
| 2,3,7,179,24323,10057325347,12523178395739983 | 5 | 3 |
| 2,3,7,179,24323,10057454579,736667018400959 | 5 | 3 |
| 2,3,11,17,79,1049,3696653 | 7 | 7 |
| 2,3,11,23,31,47059,3375982667 | 5 | 257468755 |
| 2,3,11,23,31,47059,165128325167 | 5 | 89784175 |
| 2,3,11,23,31,47059,165128325167 | 7 | 45808253 |
| 2,3,11,23,31,47147,11061526082145911 | 17 | 86259 |
| 2,3,11,23,31,211031,601432790177275 | 71 | 13 |
| 2,3,11,23,31,12017087,26715920281613179 | 7 | 965 |

Although $J_{9,3}$ is not perfectible, each of the five solutions to equation (1) with no $n_i = 3$ results in a perfectible flower of type $J_{9,6}$. They are presented in Table 4.

Solutions to equation (5), or more generally to

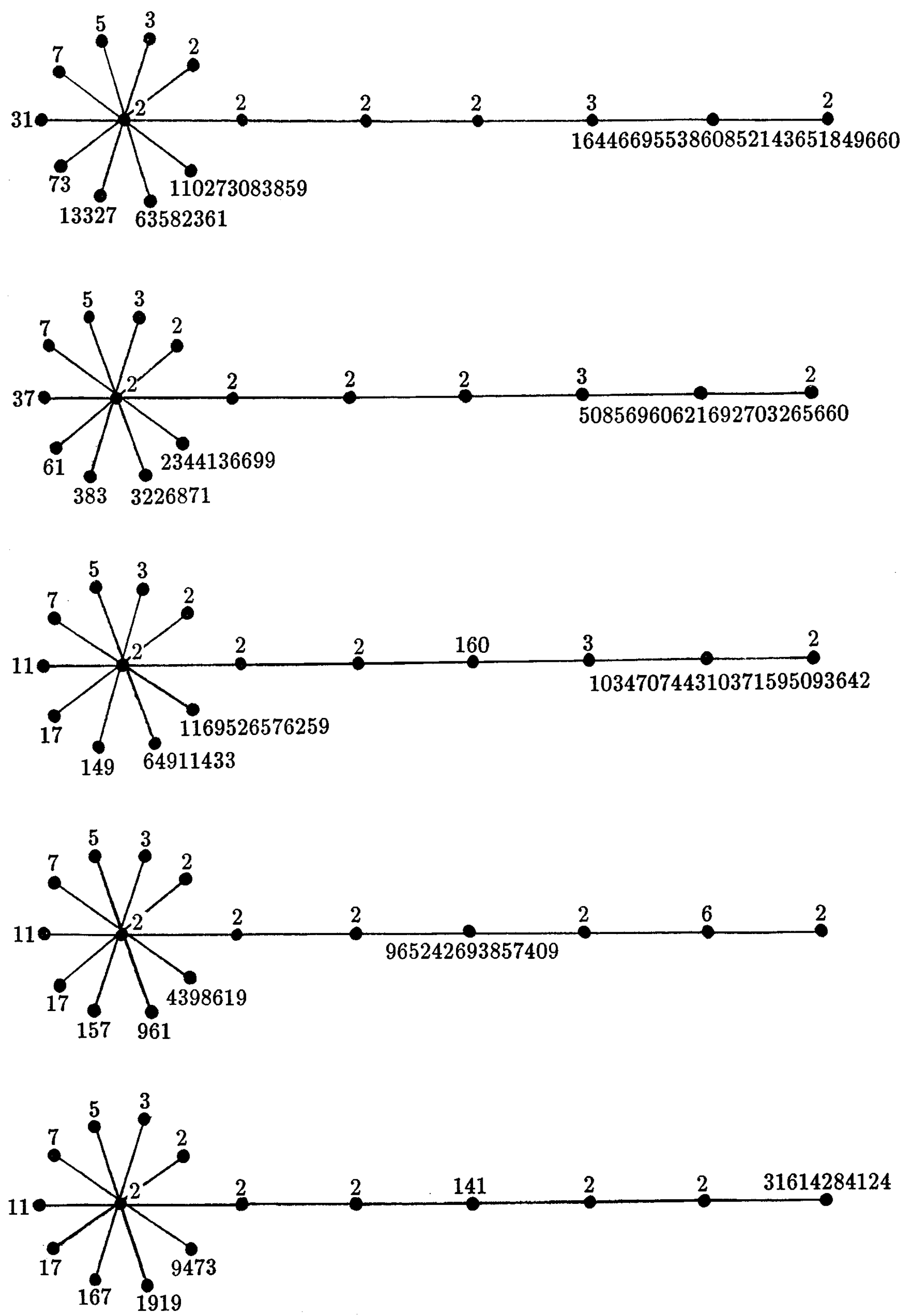
$$\sum_{i=1}^{k} \frac{1}{n_i} - \prod_{i=1}^{k} \frac{1}{n_i} = 1 \tag{9}$$

also lead to perfect weights for graphs of type $J_{k,m}$. Namely, if $n_1, \ldots, n_k$ satisfy equation (9), then the graph




is perfectly weighted for $m = \prod_{i=1}^{k} n_i - 2$. Again, Lemma 3.2 provides the proof. All solutions to equation (9) are known for $k \leq 7$ (there are 50 of them), and more than 400 are known for $k = 8$ ([6], [1]).

TABLE 4. Examples of perfect weights on $J_{9,6}$



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