ERROR ESTIMATES IN $L^2$, $H^1$ AND $L^\infty$ IN COVOLUME METHODS FOR ELLIPTIC AND PARABOLIC PROBLEMS: A UNIFIED APPROACH

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Abstract. In this paper we consider covolume or finite volume element methods for variable coefficient elliptic and parabolic problems on convex smooth domains in the plane. We introduce a general approach for connecting these methods with finite element method analysis. This unified approach is used to prove known convergence results in the $H^1$, $L^2$ norms and new results in the max-norm. For the elliptic problems we demonstrate that the error $u - u_h$ between the exact solution $u$ and the approximate solution $u_h$ in the maximum norm is $O(h^2 |\ln h|)$ in the linear element case. Furthermore, the maximum norm error in the gradient is shown to be of first order. Similar results hold for the parabolic problems.

1. Introduction

Let $\Omega$ be a convex domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$ and consider the general self-adjoint second order elliptic problem

\begin{align}
Lu & := -\sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) + qu = f, \quad x \in \Omega, \\
u & = 0, \quad x \in \partial \Omega,
\end{align}

where $q \in L^\infty$ is nonnegative, $f \in L^2(\Omega)$, and the matrix of coefficients $A := (a_{ij})$, $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$ is uniformly elliptic; i.e., there exists a positive constant $r > 0$ such that

\begin{equation}
\sum_{i,j=1}^2 a_{ij}(x)\xi_i \xi_j \geq r(\xi_1^2 + \xi_2^2) \quad \forall \xi := (\xi_1, \xi_2) \in \mathbb{R}^2 \quad a.e. \text{ in } \Omega.
\end{equation}

The natural variational problem associated with (1.1)-(1.2) is to find $u \in U := H^1_0(\Omega)$ such that

\begin{equation}
a(u, v) = (f, v) \quad \forall v \in U,
\end{equation}

where $a(\cdot, \cdot)$ is defined by

\begin{equation}
a(u, v) = \int_\Omega \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega quv \, dx.
\end{equation}
where

\begin{align}
  a(u, v) &:= \int_{\Omega} \left( \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + quv \right) dx, \\
  (f, v) &= \int_{\Omega} f v dx.
\end{align}

Since the error estimates to be derived below require that the exact solution \( u \) be in \( H^2(\Omega) \) for the \( H^1 \) norm case and be in \( H^3(\Omega) \) for the max-norm and \( L^2 \) norm cases, it is necessary to have the smooth boundary assumption on the problem domain. If instead we were to consider a polygonal problem domain, all interior angles of the domain would have to be no greater than \( \frac{\pi}{2} \) even if \( f \in C^\infty \), rendering the \( L^2 \) and max-norm estimates so obtained too limited to be useful.

Referring to Figure 1, let \( T_h = \bigcup K_Q \) be a triangulation of the polygonal domain \( \Omega_h \subset \Omega \) into a union of triangular elements, where \( K_Q \) stands for the triangle whose barycenter is \( Q \). Here \( h := \max h_K \) is the maximum of the diameters \( h_K \) over all triangles. The nodes of a triangular element are its vertices. We further require that the vertices which lie on \( \partial \Omega_h \) also lie on \( \partial \Omega \), so that there exists a constant \( C \) independent of \( h \) satisfying

\begin{equation}
  \text{dist}(x, \partial \Omega) \leq Ch^2 \quad \forall x \in \Omega \setminus \Omega_h.
\end{equation}

Associated with the primal partition \( T_h \), we define its dual partition \( T_h^* \) of \( \Omega_h \) as follows. Let \( P_0 \) be an interior node and \( P_i, i = 1, \ldots, 6 \) be its adjacent nodes, and \( M_i := M_{0i} \) be the midpoint of \( P_0 P_i \). Connect successively the points \( M_1, Q_1, M_2, Q_2, \ldots, M_6, Q_6, M_1 \) to obtain the dual polygonal element \( K_{P_0}^* \). Its nodes are defined to be \( Q_i, i = 1, \ldots, 6 \). The dual element \( K_{P_2}^* \) based at a typical boundary node \( P_2 \) is \( M_{12} Q_1 M_2 Q_2 M_{23} P_2 \). Let \( \Omega_h \) denote the set of all nodes of \( T_h \); \( \Omega_h^* := \Omega_h - \partial \Omega \) the set of all interior nodes in \( T_h \), and \( S_Q \) and \( S_{P_0}^* \) denote the areas of triangle \( K_Q \) and

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Primal and dual partitions of a convex domain}
\end{figure}
polygon \(K^*_h\), respectively. Throughout this paper we shall assume the partitions to be quasi-uniform. There exist two positive constants \(C_1, C_2\) independent of \(h\) such that
\begin{align}
(1.8) \quad C_1 h^2 & \leq S_Q \leq C_2 h^2, \quad Q \in \Omega^*_h, \\
(1.9) \quad C_1 h^2 & \leq S_{P_0} \leq C_2 h^2, \quad P_0 \in \bar{\Omega}_h.
\end{align}

Corresponding to \(T_h\), we define the trial function space \(U_h \subset H^1_0(\Omega)\) as the space of continuous functions on the closure of \(\Omega\) which vanish outside \(\Omega_h\) and are linear on each triangle \(K_Q \in T_h\). Let \(\Pi_h : U \rightarrow U_h\) be the usual linear interpolator, and thus if \(u \in W^{2,p}(\Omega)\),
\begin{equation}
|u - \Pi_h u|_{m,p} \leq Ch^{2-m} |u|_{2,p}, \quad m = 0, 1, \quad 1 \leq p \leq \infty,
\end{equation}
where \(|\cdot|_{m,p}\) is the usual seminorm of the Sobolev space \(W^{m,p}(\Omega)\). This inequality can be obtained from its “polygonal” version using standard analysis [23] in the “skin layer” with the help of (1.7). Throughout the paper \(C\) will denote a generic constant independent of \(h\) and can have different values in different places. We use \(||\cdot||_m\) and \(|\cdot|_m\) for the norm \(||\cdot||_{m,p}\) and the seminorm of \(W^{m,p}(\Omega)\), respectively, when \(p = 2\).

The test function space \(V_h \subset L^2(\Omega)\) associated with the dual partition \(T^*_h\) is defined as the set of all piecewise constants. More specifically, let \(\chi_{P_0}\) be the characteristic function of the set \(K^*_h\), we have for \(v_h \in V_h\)
\begin{align}
(1.10) \quad v_h = \sum_{P_0 \in \Omega^*_h} v_h(P_0) \chi_{P_0}.
\end{align}
Note that a test function is identically zero outside \(\Omega_h\). Define the transfer operator \(\Pi_h^* : U_h \rightarrow V_h\) connecting the trial and test spaces as
\begin{align}
(1.12) \quad \Pi_h^* w := \sum_{P_0 \in \Omega^*_h} w_h(P_0) \chi_{P_0},
\end{align}
and hence
\begin{equation}
(1.13) \quad \|w - \Pi_h^* w\|_0 \leq Ch |w|_1.
\end{equation}

The approximate problem we consider is to find \(u_h \in U_h\) such that
\begin{equation}
(1.14) \quad a^*(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,
\end{equation}
where
\begin{align}
(1.15) \quad a^*(u_h, v_h) := \sum_{P_0 \in \Omega^*_h} v_h(P_0) a^*(u_h, \chi_{P_0}),
\end{align}
and
\begin{align}
(1.16) \quad a^*(u_h, \chi_{P_0}) := -\int_{\partial K^*_0} (A \nabla u_h) \cdot \mathbf{n} \, ds + \int_{K^*_0} qu_h \, dx,
\end{align}
where \(\mathbf{n}\) is an outward unit normal to \(\partial K^*_0\), and \(a^* (\cdot, \cdot)\) is bilinear by construction. Using the facts \(n_1 ds = dx_2\) and \(n_2 ds = -dx_1\) yields
\begin{align}
(1.17) \quad a^*(u_h, \chi_{P_0}) &= -\int_{\partial K^*_0} \sum_{i,j=1}^2 a_{ij} \frac{\partial u_h}{\partial x_j} n_i ds + \int_{K^*_0} qu_h \, dx \\
&= -\int_{K^*_0} w^{(1)}_h \, dx_2 + \int_{\partial K^*_0} w^{(2)}_h \, dx_1 + \int_{K^*_0} qu_h \, dx,
\end{align}
where \( n_i \) is the \( i \)-th component of the outward unit normal to \( \partial K_p^* \), and

\[
\begin{align*}
  w_h^{(1)} &= a_{11} \frac{\partial u_h}{\partial x_1} + a_{12} \frac{\partial u_h}{\partial x_2}, \\
  w_h^{(2)} &= a_{21} \frac{\partial u_h}{\partial x_1} + a_{22} \frac{\partial u_h}{\partial x_2}. 
\end{align*}
\]

Let us relate our work to the existing literature. The basic idea of the finite volume method for general elliptic problems is to use the divergence theorem on the elliptic operator \( L \) of (1.1) to convert the double integral into a boundary integral as in (1.17). If one discretizes the boundary integral in (1.17) using finite differences, one gets the so-called finite volume difference methods [1, 22] or the generalized difference methods [15, 16, 17]. On the other hand if one uses finite element spaces in the discretization, one gets the so-called finite volume element methods [3, 4]. In both cases two grids dual to each other are used. More recently, Nicolaides [18] generalized the usual operators in vector analysis such as Div, Grad, and the Laplacian to Delaunay-Voronoi meshes. This class of methods is now termed the covolume method and has been successfully extended to practical fluid problems [13, 14, 19, 21]. See [20] for a survey of the covolume method. Porsching [25] initiated the so-called network method, which has also been extended to the Stokes problem [6, 12, 11] with rigorous analysis and to two fluid flow problems [24, 5]. In the network method the emphasis is to conserve mass or energy over control volumes. The meshes chosen do not have to be of the Delaunay-Voronoi type. In this paper we take barycenters in favor of circumcenters (the Delaunay-Voronoi mesh system uses circumcenters), since the maximum norm estimation is less amenable in the latter case. We shall refer to any finite volume method utilizing two grids as a covolume method since the last two methods mentioned above are now subsumed under the name the covolume method [20]. In all the covolume methods cited so far none has addressed maximum norm estimates for general elliptic or parabolic problems, which are crucial to studying their nonlinear counterpart where the coefficient matrix \( A \) becomes dependent on the solution. (However, some computational results in a discrete \( L^\infty \) norm were reported in [13, p. 160].) The approximation problem (1.14) has been considered by [16, 17] where convergence results in the \( H^1 \) and \( L^2 \) norms were demonstrated. However, we shall prove these results in a unified way. The main purpose of this paper is to provide convergence results in the maximum norm for (1.14) and for an accompanying approximate parabolic problem.

We now outline a central idea used in this paper to show convergence in \( L^2, H^1 \), and maximum norms. The idea, we think, is general enough to be useful for numerical analysts working in covolume methods. Our style of presenting it will follow that of the classical paper [23] on maxi-norm estimates in the finite element method. The central idea of analyzing the convergence of covolume methods is to reformulate (1.14) to find \( u_h \in \mathcal{U}_h \) such that

\[
a^*(u_h, \Pi^*_h T_h) = (f, \Pi^*_h T_h) \quad \forall T_h \in \mathcal{U}_h,
\]

which is a standard Galerkin method. With this association we can then tap into standard finite element analysis. A covolume method based on linear elements, if done properly, usually results in a system that is very close to the classical piecewise linear Galerkin method (more about this later). Comparison of the two systems then often leads to fruitful analysis. (This and similar ideas have been successfully
exploited in [6, 12, 11, 8, 9, 10].) Now if one strives to carry out this program, one is very naturally led into considering the quantity (d for “deviation”)

\[ d(v - v_h, T_h) := a(v - v_h, T_h) - a^*(v - v_h, \Pi_h^* T_h), \]

where \( v \) is a “general” function, \( v_h \in U_h, T_h \in U_h. \) The basic observation is that (see (2.11))

\[ d(v - v_h, T_h) = E_1 + E_2 + E_3 + E_4 + E_5, \]

where the \( E_i \) can be given various bounds that contain extra or “free” powers of \( h; \)  something unexpected at first glance (at the \( E_i \)). Thus, for example, the bounds on various \( E_i \) take the following forms:

\begin{align*}
\text{(B.E}_1\text{)} & \quad C_A h ||v - v_h||_1 ||T_h||_1, \\
\text{(B.E}_2\text{)} & \quad C_A h ||v - v_h||_1 + h^{1/2} ||v - v_h||_1^{1/2} ||T_h||_1.
\end{align*}

Here \( C_A \) depends on ||\( \nabla A ||_{\infty}; \) it is 0 if the coefficient matrix \( A \) is constant.

\begin{align*}
\text{(B.E}_3\text{)} & \quad C_2 h^2 ||v||_{5,p} ||T_h||_{1,p'}, \frac{1}{p} + \frac{1}{p'} = 1, \\
\text{(B.E}_4\text{)} & \quad C_A h ||v - v_h||_1 + h^{1/2} ||v - v_h||_1^{1/2} ||T_h||_1, \\
\text{(B.E}_5\text{)} & \quad C_q h ||v - v_h||_0 ||T_h||_1.
\end{align*}

Here \( C_q = 0 \) if the function \( q \equiv 0. \)

**Remark 1.1.** See (2.12)-(2.26) for the derivation of these bounds.

To give a feel for the usefulness of this observation, let us take the case of

\[ v \equiv 0, \quad A \text{ constant}, \quad q \equiv 0. \]

Then

\[ d(u_h, T_h) = 0! \]

Thus the covolume approximation is given by

\[ a(u_h, T_h) = (f, \Pi^*_h T_h) \quad \forall T_h \in U_h, \]

whereas, for the ordinary Galerkin solution, \( \tilde{u}_h, \)

\[ a(\tilde{u}_h, T_h) = (f, T_h) \quad \forall T_h \in U_h. \]

Hence it is obvious that the covolume approximation can be viewed as a Galerkin method with a variational crime. In the general case,

\[ a(u_h, T_h) + d(u_h, T_h) = (f, \Pi^*_h T_h) \]

with similar interpretation as two variational crimes. This view is very useful when dealing with the generalized Stokes problem (see [6, 12, 11] for more detail).

Now back to the issues of general estimates; take \( v \equiv 0, v_h \in U_h \) and \( T_h = v_h \) and apply (B.E_1), (B.E_2), (B.E_4), and (B.E_5) ((B.E_3) is void since \( v \equiv 0 \)): 

\[ |d(v_h, v_h)| \leq C h ||v_h||_1^2. \]

From this the coercivity (for \( h \) small enough) and boundedness of \( a^*(\cdot, \Pi^*_h \cdot) \) follow (see Lemma 2.3).
Next, take $v \equiv 0$, $v_h = e_h := \tilde{u}_h - u_h$ (ordinary Galerkin) to find

$$
\|e_h\|_2^2 \leq C a^*(\tilde{u}_h - u_h, \Pi_h^* e_h)
= C [a^*(\tilde{u}_h, \Pi_h^* e_h) - a^*(u_h, \Pi_h^* e_h)]
= C [(f, e_h) - (f, \Pi_h^* e_h) - d(\tilde{u}_h, e_h)],
$$
and it follows immediately that

$$
\|e_h\|_2^2 \leq C h (\|f\|_0 + \|\tilde{u}_h\|_1) \|e_h\|_1
$$
so that, by the triangle inequality, $\|u - u_h\|_1 \leq C h \|f\|_0$, which proves the $H^1$ convergence (see Lemma 2.5).

Similarly, we can derive $L^2$ convergence via a duality argument as follows. Note that

$$
\|e_h\|_0 = \sup_{\|\phi\|_0 = 1} (e_h, \phi).
$$

Given a $\phi$ with unit $L^2$-norm, let $L\psi = \phi, \psi = 0$ on $\partial\Omega$ and let $\tilde{\psi}_h$ be the Ritz projection of $\psi$. Thus

$$
(e_h, \phi) = a(e_h, \psi) = a(e_h, \tilde{\psi}_h)
= a(u - u_h, \tilde{\psi}_h)
= d(u - u_h, \tilde{\psi}_h)
\leq C_A (h\|u - u_h\|_1 + h^{3/2}\|u - u_h\|_1^{1/2}\|u\|_2^{1/2})
+ C_2 h^2 \|u\| \|\tilde{\psi}_h\|_{1,p'} (\frac{1}{2} + \frac{1}{p'} = 1)
+ C_q h \|u - u_h\|_0.
$$

Here, $\|\tilde{\psi}_h\|_{1,p'} \leq C \|\psi\|_{1,p'} \leq C_p \|\psi\|_2$ (stability and Sobolev). Clearly, after some trivial manipulations, we obtain convergence in the $L^2$ norm.

The $W^{1,\infty}$ and $L^\infty$ norm estimation follows the same vein but is more involved. The details can be found in Section 3. The organization of this paper is as follows. In Section 2 we list and prove preliminary lemmas and the $H^1, L^2$ norm convergence results. In Section 3 we derive maximum norm error estimates for the elliptic problems. The main results are contained in Theorem 3.1 (the max-norm error in the approximate solution is $O(h^2|\ln h|)$) and Theorem 3.2 (the max-norm error in the gradient is $O(h)$). The method of proof uses the above-mentioned central idea with the aid of the discrete Green’s function. In Section 4 we give similar maximum norm estimates for parabolic equations.

2. Preliminaries

Define the discrete $L^2$ norm:

$$
\|u_h\|_{0,h} := \|\Pi_h^* u_h\|_0 = \left\{ \sum_{K \in T_h} u_h^2(P) S_{P}^* \right\}^{1/2}.
$$

Referring to Figure 2 and using the fact that $Q$ are centers and $M_i$ are midpoints, we have

$$
\|u_h\|_{0,h} = \frac{1}{3} \sum_{K_Q \in T_h} [u_h^2(P_1) + u_h^2(P_2) + u_h^2(P_3)] S_Q^{1/2}.
$$
Next define the discrete $H^1$ seminorm and norm:

$$
|u_h|_{1,h} := \left( \sum_{K_Q \in T_h} |u_h|_{h,K_Q}^2 \right)^{1/2},
$$

(2.3)

$$
|u_h|_{1,h,K} := \left\{ \left( \left( \frac{\partial u_h}{\partial x_1}(Q) \right)^2 + \left( \frac{\partial u_h}{\partial x_2}(Q) \right)^2 \right)^{1/2} \right\} S_Q
$$

(2.4)

$$
||u||_{0,h} := \left\{ ||u_h||_{0,h}^2 + |u_h|^2_{1,h} \right\}^{1/2},
$$

(2.5)

**Lemma 2.1.** The two norms $\cdot |_{1,h}$ and $\cdot |_1$ are consistent, i.e., $\cdot |_{1,h} = \cdot |_1$, and $\cdot ||_{0,h}$ and $\cdot ||_{1,h}$ are equivalent to $\cdot ||_0$ and $\cdot ||_1$, respectively. Here the equivalence constants are independent of $h$.

**Proof.** The first statement is easy to see since $\nabla u_h$ is constant over $K_Q$. As for the second statement, it suffices to show the equivalence of the $L^2$ norms. In reference to Figure 2, we have with $K = K_Q$

$$
\int_K |u_h|^2 dx = \frac{1}{3} \left[ u_h^2(M_1) + u_h^2(M_2) + u_h^2(M_3) \right] S_Q = \frac{1}{12} \left[ u_h^2(P_1) + u_h^2(P_2) + u_h^2(P_3) + (u_h(P_1) + u_h(P_2) + u_h(P_3))^2 \right] S_Q.
$$

Summing over $K$ yields

$$
\frac{1}{4} ||u_h||_{0,h}^2 \leq ||u_h||_{0,h}^2 \leq ||u_h||_{0,h}^2.
$$

**Lemma 2.2.** $\Pi_h^*$ is self-adjoint with respect to the $L^2$ inner product.

$$
(u_h, \Pi_h^* u_h) = (\bar{u}_h, \Pi_h^* u_h), \quad \forall u_h, \bar{u}_h \in U_h.
$$

**Define**

$$
|||u_h|||_0 := (u_h, \Pi_h^* u_h)^{1/2}.
$$

(2.7)

Then $|||\cdot|||_0$ and $||\cdot||_0$ are equivalent. Here the equivalence constants are independent of $h$. 
Proof. In reference to Figure 3, for $i = 1, \ldots, 3$, let $e_i$ be the quadrilateral $P_i M_i Q M_{i+2}$, ($M_5 = M_2, M_4 = M_1$) and $\lambda_i$ be the Lagrange nodal basis functions associated with $P_i$, i.e., $\lambda_1, \lambda_2$, and $\lambda_3$ are the barycentric coordinates. Over a typical $K$ write

$$u_h = \sum_{i=1}^{3} u_h(P_i) \lambda_i$$

(we will use local indices when there is no danger of confusion), and use (1.12) to obtain

$$(u_h, \Pi_k^h \bar{u}_h) = \sum_{K \in T_h} \int_K u_h \Pi_k^h \bar{u}_h \, dx$$

$$= \sum_{K \in T_h} \sum_{l=1}^{3} \bar{u}_h(P_l) \int_{e_l} u_h \, dx$$

$$= \sum_{K \in T_h} \sum_{l=1}^{3} \bar{u}_h(P_l) \sum_{k=1}^{3} u_h(P_k) \int_{e_l} \lambda_k \, dx$$

$$= \sum_{K \in T_h} \sum_{k=1}^{3} \sum_{l=1}^{3} \bar{u}_h(P_l) u_h(P_k) \int_{e_k} \lambda_l \, dx$$

$$= \sum_{K \in T_h} \sum_{k=1}^{3} u_h(P_k) \sum_{l=1}^{3} \bar{u}_h(P_l) \int_{e_k} \lambda_l \, dx$$

where we have interchanged the summations and used the fact that

$$\int_{e_k} \lambda_l \, dx = \int_{e_l} \lambda_k \, dx.$$ 

This last equality can be shown as follows. First it is easy to see that the triangle $K$ is divided into six equal-area subtriangles. Use the three-vertices quadrature rule.
on linears to evaluate:

\[
\int_{e_1} \lambda_2 dx = \int_{e_1^+} \lambda_2 dx + \int_{e_1^-} \lambda_2 dx
\]

\[
= \frac{1}{3}(\lambda_2(P_1) + \lambda_2(Q) + \lambda_2(M)) S_{e_1^+}
\]

\[
+ \frac{1}{3}(\lambda_2(P_1) + \lambda_2(Q) + \lambda_2(M)) S_{e_1^-}
\]

\[
= \frac{1}{3}(0 + 1/3 + 1/2) S_{e_1^+} + \frac{1}{3}(0 + 1/3 + 0) S_{e_1^-},
\]

where \(e_1^+\) and \(e_1^-\) are the two subtriangles that make up \(e_1\). Since \(S_{e_1^-}, S_{e_1^+}, S_{e_2^-}\)
and \(S_{e_2^+}\) are the same, we see that

\[
\int_{e_1} \lambda_2 dx = \int_{e_2} \lambda_1 dx.
\]

The other cases can be handled similarly since the underlying integrals only depend
on the two areas as shown above. Finally, as a by-product, the equivalence of the
two norms now follows by direct computation.

Now let us derive the important relation (1.21) mentioned in Section 1. For
\(v \in H^2(\Omega), v_h, T_h \in U_h\), we have by Green’s formula and the fact that \(T_h\) vanishes
outside \(\Omega_h\) that (see Figure 2)

\[
a(v - v_h, T_h) = \sum_{K \in T_h} \int_K \sum_{i,j=1}^2 \left[ a_{ij}(x) - a_{ij}(Q) \right] \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial T_h}{\partial x_i} dx
\]

\[
+ \int_{\Omega_h} q(v - v_h) T_h dx
\]

\[
= \sum_{K} \int_K \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial T_h}{\partial x_i} dx
\]

\[
- \sum_{K} \int_K \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial^2 v}{\partial x_i \partial x_j} T_h dx
\]

\[
+ \sum_{K} \int_{\partial K} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial^2 v}{\partial x_i \partial x_j} \cos(n, x_i) T_h ds
\]

\[
+ \int_{\Omega_h} q(v - v_h) T_h dx,
\]

(2.8)

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where $Q$ is the center of $K$. Let $K^v$ denote the set of all vertices of $K$. By Green’s formula we have for $w \in H^2, w_h \in V_h$

\[
\sum_{K \in T_h} \int_K \frac{\partial^2 w}{\partial x_i \partial x_j} w_h dx
\]

\[
= \sum_{K} \sum_{P \in K^v} \int_{K \cap K} \frac{\partial^2 w}{\partial x_i \partial x_j} w_h dx
\]

\[
= \sum_{K} \sum_{P \in K^v} \int_{\partial(K \cap K^v)} \frac{\partial w}{\partial x_i} \cos\langle n, x_i \rangle w_h ds
\]

\[
= \sum_{K} \sum_{P \in K^v} \left\{ \int_{\partial(K \cap K^v)} \frac{\partial w}{\partial x_i} \cos\langle n, x_i \rangle w_h ds \right\}.
\]

Hence, with $a_{ij}(Q)w$ in place of $w$ in the above equation,

\[
\sum_{K \in T_h} \int_K a_{ij}(Q) \frac{\partial^2 w}{\partial x_i \partial x_j} w_h dx
\]

\[
= \left\{ \sum_{K} \sum_{P \in K^v} \int_{\partial(K \cap K^v)} a_{ij}(Q) \frac{\partial w}{\partial x_j} \cos\langle n, x_i \rangle w_h ds \right\}
\]

\[
+ \left\{ \sum_{K} \int_{\partial K} a_{ij}(Q) \frac{\partial w}{\partial x_j} \cos\langle n, x_i \rangle w_h ds \right\}.
\]

Now argue as in deriving (2.8) and use (2.9) with $w = v - v_h$ and $w_h = \Pi_h^T$ to obtain

(2.10)

\[
a^*(v - v_h, \Pi_h^T)
\]

\[
= - \sum_{K} \sum_{P \in K^v} \int_{\partial(K \cap K^v)} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i \rangle \Pi_h^T ds
\]

\[
+ \int_{\Omega_h} q(v - v_h) \Pi_h^T dx
\]

\[
= - \sum_{K} \sum_{P \in K^v} \int_{\partial(K \cap K^v)} \sum_{i,j=1}^2 (a_{ij}(x) - a_{ij}(Q)) \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i \rangle \Pi_h^T ds
\]

\[
- \sum_{K} \int_{K} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial^2 v}{\partial x_i \partial x_j} \Pi_h^T ds
\]

\[
+ \sum_{K} \int_{\partial K} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i \rangle \Pi_h^T ds
\]

\[
+ \int_{\Omega_h} q(v - v_h) \Pi_h^T dx.
\]
Hence

\begin{equation}
(2.11) \quad a(v - v_h, T_h) - a^*(v - v_h, \Pi_h^* T_h) = \sum_{i=1}^{5} E_i(v - v_h, T_h),
\end{equation}

where

\begin{equation}
(2.12) \quad E_1(v - v_h, T_h) = \sum_K \int_K \sum_{i,j=1}^{2} [a_{ij}(x) - a_{ij}(Q)] \frac{\partial(v - v_h)}{\partial x_j} \frac{\partial T_h}{\partial x_i} \, dx,
\end{equation}

\begin{equation}
(2.13) \quad E_2(v - v_h, T_h) = \sum_K \sum_{P \in K} \int_{\partial K_P \cap K} \sum_{i,j=1}^{2} [a_{ij}(x) - a_{ij}(Q)]
\cdot \frac{\partial(v - v_h)}{\partial x_j} \cos(n, x_i) \Pi_h^* T_h \, ds,
\end{equation}

\begin{equation}
(2.14) \quad E_3(v - v_h, T_h) = -\sum_K \int_K \sum_{i,j=1}^{2} a_{ij}(Q) \frac{\partial^2 v}{\partial x_i \partial x_j} (T_h - \Pi_h^* T_h) \, dx,
\end{equation}

\begin{equation}
(2.15) \quad E_4(v - v_h, T_h) = \sum_K \int_{\partial K} \sum_{i,j=1}^{2} a_{ij}(Q) \frac{\partial(v - v_h)}{\partial x_j} \cos(n, x_i) (T_h - \Pi_h^* T_h) \, ds,
\end{equation}

\begin{equation}
(2.16) \quad E_5(v - v_h, T_h) = \int_{\Omega_h} q(v - v_h)(T_h - \Pi_h^* T_h) \, dx.
\end{equation}

We are now in a position to show various bounds for \(E_i\)'s introduced in the previous section. In view of the definition (2.12), bound (B.E_1) is straightforward since \(a_{ij}\) is in \(W^{1,\infty}\). As for (B.E_2), from (2.13) \(E_2(v - v_h, T_h)\) can be rewritten (see Figure 2)

\begin{equation}
(2.17) \quad E_2(v - v_h, T_h) = \sum_K \sum_{l=1}^{3} \int_{M_l Q} \sum_{i,j=1}^{2} [a_{ij}(x) - a_{ij}(Q)] \frac{\partial(v - v_h)}{\partial x_j} \times \cos(n, x_i) ds \, [T_h(P_l) - T_h(P_{l+1})],
\end{equation}

where \(P_1 := P_1\). The equality is obtained by noticing that each line segment \(M_l Q\) is traveled twice but in opposite orientations (once as \(M_l Q\), once as \(QM_l\)) and then collecting the like-terms. By Taylor’s expansion and the fact \(T_h\) is linear in \(K\),

\begin{equation}
(2.18) \quad |T_h(P_l) - T_h(P_{l+1})| = \sum_{i=1}^{2} \left| \frac{\partial T_h}{\partial x_i} [x_i(P_l) - x_i(P_{l+1})] \right|
\leq h(\left| \frac{\partial T_h}{\partial x_1} \right| + \left| \frac{\partial T_h}{\partial x_2} \right|) \leq C |T_h|_{1,h,K}.
\end{equation}

On the other hand, by the Cauchy-Schwarz inequality

\begin{equation}
(2.19) \quad \int_{M_l Q} \frac{\partial(v - v_h)}{\partial x_i} \, ds \leq Ch^{1/2} \left( \int_{M_l Q} |\phi_i|^2 ds \right)^{1/2},
\end{equation}
where \( \phi_i := \frac{\partial(v-v_h)}{\partial x_i} \). Use the trace theorem ([2, p. 37]) and a scaling argument to obtain

\[
\int_{M_{i'}Q} |\phi_i|^2 \, ds \leq C(h^{-1} ||\phi_i||_{0,K} + ||\nabla \phi_i||_{0,K} ||\phi_i||_{0,K}) 
\]

\[
\leq C(h^{-1} |v - v_h|_{1,K}^2 + |v - v_h|_{2,K}|v - v_h|_{1,K}) 
\]

\[
= C(h^{-1} |v - v_h|_{1,K}^2 + |v|_{2,K}|v - v_h|_{1,K}).
\]

Collecting estimates, using Lemma 2.1 and the generalized Hölder’s inequality, we have

\[
|E_2(v - v_h, T_h)| \leq Ch \{ |v - v_h|_{1,T_h} + h^{1/2} |v - v_h|_{2,T_h} \} 
\]

which implies (B.E2).

Using proper quadratures for the two integrands and the fact that the quadrilaterals \( e_i \) of Figure 3 have equal area, it is easy to see that

\[
\int_K (T_h - \Pi_h T_h) \, dx = 0 \quad \forall T_h \in U_h.
\]

Hence

\[
|E_3| = \sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q) \left[ \frac{\partial^2 v(x)}{\partial x_i \partial x_j} - \mathcal{P}_K \frac{\partial^2 v}{\partial x_i \partial x_j} \right] (T_h - \Pi_h T_h) \, dx,
\]

where \( \mathcal{P}_K \) is the local \( L_2 \) projection to the space of piecewise constants. (Note that using \( \mathcal{P}_K \frac{\partial^2 v}{\partial x_i \partial x_j} \) instead of \( \frac{\partial^2 v}{\partial x_i \partial x_j} (Q) \) avoids asking \( v \) to be in \( C^2 \), as is done in some literature.) From this bound (B.E3) follows easily.

As for the estimation of \( E_4 \), first note that \( \frac{\partial v}{\partial x_i} \cos\langle n, x_i \rangle \) is constant along an edge \( L \) of the element \( K \) and that

\[
\int_L (T_h - \Pi_h T_h) \, ds = 0.
\]

Thus

\[
E_4(v - v_h, T_h) 
\]

\[
= \sum_K \int_{\partial K} \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i \rangle (T_h - \Pi_h T_h) \, ds 
\]

\[
= \sum_K \int_{\partial K} \left[ \sum_{i,j=1}^2 a_{ij}(Q) \frac{\partial v}{\partial x_j} \cos\langle n, x_i \rangle \right] (T_h - \Pi_h T_h) \, ds.
\]

Let \( E \) be the collection of all the interior edges in the primal triangulation \( T_h \). (An interior edge does not lie on \( \partial \Omega_h \).)
Using the boundary condition of $T_h$ on $\partial \Omega_h$, continuity of $T_h - \Pi_h^* T_h$ and continuity of $\partial_{x_j} Q_h(x_i)$ across the edges in $\mathcal{E}$ (guaranteed by $v \in H^3(\Omega)$), we have

$$E_4 = \sum_{L \in \mathcal{E}} \int_L \sum_{i,j=1}^{2} (a_{ij}(Q^+_L) - a_{ij}(Q^-_L))$$

$$\times \left( \frac{\partial v}{\partial x_j} \cos(n, x_i)(T_h - \Pi_h^* T_h) \right) ds$$

(2.25)

$$= \sum_{L \in \mathcal{E}} \int_L \sum_{i,j=1}^{2} (a_{ij}(Q^+_L) - a_{ij}(Q^-_L))$$

$$\times \left( \frac{\partial v}{\partial x_j} - v_j \right) \cos(n, x_i)(T_h - \Pi_h^* T_h) ds,$$

where $Q^+_L$ and $Q^-_L$ are the centers of the two triangles sharing $L$ as a common edge, and the addition of a constant $v_j$ is due to (2.23). Now we choose $v_j$ as

$$v_j := \frac{1}{2} \left( \frac{\partial v^+_h}{\partial x_j} + \frac{\partial v^-_h}{\partial x_j} \right),$$

where $v^+_h$ (resp. $v^-_h$) is the restriction of $v_h$ to the left (resp. right) triangle $K_L$ (resp. $K_R$).

Observe that

$$\sum_{L \in \mathcal{E}} \int_L \sum_{i,j=1}^{2} (a_{ij}(Q^+_L) - a_{ij}(Q^-_L)) \left( \frac{\partial v}{\partial x_j} - \frac{\partial v^+_h}{\partial x_j} \right) \cos(n, x_i)(T_h - \Pi_h^* T_h) ds,$$

where $\sigma = +$ or $-$ resembles $E_2$. The technique used in deriving (2.20) yields

$$\left( \int_L (T_h - \Pi_h^* T_h)^2 ds \right)^{1/2} \leq Ch^{1/2} ||T_h||_{1,K}.$$

Thus as in deriving out (2.21), we have bound (B.E4)

(2.26) $|E_4(v - v_h, T_h)| \leq Ch ||v - v_h||_1 ||T_h||_1 + h^{1/2} ||v - v_h||_2^2 ||T_h||_1 \}.$

Finally, bound (B.E5) follows from (2.16) easily. The following lemma is now proved in view of the central observation in Section 1.

**Lemma 2.3.** There exist positive constants $h_0, \alpha, M$ such that for $0 \leq h \leq h_0$

(2.27) $a^*(u_h, \Pi_h^* u_h) \geq \alpha ||u_h||_1^2, \quad \forall u_h \in U_h.$

(2.28) $|a^*(u_h, \Pi_h^* T_h)| \leq M ||u_h||_1 ||T_h||_1, \quad \forall u_h, T_h \in U_h.$

For covolume methods we seldom have a symmetric bilinear form $a^*(\cdot, \Pi_h^* \cdot)$ even though $a(\cdot, \cdot)$ is. However, we have a lemma which measures how far the bilinear form $a^*(\cdot, \Pi_h^* \cdot)$ is from being symmetric. This lemma will be used in the parabolic problem.

**Lemma 2.4.** There exist positive constants $M, h_0$ such that for $0 < h \leq h_0$

(2.29) $|a^*(u_h, \Pi_h^* T_h) - a^*(T_h, \Pi_h^* u_h)| \leq Mh ||u_h||_1 ||T_h||_1 \quad \forall u_h, T_h \in U_h.$

Proof. Use (1.20) and the triangle inequality to derive

$|a^*(u_h, \Pi_h^* T_h) - a^*(T_h, \Pi_h^* u_h)| \leq |d(u_h, T_h) - d(T_h, u_h)|.$

Invoking proper bounds for $d(\cdot, \cdot)$ completes the proof. □
Theorem 3.1. Let 

\[ |u - u_h|_1 \leq C h |u|_2, \]  
\[ |u - u_h|_0 \leq C h^2 |u|_3, \quad (p > 1), \]

whenever the right-hand sides make sense.

Given any \( z \in \bar{\Omega} \), we define \( G_z^h \in U_h \) to be the discrete Green’s function associated with the form \( a(\cdot, \cdot) \) if

\[ a(G_z^h, w_h) = w_h(z) \quad \forall w_h \in U_h. \]

Lemma 2.6. The function \( G_z^h \) possesses the following properties [26, 27]:

\[ \|G_z^h\|_1 \leq C \ln h^{1/2}. \]

Let \( v \) be a given unit vector (direction) and let \( \Delta z \) be any vector parallel to \( v \). Then we define

\[ \partial_z G_z^h := \lim_{\Delta z \to 0} \frac{G_z^{h+\Delta z} - G_z^h}{|\Delta z|}. \]

Lemma 2.7. The derivative \( \partial_z G_z^h \in U_h \) has the following properties [27]:

\[ a(\partial_z G_z^h, v_h) = \partial_z v_h(z) \quad \forall v_h \in U_h, \]
\[ \|\partial_z G_z^h\|_1 \leq C h^{-1}. \]

Lemma 2.8. Let \( u \) and \( u_h \) be the solutions of (1.1) and (1.14), respectively. Then

\[ a^*(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \]

3. Maximum norm estimates for an elliptic problem

Theorem 3.1. Let \( u \) be the solution of (1.1) and \( u_h \) be the solution of (1.14). Then

\[ |u - u_h|_{0, \infty} \leq C h^2 |\ln h| \left[ |u|_3 + |u|_{2, \infty} \right], \]

provided that \( u \in H^3(\Omega) \cap W^{2, \infty}(\Omega) \cap H^3(\Omega). \)

Proof. Let \( \tilde{u}_h \) be the ordinary Galerkin of (1.1).

\[ |u - u_h|_{0, \infty} \leq |u - \tilde{u}_h|_{0, \infty} + ||u_h - u_h||_{0, \infty}. \]

Since it is well known [26] that the maximum norm error in \( \tilde{u}_h \) is bounded by \( C h^2 |\ln h| |u|_{2, \infty} \), it suffices to estimate \( e_h := \tilde{u}_h - u_h \). By the definition of the discrete Green’s function and (2.37)

\[ e_h(z) = a(e_h, G_z^h) = a(u - u_h, G_z^h) = d(u - u_h, G_z^h). \]

Now we estimate \( E_i(u - u_h, G_z^h), i = 1, \ldots, 5 \). By (B.E1), (2.33) and Lemma 2.5,

\[ |E_i(u - u_h, G_z^h)| \leq Ch ||u - u_h||_1 |G_z^h|_1 \leq Ch^2 |\ln h|^{1/2} ||u||_2. \]
By (B.E\textsubscript{2}), (2.33) and Lemma 2.5,
\begin{align*}
|E_2(u - u_h, G^h_z)| & \leq C_A h [[u - u_h]|_1 + h^{1/2}[[u - u_h]|_1^{1/2}[[u]|_2^{1/2}||G^h_z||_1 \\
& \leq Ch^2 |\ln h|^\frac{1}{2}||u||_2.
\end{align*}
(3.4)

By (B.E\textsubscript{3}) and (2.33),
\begin{align*}
|E_3(u - u_h, G^h_z)| & \leq Ch^2 ||u||_3 ||G^h_z||_1 \\
& \leq Ch^2 |\ln h|^\frac{1}{2}||u||_3,
\end{align*}
(3.5)

Finally
\begin{align*}
|E_4(u - u_h, G^h_z)| & \leq Ch^2 |\ln h|^\frac{1}{2}||u||_2.
\end{align*}
(3.6)

**Theorem 3.2.** Under the hypotheses of Theorem 3.1
\begin{align*}
||u - u_h||_{1, \infty} \leq Ch \left[||u||_3 + ||u||_{2, \infty}\right].
\end{align*}
(3.7)

**Proof.** The proof parallels the development in Theorem 3.1. Since it is well known [26] that the error in \(\tilde{u}_h\) is bounded by \(Ch||u||_{2, \infty}\), it suffice to estimate \(\varepsilon_h := \tilde{u}_h - u_h\) in \(W^{1, \infty}\). As before
\begin{align*}
\partial_z \varepsilon_h(z) & = a(e_h, \partial_z G^h_z) \\
& = a(u - u_h, \partial_z G^h_z) \\
& = d(u - u_h, \partial_z G^h_z) \\
& = \sum_{i=1}^5 E_i(u - u_h, \partial_z G^h_z),
\end{align*}
where
\begin{align*}
|E_1(u - u_h, \partial_z G^h_z)| & \leq Ch ||u - u_h||_1 ||\partial_z G^h_z||_1 \\
& \leq Ch ||u||_2, \\
|E_2(u - u_h, \partial_z G^h_z)| & \leq Ch^2 ||u||_2 ||\partial_z G^h_z||_1 \\
& \leq Ch ||u||_2, \\
|E_3(u - u_h, \partial_z G^h_z)| & \leq Ch^2 ||u||_3 ||\partial_z G^h_z||_1 \\
& \leq Ch ||u||_3, \\
|E_4(u - u_h, \partial_z G^h_z)| & \leq Ch ||u||_2, \\
|E_5(u - u_h, \partial_z G^h_z)| & \leq C ||u - u_h||_1 ||\partial_z G^h_z||_1 \\
& \leq Ch ||u||_3.
\end{align*}
(3.8)

Combining all the above inequalities completes the proof.
4. Maximum norm estimates for parabolic problems

Consider the associated parabolic problem to (1.1)-(1.2):

\[ (4.1) \quad u_t + Lu = f(x,t), \quad (x,t) \in \Omega \times (0,T) \]
\[ (4.2) \quad u = 0, \quad (x,t) \in \partial \Omega \times (0,T) \]
\[ (4.3) \quad u = u_0(x), \quad t = 0, x \in \Omega, \]

where \( L \) is the elliptic operator of (1.1) and \( u_t := \frac{\partial u}{\partial t} \). The domain \( \Omega \) has the primal partition \( T_h \) and dual partition \( T_h^* \) of the types specified in Section 1. The trial and test spaces are still \( U_h \subset H^1_0(\Omega) \) and \( V_h \subset L^2(\Omega) \), respectively. Consider the time-continuous approximation to (4.1)-(4.3):

Find \( u_h := u_h(\cdot, t) \in U_h, 0 \leq t \leq T \) such that

\[ (4.4) \quad (u_{h,t}, v_h) + a^*(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, t > 0, \]
\[ (4.5) \quad u_h(x,0) = u_{0h}(x), \quad x \in \Omega, \]

where the approximate initial condition \( u_{0h} \) is the elliptic projection (see (4.8)) of the exact initial function to be specified in (4.15).

**Theorem 4.1.** Let \( u \) and \( u_h \) be the solutions of (4.1)-(4.3) and (4.4)-(4.5), respectively. Then for \( p > 1 \)

\[ (4.6) \quad ||u - u_h||_{L^\infty(L^\infty)} \leq C h^2 |\ln h| \{ ||u||_{L^\infty(H^3)} + ||u||_{L^\infty(W^{2,p})} + ||u_t||_{L^2(W^{3,p})} \}, \]
\[ (4.7) \quad ||u - u_h||_{L^\infty(W^{1,p})} \leq C h |\ln h| \{ ||u||_{L^\infty(H^3)} + ||u||_{L^\infty(W^{2,p})} + ||u_t||_{L^2(W^{3,p})} \}, \]

where \( L^\infty(L^\infty) := L^\infty(0,T; L^\infty(\Omega)), L^\infty(H^3) := L^\infty(0,T; H^3(\Omega)) \).

**Proof.** Introduce the self-adjoint operator \( R_h : H^2(\Omega) \cap H^1_0(\Omega) \rightarrow U_h \) defined by

\[ (4.8) \quad a^*(R_h u, v_h) = a^*(u, v_h) \quad \forall v_h \in V_h. \]

By Lemma 2.5, and Theorems 3.1 and 3.2,

\[ (4.9) \quad ||(u - R_h u)_t||_0 \leq C h^2 ||u_t||_{3,p}, \quad p > 1, \]
\[ (4.10) \quad ||(u - R_h u)||_{0,\infty} \leq C h^2 |\ln h| \{ ||u||_3 + ||u||_{2,\infty} \}, \]
\[ (4.11) \quad ||(u - R_h u)||_{1,\infty} \leq C h ||u||_3 + ||u||_{2,\infty}. \]

Write \( u - u_h = (u - R_h u) + (R_h u - u_h) := \eta + \xi \). It suffices to estimate \( \xi \). By (4.1)-(4.4) and (4.8),

\[ (4.12) \quad (\xi_t, v_h) + a^*(\xi, v_h) = -(\eta_t, v_h), \forall v_h \in V_h. \]

Set \( v_h = \Pi^*_h \xi_t \) and use (2.7) to obtain

\[ (4.13) \quad ||\xi_t||_{0}^2 + \frac{1}{2} \frac{d}{dt} a^*(\xi, \Pi^*_h \xi) \]
\[ = - (\eta_t, \Pi^*_h \xi_t) + \frac{1}{2} [a^*(\xi_t, \Pi^*_h \xi_t) - a^*(\xi, \Pi^*_h \xi_t)]. \]

By Lemma 2.4, an inverse inequality, and Lemma 2.2,

\[ |a^*(\xi_t, \Pi^*_h \xi) - a^*(\xi, \Pi^*_h \xi)| \leq C h ||\xi_t||_1 ||\xi||_1 \]
\[ \leq C ||\xi_t||_0 ||\xi||_1 \leq ||\xi_t||_0^2 + C ||\xi||_1^2. \]
where we have used the $\epsilon$-inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$ for positive $\epsilon, a, b$. Taking $\epsilon$ small enough to absorb the $\xi_t$ term on the right-hand side into the left-hand side, we have

$$\frac{d}{dt} a^*(\xi, \Pi_h^* \xi) \leq C(\|\eta_t\|_0^2 + \|\xi\|_1^2).$$

Set

$$u_{0h} = R_h u(0)$$

so that $\xi(0) = 0$. Integrate (4.14) and use Lemma 2.3 to get

$$\alpha\|\xi\|_1^2 \leq a^*(\xi, \Pi_h^* \xi) \leq C \int_0^t (\|\eta_t\|_0^2 + \|\xi\|_1^2) d\tau.$$

Use (4.9) and the Gronwall’s inequality to get

$$\|\xi\|_1 \leq Ch^2\|u_t\|_{L^2(W^{3,p})}, \quad p \geq 1.$$  

From the asymptotic Sobolev inequality ([23, p. 274]), we have

$$\|\xi\|_{0, \infty} \leq C|\ln h^\frac{1}{2}\|\nabla \xi\|_0 \leq C|\ln h^\frac{1}{2}\|u_t\|_{L^2(W^{3,p})}.$$  

Combine (4.10) and (4.18) to get (4.6) and then use an inverse inequality to get

$$\|\xi\|_{1, \infty} \leq Ch^{-1}\|\xi\|_1 \leq C h^2\|u_t\|_{L^2(W^{3,p})}.$$  

Noting (4.11) derives (4.7) completes the proof.

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**References**


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