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# THE POSTAGE STAMP PROBLEM: AN ALGORITHM TO DETERMINE THE $h$-RANGE ON THE $h$-RANGE FORMULA ON THE EXTREMAL BASIS PROBLEM FOR $k = 4$

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Abstract. Given an integral "stamp" basis $A_k$ with $1 = a_1 < a_2 < \ldots < a_k$ and a positive integer $h$, we define the *h-range* $n(h, A_k)$ as

$$n(h, A_k) = \max\{N \in \mathbf{N} \mid n \leq N \Longrightarrow n = \sum_1^k x_i a_i, \sum_1^k x_i \leq h, \;\; n, \; x_i \in \mathbf{N}_0\}.$$

$\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For given $h$ and $k$, the *extremal basis* $A_k^*$ has the largest possible *extremal h-range*

$$n(h, k) = n(h, A_k^*) = \max_{A_k} n(h, A_k).$$

We give an algorithm to determine the $h$-range. We prove some properties of the $h$-range formula, and we conjecture its form for the extremal $h$-range. We consider *parameter bases* $A_k = A_k(h)$, where the basis elements $a_i$ are given functions of $h$. For $k = 4$ we conjecture the extremal parameter bases for $h \geq 11385$.

## 1. Background

Given an integral basis $A_k = \{a_1, a_2, \ldots, a_k\}$ with $a_1 = 1 < a_2 < \ldots < a_k$ and a positive integer $h$, we define the *h-range* $n(h, A_k)$ as

$$n(h, A_k) = \max\{N \in \mathbf{N} \mid n \leq N \Longrightarrow n = \sum_1^k x_i a_i, \sum_1^k x_i \leq h, \;\; n, \; x_i \in \mathbf{N}_0\}.$$

$\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The integer $n \in \mathbf{N}$ has an *h-representation* by $A_k$ if

$$n = \sum_1^k x_i a_i \mid \;\; \sum_1^k x_i \leq h, \;\; x_i \in \mathbf{N}_0.$$

We consider only bases $A_k$ which are *h-admissible*, that is,

$$a_k \leq n(h, A_k).$$

For given $h$ and $k$, the *extremal basis* $A_k^*$ has the largest possible *extremal h-range*

$$n(h, k) = n(h, A_k^*) = \max_{A_k} n(h, A_k).$$

A popular interpretation arises if we consider the integers $a_i$ as *stamp* denominations and $h$ as the "size of the envelope." More information about the "postage

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stamp problem" can be found in E. S. Selmer's comprehensive research monograph [17]. Here we mainly use Selmer's notation and presentation.

In the beginning, the main interest was centered around the *global* aspect, to find an *extremal basis* $A_k^*$ with extremal $h$-range. The "*local*" aspect is: Determine $n(h, A_k)$ when $h, k$ and a particular basis $A_k$ are given.

In the global case, a convenient approach is to keep $k$ fixed and let $h$ increase, asking for *asymptotic values* of the extremal $h$-range $n(h, k)$. We can also ask for asymptotic values of "local" $h$-ranges $n(h, A_k) = n(h, A_k(h))$, when the basis elements $a_i$ are given functions of $h$. We shall call such bases $A_k(h)$ *parameter bases*.

Let $\varphi$ be the *prefactor* defined by

$$n(h, A_k(h)) = \varphi(\frac{h}{k})^k(1 + \mathrm{o}(1)). \tag{1}$$

Both the local and the global problems are trivial for $k = 2$, Stöhr [20]. The extremal bases $A_3^*$ were determined by Hofmeister [4], [5]. For $k \geq 4$, our knowledge is much more limited. The best known general upper bound is due to Rødseth [15]:

$$n(h, k) \leq \frac{(k-1)^{k-2}}{(k-2)!}(\frac{h}{k})^k + \mathcal{O}(h^{k-1}).$$

For $k = 4$, the prefactor $\varphi = 4.5$ is far too large, and Kirfel [7] has the strongest published result:

$$n(h, 4) \leq 2.35\left(\frac{h}{4}\right)^4 + \mathcal{O}(h^3).$$

In [12] the author proved the lower bound

$$n(h, 4) \geq 2.008\left(\frac{h}{4}\right)^4 + \mathcal{O}(h^3).$$

The proof consists in determining a parameter basis $A_4 = A_4(h)$ whose $h$-range equals the bound given. However (May 1991, unpublished), Kirfel and the author have shown that the lower bound 2.008... (more decimals in (32)) is really *sharp*. Hence, it is natural to investigate the local extremal parameter bases for $k = 4$.

For $k = 5$, Kolsdorf in [6] has given a parameter basis with asymptotic $h$-range $3.06(h/5)^5$.

It was shown by Kirfel [8] that the limit

$$c_k = \lim_{h\to\infty} n(h, k)/(h/k)^k \tag{2}$$

really exists for all $k \geq 2$. It is known that $c_2 = 1$, $c_3 = 4/3$, and $c_4 = 2.008...$.

Looking for the extremal bases, we consider parameter bases $A_k(h)$ for which

$$n(h, A_k(h)) \text{ has order of magnitude } h^k. \tag{3}$$

For the basis elements, this implies that $a_i(h)$ has order of magnitude $h^{i-1}$, $i = 2,\ 3,\ \ldots,\ k$.

**Representations and gain.** The *regular representation* of $n$ by $A_k$,

$$n = \sum_1^k e_i a_i, \tag{4}$$

satisfies the conditions

$$e_1 + e_2a_2 + \ldots + e_ja_j < a_{j+1}, \qquad j = 1, 2, \ldots , k-1. \tag{5}$$

A representation of $n$ is *minimal* if the number of addends is the smallest possible among all representations. For the elements $a_i \in A_k,\ i = 2, 3, \ldots , k$, we write

$$a_i = \gamma_{i-1}a_{i-1} - \sum_{j=1}^{i-2} \beta_j^{(i)} a_j, \tag{6}$$

where $\gamma_{i-1} = \lceil a_i/a_{i-1} \rceil \ \geq 2$, and $\sum_{j=1}^{i-2} \beta_j^{(i)} a_j = \gamma_{i-1}a_{i-1} - a_i$ is the regular representation by $A_{i-2}$. As usual, $\lceil x \rceil$ denotes the smallest integer $\geq x \in \mathbf{R}$. Hofmeister [5] calls (6) the *normal form* of the basis $A_k$. Let $n \in \mathbf{N}$ have a regular representation (4) by $A_k$, and let $s_i \in \mathbf{Z}$, $i = 2, 3, \ldots , k$. From (6) we get a new representation $n = \sum z_ja_j$ by an $(s_2, s_3, \ldots , s_k)$-*transfer*:

$$\begin{aligned} n &= \sum_{i=1}^{k} e_ia_i + \sum_{i=2}^{k} s_i \left( \gamma_{i-1}a_{i-1} - a_i - \sum_{j=1}^{i-2} \beta_j^{(i)} a_j \right) \\ &= \sum_{j=1}^{k} \left( e_j - s_j + s_{j+1}\gamma_j - \sum_{i=j+2}^{k} s_i\beta_j^{(i)} \right) a_j = \sum_{j=1}^{k} z_ja_j, \end{aligned} \tag{7}$$

with $s_1 = s_{k+1} = \gamma_k = 0$. We say that the transfer is *possible* if $z_j \geq 0, \quad j = 1, \ldots , k$.

The sum of the reductions in the coefficients is the *gain* $G(s_2, s_3, \ldots , s_k)$ in the transfer:

$$G(s_2, s_3, \ldots , s_k) = \sum_{j=1}^{k} (e_j - z_j). \tag{8}$$

The usefulness of such transfers stems from the following result of Hofmeister [5]: Every "legal" representation $n = \sum z_ia_i$ $(z_i \geq 0)$ can be *obtained from the regular representation* by a suitable $(s_2, s_3, \ldots , s_k)$-transfer with all $s_i \geq 0$. We also cite another result of Hofmeister [5]: If a parameter basis $A_k(h)$ satisfies (3) and is expressed in *normal* form (6), then the $s_i$ of any possible $(s_2, s_3, \ldots , s_k)$-transfer are *bounded* as $h \to \infty$. See also Kirfel [7].

In 1963, Hofmeister [5], [3] gave formulas for the *regular* $h$-range of a basis. If only regular $h$-representations are allowed, we get the regular $h$-range. He also conjectured the formula for the extremal regular $h$-range, later proved by Mrose [14].

Let

$$h_0 = h_0(A_k) = \min\{h \in \mathbf{N} \mid a_k \leq n(h, A_k)\}.$$

For all $k$ and $h \geq h_0$ we trivially have

$$n(h+1, A_k) \geq n(h, A_k) + a_k. \tag{9}$$

Furthermore, Selmer [17] proved that, for arbitrary $k$ and $h \geq h_0$,

$$n(h, A_k) \geq (h+1)a_{k-1} - a_k \tag{10}$$

implies

$$n(h+1, A_k) = n(h, A_k) + a_k. \tag{11}$$

If $h$ is increased by 1, the right-hand side of (10) increases with $a_{k-1}$, while the left-hand side increases with at least $a_k$. There is consequently an $h_1$ ($\geq h_0$) such that (10) and hence (11) are satisfied for all $h \geq h_1$. This means that for given $h$, $h \geq h_1$, we have

$$n(h, A_k) = n(h_1, A_k) + (h - h_1)a_k. \tag{12}$$

We see that for a basis $A_k$ there may be different $h$-range formulas according to the value of $h$, $h_0 \leq h \leq h_1$. From (12), the $h$-range formula is the same for all $h \geq h_1$. In looking for bases with large $h$-range, we often have the same $h$-range formula for all $h \geq h_0$.

**Lemma 1.** *Let the basis $A_k$ and the possible transfers $T^{(i)} = (s_2^{(i)}, s_3^{(i)}, \dots, s_k^{(i)})$, $i = 1, 2, \dots, \eta$, be given. Let*

$$h_2 = \min\{h | n(h, A_k) \geq \max_i\{s_k^{(i)}\}a_k\}.$$

*Then for $h \geq h_2$*

$$n(h, A_k) = n(h_2, A_k) + (h - h_2)a_k.$$

*Proof.* The minimal representation of a positive integer is independent of the value of $h$. For $h \geq h_2$ we can use all the transfers. From above we know that for $h \geq h_1$ the $h$-range is determined by (12) and we have $h_1 \leq h_2$. Note that only the transfers actually used determine $h_2$.

## 2. The $h$-range algorithm

In the literature we find more or less general $h$-range algorithms by Lunnon [9], Riddell and Chan [16], Mossige [10], and Challis [2].

Let the basis $A_4$ and the possible transfers be given. For each integer $n \in [1, n(h, A_4)]$ given in a regular representation $\sum e_j a_j$, we use the possible transfer with the largest gain to give the minimal representation of $n$, $\sum z_j a_j$. It satisfies the inequality $\sum e_j - gain = \sum z_j \leq h$. The algorithm gives sufficient such inequalities that express the conditions that all the integers $n$ have an $h$-representation. The least integer $n$ with $n+1$ not having an $h$-representation is the $h$-range. For a given basis, the algorithm determines $h_0$ and from which $h \geq h_0$ the $h$-range formula is the same. The result is valid for all $h \geq h_0$.

We give the algorithm for $k = 4$, but it may be generalized to $k > 4$.

Now, let the possible transfers $T^{(i)} = (s_2^{(i)}, s_3^{(i)}, s_4^{(i)})$, $i = 1, \dots, \eta$, for the basis $A_4$ be given. Then the minimal representation of an integer $n > 0$ is independent of $h$.

The upper bounds for the $e_j$'s are given such that the representation (4) is regular. The conditions for the transfers to be possible give lower bounds for the $e_j$'s. The coefficients $z_j$ of (7) must be $\geq 0$, giving lower bounds on the $e_j$'s. The gain (reduction of coefficient sum) must be positive.

We get the following values of the gain and the lower bounds for $e_j$'s:

$$\begin{aligned} G_i &= s_2^{(i)}(-\gamma_1 + 1) + s_3^{(i)}(\beta_1^{(3)} - \gamma_2 + 1) \\ &\quad + s_4^{(i)}(\beta_1^{(4)} + \beta_2^{(4)} - \gamma_3 + 1) \geq 1, \end{aligned} \tag{13}$$

$$(14)\qquad \begin{aligned} e_1 &\geq -s_2^{(i)}\gamma_1 + s_3^{(i)}\beta_1^{(3)} + s_4^{(i)}\beta_1^{(4)} = L_i^{'},\\ e_2 &\geq s_2^{(i)} - s_3^{(i)}\gamma_2 + s_4^{(i)}\beta_2^{(4)} = M_i^{'},\\ e_3 &\geq s_3^{(i)} - s_4^{(i)}\gamma_3 = N_i^{'},\\ e_4 &\geq s_4^{(i)} = Q_i. \end{aligned}$$

We may, however, find $L_i^{'}$ and/or $M_i^{'}$ and/or $N_i^{'} < 0$, and operate instead with lower bounds

$$e_1 \geq L_i = \max\{0, L_i^{'}\},\ e_2 \geq M_i = \max\{0, M_i^{'}\},\ e_3 \geq N_i = \max\{0, N_i^{'}\}.$$

There may be repetitions among the $L_i,\ M_i,\ N_i,$ or $Q_i$. We sort them first *without* repetitions:

$$\begin{array}{l} 0 = L_0 < L_1 < \ldots < L_{r_1} < L_{r_1+1} = U_1 + 1\\ 0 = M_0 < M_1 < \ldots < M_{r_2} < M_{r_2+1} = U_2 + 1\\ 0 = N_0 < N_1 < \ldots < N_{r_3} < N_{r_3+1} = U_3 + 1\\ 0 = Q_0 < Q_1 < \ldots < Q_{r_4} < Q_{r_4+1} = U_4 + 1. \end{array}$$

The numbers $L_0,\ L_{r_1+1}$, and so forth, are added. Here $L_0 = M_0 = N_0 = Q_0 = G_0 = 0$ corresponds to using the regular representation itself, hence no transfer. The upper bounds for $e_j$, say $U_j,\quad j = 1, 2, 3$, are given such that the representation (4) is regular. For $e_4$ we note that the largest $s_4^{(i)}$ is $< h$, and we put $U_4 = h$. Then we sort *all* the gains $G_i$,

$$G^{(1)} \geq G^{(2)} \geq \ldots \geq G^{(\eta)} > 0,$$

without registering possible equalities. This gives a sequence of quintuples

$$(G^{(i)},\ L^{(i)},\ M^{(i)},\ N^{(i)},\ Q^{(i)}), \qquad i = 1, 2, \ldots, \eta,$$

to which we add $(0, 0, 0, 0, 0)$, corresponding to no transfer. Assume that $p, q, r$ and $s$ are given such that

$$(15)\qquad 0 \leq p \leq r_1, \quad 0 \leq q \leq r_2, \quad 0 \leq r \leq r_3, \quad 0 \leq s \leq r_4.$$

Let $e_1, e_2, e_3$ and $e_4$ be given such that

$$(16)\qquad L_p \leq e_1 \leq L_{p+1} - 1,\ M_q \leq e_2 \leq M_{q+1} - 1,$$

$$(17)\qquad N_r \leq e_3 \leq N_{r+1} - 1,\ Q_s \leq e_4 \leq Q_{s+1} - 1.$$

We then scan the quintuples $(G^{(i)},\ L^{(i)},\ M^{(i)},\ N^{(i)},\ Q^{(i)}), i = 1, 2, \ldots, \eta + 1$, and register the first time (largest gain) such that

$$L^{(i)} < L_{p+1}, \quad M^{(i)} < M_{q+1}, \quad N^{(i)} < N_{r+1}, \quad Q^{(i)} < Q_{s+1}.$$

The corresponding gain $G^{(i)} = G_{pqrs}$ is then the largest one which can be used in the case (16), (17). We must always have

$$e_1 + e_2 + e_3 + e_4 - G^{(i)} \leq h.$$

In the "worst" case $e_1 = L_{p+1} - 1,\ e_2 = M_{q+1} - 1,\ e_3 = N_{r+1} - 1$, and the corresponding integer $n$ has the regular representation

$$(18)\qquad n = L_{p+1} - 1 + (M_{q+1} - 1)a_2 + (N_{r+1} - 1)a_3 + e_4^{'}a_4,$$

with

$$(19)\qquad L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 + e_4^{'} - G_{pqrs} \leq h.$$

If $Q_{s+1} < U_4 + 1$, then $e_4^{'} = Q_{s+1} - 1$, and we must have

$$L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 + Q_{s+1} - 1 - G_{pqrs} \leq h. \tag{20}$$

The inequality defines a lower bound for $h$. If $Q_{s+1} = U_4 + 1$, then

$$e_4^{'} = h - (L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 - G_{pqrs}) \tag{21}$$

gives an upper bound for $e_4^{'}$.

Each subset with $Q_{s+1} = U_4 + 1$ determines a value $e_4^{'}$ such that all values $n = \sum e_j a_j$ satisfying (16), (17) have $h$-representations, and the value

$$n^{'} = L_{p+1} - 1 + (M_{q+1} - 1)a_2 + (N_{r+1} - 1)a_3 + (e_4^{'} + 1)a_4 \tag{22}$$

does not, but all other values

$$m^{'} = e_1 + e_2 a_2 + e_3 a_3 + (e_4^{'} + 1)a_4,$$

where $e_1,\ e_2,\ e_3$ satisfy (16), (17), do. Let

$$m = \min_{pqr}\{n^{'}\}; \tag{23}$$

then $m$ has no $h$-representation, but all values less than $m$ do, so the $h$-range $n(h, A_k) = m - 1$.

If $Q_{s+1} < U_4 + 1$, then $e_4^{'} = Q_{s+1} - 1 \geq 0$, and the inequality (20) defines a lower bound for $h$. Then $h_0$ is the minimal value of $h$ that satisfies the inequalities (20) in all the cases with $e_4^{'} = Q_{s+1} - 1 = 0$. Let $h_3$ be the minimal value of $h$ such that all the inequalities (20) are satisfied. Then for $h \geq h_3$ the $h$-range $n(h, A_4) \geq \max_i\{s_4^{(i)}\}a_4$, with the index running over all used transfers. From Lemma 1, the basis has the same $h$-range formula for all $h \geq h_3$. If $h_3 > h_0$ and $h_0 \leq h \leq h_3$, then the $h$-range is $m - 1$, (23).

For given $h \geq h_3$, the upper bound on $e_4^{'}$ is

$$h - \max(L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 - G_{pqrs}), \tag{24}$$

where the maximum is taken over all the cases with $Q_{s+1} = U_4 + 1$, see (21). One may also use $e_4^{'}$ to determine the prefactor of the basis; see [12] and Selmer [19].

Let $h \geq h_3$. The integers $n \in [0, ha_4]$ given in regular representation with an $h$-representation may be split into disjoint sets. For each set of integers we perform the procedure above. Let $N$ be the smallest one of the integers $m - 1$, (23) with $Q_{s+1} = U_4 + 1$. Since we have used the possible transfer with the largest gain for each integer, $N$ is the $h$-range of the basis.

The algorithm may be easily modified for a parameter basis $A_4(h)$ where $\gamma$, $\beta$, $G$, $L$, $M$, $N$ and $U$ are either linear expressions in $h$ of the form $ch + d$ or constants. This means that the comparisons may have to be done in two steps. Let $L_1 = c_1 h + d_1$ and $L_2 = c_2 h + d_2$. Then if $c_1 \neq c_2$ we are finished with one comparison. If $c_1 = c_2$ we have to compare $d_1$ with $d_2$ also.

We have described a constructive procedure to determine the $h$-range of a given explicit basis $A_4$ or a parameter basis $A_4(h)$ with a given set of transfers.

In [12] the author used the algorithm for $k = 4$ to determine the $h$-range formulas of the parameter basis that by optimization gave the asymptotic prefactor $c_4$. Also it contributes to the characterization of the $h$-range formulas.

The algorithm requires that all the subsets (16), (17) must be considered in turn. A slightly different approach might reduce the number of subsets which need to be considered.

First, choose $Q_s \leq e_4 \leq Q_{s+1}-1$. Now extract from the set of quintuples $(G^{(i)},\ L^{(i)},\ M^{(i)},\ N^{(i)},\ Q^{(i)})$ just those which satisfy $Q^{(i)} < Q_{s+1}$. We do not need to consider other transfers, because they are not possible for these values of $e_4$. This set of quintuples defines new subdivisions for $e_3$, and there will in general be fewer subdivisions than before. Next, we choose one of these subdivisions $N_r \leq e_3 \leq N_{r+1}-1$, and repeat the process. Finally, when we have chosen subdivisions for $e_4,\ e_3,\ e_2$ and $e_1$ we will have a set of quintuples that describes precisely those transfers which are possible for the subset, and so we have only to choose the one with highest gain.

**Properties of the $h$-range formula.** Since Hofmeister [5] gave explicit formulas for the regular $h$-range of a basis, we assume that at least one transfer must be applied.

**Theorem 1.** *Let $h,k \geq 3$, and let the admissible basis $A_k$ be given in normal form (6). Let $\sum_1^k \epsilon_i a_i = n(h,A_k),\ \epsilon_i \in \mathbf{N}\cup\{0\}$, be the regular representation of the $h$-range. Let us assume $\epsilon_1 < a_2 - 2$. Then*

$$\epsilon_1 = \sum_{i=3}^{k} s_i\beta_1^{(i)} - s_2\gamma_1 - 2,$$

*where $(s_2,\dots,s_k)$ is one of the transfers used for $A_k$. For this transfer to be possible for an integer with regular representation $\sum e_j a_j$, it is at least necessary that*

$$e_1 \geq \sum_{i=3}^{k} s_i\beta_1^{(i)} - s_2\gamma_1.$$

*Proof.* Let $n(h,A_k)=N$. The integer $N+1$ has no $h$-representation. Consider the integer $N+2=\sum \epsilon_i a_i + 2$. Since the basis is admissible, we have one coefficient $\epsilon_j \geq 1$, $j \in [2,k]$. Then the integer

$$M = N+2-a_j = \epsilon_1 + 2 + \sum_{2}^{j-1}\epsilon_i a_i + (\epsilon_j - 1)a_j + \sum_{j+1}^{k}\epsilon_i a_i = \sum_{1}^{k} z_i^{'} a_i,$$

and the representation is regular with $\sum_1^k z_i^{'} > h$. $M$ has an $h$-representation

$$M = \sum_{1}^{k} z_i a_i \text{ with } \sum_{1}^{k} z_i \leq h, \text{ since } M \leq N.$$

If $(s_2,\dots,s_k)$ is the transfer between the two representations for $M$, we have at least one $s_j > 0$, $j\in[2,k]$. The $h$-representation of $M$ can not be used for $M-1$, since $N+1 = M-1+a_j$ would then have an $h$-representation. Hence the representation of $M$ must have $z_1 = 0$, and thus from (7) (with $e_j$ replaced by $z_j^{'}$)

$$0 = z_1^{'} - 0 + s_2\gamma_1 - \sum_{3}^{k} s_i\beta_1^{(i)} = \epsilon_1 + 2 + s_2\gamma_1 - \sum_{3}^{k} s_i\beta_1^{(i)}.$$

From the $h$-range algorithm and Theorem 1 we have

**Theorem 2.** *Let $k=4$, $h\geq 3$, and let the admissible basis $A_4$ be given in normal form (6) with $\gamma_2 \geq 3$, $\beta_2^{(4)} \geq 1$ and $2a_2 > \beta_1^{(3)} + \beta_1^{(4)} > a_2$. Let the used transfers* *of the basis be* $T^{(j)} = (s_2^{(j)}, s_3^{(j)}, s_4^{(j)})$, $j = 1, \dots, \eta$. *Let the regular representation of the h-range of the basis be*

$$n = \epsilon_1 + \epsilon_2 a_2 + \ldots + \epsilon_4 a_4. \tag{25}$$

*Then*

$$\epsilon_1 = \begin{cases} \sum_{i=3}^{4} s_i^{(j_1)} \beta_1^{(i)} - s_2^{(j_1)} \gamma_1 - 2, \\ \gamma_1 - 2, \end{cases}$$

$$\epsilon_2 = \begin{cases} s_4^{(j_2)} \beta_2^{(4)} - s_3^{(j_2)} \gamma_2 + s_2 - 1, \\ \gamma_2 - \beta_2^{(4)} - 2 - \delta, \\ \gamma_2 - 1 - \delta, \end{cases}$$

$$\epsilon_3 = \begin{cases} s_3^{(j_3)} - s_4^{(j_3)} \gamma_3 - 1, \\ \gamma_3 - 1 - \delta, \end{cases}$$

$$\epsilon_4 = h - \sum_{i=1}^{3} \epsilon_i + g,$$

*where* $g$ *is the gain of the possible transfer of* $n$ *with the largest gain. For at most one value of* $l \in \{2, 3\}$ *we have* $\epsilon_l = \gamma_l - 1$. *Here* $\delta = 0$ *or* $\delta = 1$, $j_1, j_2, j_3 \in \{1, 2, \dots, \eta\}$.

*Proof.* From the $h$-range algorithm we have that the values of $\epsilon_l$ are given by either the conditions for the transfers to be possible or the conditions for $n$ to be in regular representation, [12]. In the algorithm we may have $p = r_1$, giving $L_{r_1+1} = \gamma_1$ and, from (23), $\epsilon_1 = \gamma_1 - 2$. If $p < r_1$ we find $\epsilon_1$ from the algorithm or Theorem 1. The possible transfer of $n$ with the largest gain and with the conditions on the $e_j$ such that we can have $e_j \le \epsilon_j$, $j = 1, 2, 3$, gives the gain $g \ge 0$. If no possible transfer for $n$ exists, then $g = 0$.

**Conjecture 1.** *For* $k \ge 3$, *there exist an* $h_s \in \mathbf{N}$, *a set of transfers* $T^{(j)} = (s_2^{(j)}, \dots, s_k^{(j)})$, $j = 1, \dots, \eta$, *and a* $\sigma \in [1, \eta]$ *such that for* $h > h_s$ *we have the extremal parameter basis* $A_k^*(h)$ *given in normal form* (6) *uses the transfers* $T^{(j)}$, $j = 1, \dots, \eta$. *If the regular representation of the h-range of the basis is*

$$n = \epsilon_1 + \epsilon_2 a_2 + \ldots + \epsilon_k a_k,$$

*then*

$$\epsilon_1 = \sum_{i=3}^{k} s_i^{(\sigma)} \beta_1^{(i)} - s_2^{(\sigma)} \gamma_1 - 2,$$

$$\epsilon_l = \gamma_l - 2,$$

*for* $l = 2, 3, \dots, k-1$, *and*

$$\epsilon_k = h - \sum_{i=1}^{k-1} \epsilon_i + g,$$

*where* $g$ *is the gain of the possible transfer of* $n$ *with the largest gain.*

Also from a numerical point of view the conjecture is quite interesting, to find a upper bound for a given basis.

For $k = 3$, $h > 22$, the (0,1)-transfer with the condition $e_1 \ge \beta_1^{(3)} = \beta$, see (7), is the only transfer used for the $A_3^*$ basis, Hofmeister [4]. But with $\epsilon_1 = \beta - 2$, we cannot apply it on $n(h, 3)$. Hence, the extremal $h$-range $n(h, 3)$ is a minimal regular $h$-representation. The extremal bases for $k = 3$ and $h \ge 6$ have $\epsilon_2 = \gamma_2 - 2$.

All the known extremal bases for $k = 4$ are determined numerically and have for $h \geq 43$ and 23 other values, $6 \leq h < 42$, $\epsilon_2 = \gamma_2 - 2$ and $\epsilon_3 = \gamma_3 - 2$. See Challis [2], Mossige [10] and [11].

## 3. The conjecture in the case $k = 4$

Let $k = 4$, $h = 12\alpha t + i$, $i \in [0, 11]$, $\alpha \geq 1$, $\alpha \in \mathbf{Q}$, $t \in \mathbf{N}$. The parameter basis $A_M = A_M(h, b, p)$ we are going to use in normal form is ($a_1 = 1$)

$$\begin{aligned} a_2 &= 9\alpha t + b_1 t + p_1, \\ a_3 &= (3\alpha t + b_3 t + p_3)a_2 - (5\alpha t + b_2 t + p_2), \\ a_4 &= (2\alpha t + b_6 t + p_6)a_3 - (\alpha t + b_5 t + p_5)a_2 - (6\alpha t + b_4 t + p_4), \end{aligned} \tag{26}$$

where $b_l, p_l \in \mathbf{Z}$ (to be chosen suitablely) and where we put $b = (b_1, b_2, \ldots, b_6)$ and $p = (p_1, p_2, \ldots, p_6)$. We shall also consider the basis $A_S = A_S(h, b, p)$, given by replacing the coefficient 5 in (26) by 7 and the coefficient 6 by 4. Let $A_M = A_M(h, b)$ be the basis (26) with $p = (0, \ldots, 0)$, and similarly, $A_S = A_S(h, b)$.

Since 1971, the "record" prefactor $\varphi = 2$ was held by the parameter basis $A_M(h, b)$ discovered by Hofmeister and Schell [5] with

$$b = (0, 0, \ldots, 0), \quad \alpha = 1 \tag{27}$$

and the transfers that give a positive gain

$$T_1 = (0, 1, 0)\,, \quad T_2 = (0, 0, 1)\,, \quad T_3 = (1, 1, 2)\,, \quad T_4 = (1, 0, 2). \tag{28}$$

In 1988 Braunschädel [1] gave the basis $A_S$ with (27), using the transfers

$$T_1,\ T_2,\ T_3,\ T_4^{'} = (0, 0, 2). \tag{29}$$

He examined (on a computer) all bases $A_4(h)$ of the form,

$$h = Ht;\ a_2 = c_1 t,\ a_3 = c_2 t a_2 - c_3 t,\ a_4 = c_4 t a_3 - c_5 t a_2 - c_6 t, \tag{30}$$

with $c_l \in \mathbf{N}$, allowing only $(s_2, s_3, s_4)$-transfers with $s_2, s_3, s_4 \leq 2$. He then always found $\varphi \leq 2$, and $\varphi = 2$ only for the bases $A_M(h, b)$ and $A_S(h, b)$ with (27) (see also Selmer [19]).

The author's idea was to make small variations of the leading coefficients of the elements of the basis (26), by varying $b$ around the six-tuple $(0, \ldots, 0)$, to see whether an increase of the prefactor is possible. Let

$$b_M = (15, 1, -15, 6, -13, -20). \tag{31}$$

In 1985 he found a basis $A_M(h, b_M)$ with $\varphi > 2$ (see [12]):

To get the prefactor $\varphi$ of this basis we consider the polynomial

$$g(\gamma) = -32\gamma - 168\gamma^2 - 22\gamma^3 + 3\gamma^4$$

and determine the solution $\gamma_1$ of $g^{'}(\gamma_1) = 0$, where

$$\gamma_1 = \frac{11}{6} + \frac{1}{3}\sqrt{457}\cos\frac{\xi + 4\pi}{3},\ \cos\xi = \frac{7163}{\sqrt{457^3}},\ \text{with } 0 < \xi < \pi/2,$$

$$\text{giving } \gamma_1 = -0.09712372...$$

With this $\gamma_1$ we put $\alpha_1 = -20/\gamma_1$ and

$$\sigma = 2 + 3^{-1}2^{-6}g(\gamma_1) = 2.0080397... \tag{32}$$

For given $\varepsilon > 0$ we can choose $t$ so large that for $h = 12\lfloor \alpha_1 t \rfloor$ the basis $A_M(h, b_M)$ has the prefactor

$$\varphi > \sigma - \varepsilon.$$

In fact, here $\sigma$ is a *cubic irrationality*, and can only be *approximated* by "rational" bases (26). We obtain a very good approximation if we put $\alpha = 206$, that is, $h = 2472t$, giving $\sigma$ of (32) with all seven decimals correct. As usual, $\lfloor x \rfloor$ denotes the largest integer $\leq x \in \mathbf{R}$.

In [11] we developed formulas for the possible $h$-ranges of the parameter bases $A_M(h, b, p)$, and based the optimization on the determination of the local $h$-range $n(h, A_M)$. In addition to the transfers (28), we discovered that it was possible to use

$$T_5 = (1, 2, 1)\ , \quad T_6 = (1, 0, 3). \tag{33}$$

In spite of the very small improvement on $\varphi = 2$, this result gave quite a new situation. Let

$$b_S = (15, -1, -15, 2, -13, -20). \tag{34}$$

In 1988 Selmer [19] showed that also the basis $A_S(h, b_S)$ has the prefactor (32). In [13] we show that my cited result (31), (32) for the basis$A_M(h, b_M)$ is valid also for the basis $A_S(h, b_S)$ with $b_M$ replaced by $b_S$. Selmer [18] calls the two bases an *associate* pair of bases.

**Computational results.** When we apply our $h$-range algorithm to the parameter bases $A_M$, (26) and $A_S$, it gives for each basis the sufficient inequalities that express the conditions that all the integers $n \in [1, n(h, A)]$ have an $h$-representation and it gives all the $h$-range formula candidates. By extensive computations for $h \leq 620000$ we came to two constructions of two bases. For details see [13].

**Construction 1.** Given $h = 12\alpha t + i \geq 1236$, where $\alpha \in \mathbf{Q}$, $t \in \mathbf{N}$ and $0 \leq i \leq 11$. Let $b_M = (15, 1, -15, 6, -13, -20)$. Let $P_i = (p_1, p_2, \ldots, p_6)$ and $r$ be given by Table 1. Let $\beta = \alpha t = \lfloor h/12 \rfloor$, $i = h - \lfloor h/12 \rfloor 12$ and $q = r + i$. The basis $A(t)$ has the elements ($a_1 = 1$)

$$\begin{aligned}
a_2 &= 9\beta + 15t + p_1, \\
a_3 &= (3\beta - 15t + p_3)a_2 - (5\beta + t + p_2), \\
a_4 &= (2\beta - 20t + p_6)a_3 - (\beta - 13t + p_5)a_2, -(6\beta + 6t + p_4),
\end{aligned}$$

and $h$-range formula

$$\begin{aligned}
n(t) &= (3\beta + 45t + q + 1)a_4 + (2\beta - 20t + p_6 - 2)a_3 \\
&\quad +(3\beta - 15t + p_3 - 2)a_2 + 5\beta + t + p_2 - 2.
\end{aligned}$$

Let $j = \lfloor \beta/\alpha_1 \rfloor$, where $\alpha_1$ is given in the cited result (31), (32). If $i = 0$ then put $j = \lfloor \beta/\alpha_1 \rfloor + 1$. If $n(j+1) > n(j)$ then $t = j + 1$, else $t = j$.

Then the basis $A_M^* = A_M(h, b_M, P_i) = A(t)$ has $h$-range

$$n(h, A_M^*) = n(t).$$

For $i$ even, $h - h_0 = 1$. For $i$ odd, $h - h_0 = 0$.

**Construction 2.** Given $h = 12\alpha t + i \geq 1236$, where $\alpha \in \mathbf{Q}$, $t \in \mathbf{N}$ and $0 \leq i \leq 11$. Let $b_S = (15, -1, -15, 2, -13, -20)$. Let $P_i = (p_1, p_2, \ldots, p_6)$ and $r$ be given by

TABLE 1

| $i$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $r$ |
|---|---|---|---|---|---|---|---|
| 0 | $-8$ | 0 | 10 | $-3$ | 7 | 13 | $-25$ |
| 1,2 | $-4$ | 1 | 8 | $-1$ | 5 | 10 | $-19$ |
| 3,4 | 0 | 2 | 6 | 1 | 3 | 7 | $-13$ |
| 5,6 | 4 | 3 | 4 | 3 | 1 | 4 | $-7$ |
| 7,8 | 8 | 4 | 2 | 5 | $-1$ | 1 | $-1$ |
| 9,10 | 12 | 5 | 0 | 7 | $-3$ | $-2$ | 5 |
| 11 | 16 | 6 | $-2$ | 9 | $-5$ | $-5$ | 11 |

TABLE 2

| $i$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $r$ |
|---|---|---|---|---|---|---|---|
| 0 | $-12$ | 1 | 14 | $-1$ | 11 | 19 | $-38$ |
| 1,2 | $-8$ | 2 | 12 | 0 | 9 | 16 | $-32$ |
| 3,4 | $-4$ | 3 | 10 | 1 | 7 | 13 | $-26$ |
| 5,6 | 0 | 4 | 8 | 2 | 5 | 10 | $-20$ |
| 7,8 | 4 | 5 | 6 | 3 | 3 | 7 | $-14$ |
| 9,10 | 8 | 6 | 4 | 4 | 1 | 4 | $-8$ |
| 11 | 12 | 7 | 2 | 5 | $-1$ | 1 | $-2$ |

Table 2. Let $\beta = \alpha t = \lfloor h/12 \rfloor$, $i = h - \lfloor h/12 \rfloor 12$ and $q = r + i$. The basis $A(t)$ has the elements ($a_1 = 1$)

$$\begin{aligned} a_2 &= 9\beta + 15t + p_1, \\ a_3 &= (3\beta - 15t + p_3)a_2 - (7\beta - t + p_2), \\ a_4 &= (2\beta - 20t + p_6)a_3 - (\beta - 13t + p_5)a_2, -(4\beta + 2t + p_4), \end{aligned}$$

and $h$-range formula

$$\begin{aligned} n(t) &= (3\beta + 45t + q + 1)a_4 + (2\beta - 20t + p_6 - 2)a_3 \\ &\quad + (3\beta - 15t + p_3 - 2)a_2 + 4\beta + 2t + p_4 - 2. \end{aligned}$$

Let $j = \lfloor \beta/\alpha_1 \rfloor$, where $\alpha_1$ is given in the cited result (31), (32). If $i \leq 4$, then put $j = \lfloor \beta/\alpha_1 \rfloor + 1$. If $n(j+1) > n(j)$, then $t = j + 1$, else $t = j$.

Then the basis $A_S^* = A_S(h, b_S, P_i) = A(t)$ has $h$-range

$$n(h, A_S^*) = n(t).$$

For $i$ even, $h - h_0 = 1$. For $i$ odd, $h - h_0 = 0$.

Two parameter bases $A^{(1)}(h)$ and $A^{(2)}(h)$ is said to be asymptotically equal if $a_j^{(1)}/a_j^{(2)} \to 1$ when $h \to \infty$ for $2 \leq j \leq 4$.

In an unpublished work from 1991, Kirfel and the author have shown the following result. For $h \to \infty$, all the bases $A = A(h)$ with prefactor $\varphi > 2.008$ are either asymptotically equal to $A_M(h, b_M)$ or equal to $A_S(h, b_S)$. The sets $b_M$ and $b_S$ are given in (31) and (34) respectively. In [12] it is shown that such bases exist.

Let $\mathcal{A}_M(h, b_S)$ denote the class of all bases $A = A(h)$ that are asymptotically equal to $A_M(h, b_M)$, and let $\mathcal{A}_S(h, b_S)$ be the similar class for $A_S(h, b_S)$.

We now ask for the best choice of the basis $A = A(h)$ in the class $\mathcal{A}_M(h, b_M)$ and the best choice of the basis in $\mathcal{A}_S(h, b_S)$.

Extensive computations for $h \leq 620000$ give the following results.

*Result A.* For $9793 \leq h \leq 620000$, $A_M^* = A_M(h)$ in the class $\mathcal{A}_M(h, b_M)$ is the basis with the largest $h$-range.

*Result B.* For $10653 \leq h \leq 620000$, $A_S^* = A_S(h)$ in the class $\mathcal{A}_S(h, b_S)$ is the basis with the largest $h$-range.

*Result C.* For $11385 \leq h \leq 620000$, $A_M^*$ has larger $h$-range than $A_S^*$.

For both the bases $A_M^*$ and $A_S^*$ the $h$-range formulas are of the type stated in Conjecture 1.

**Conjecture 2.** *For* $h \geq 11385$, $A_M^* = A_M(h)$ *is the extremal basis.*

For $h < 11385$ we may get better bases when we replace the set $b_M$ or $b_S$ by other sets $b$ and suitable sets $p$.

Using a result of Selmer [18], we prove in [13] that if $b = (b_1, \dots, b_6)$ and

$$b^{'} = (b_1, b_1 - b_2 + b_3, b_3, b_1 - b_4 - b_5 + b_6, b_5, b_6),$$

then for each basis $A_M(h, b)$ with prefactor $\varphi$ there is a basis $A_S(h, b^{'})$ with the same prefactor, and vice versa. If $b = b^{'}$, then

$$2b_2 = b_1 + b_3, \qquad 2b_4 = b_1 - b_5 + b_6.$$

For further details see [13].

## References

[1] R. Braunschädel, Zum Reichweitenproblem, Diplomarbeit, Math. Inst., Joh. Gutenberg-Univ., Mainz 1988.

[2] M.F.Challis, Two new techniques for computing extremal $h$-bases $A_k$, Computer J. 36 (1993), 117-126.

[3] G. Hofmeister, Über eine Menge von Abschnittbasen, J. Reine Angew. Math. 213 (1963) 43-57. MR **31:**149

[4] G. Hofmeister, Asymptotische Abschätzungen für dreielementige Extremalbasen in natürlichen Zahlen, J. Reine Angew. Math. 232 (1968) 77-101. MR **38:**1068

[5] G. Hofmeister, Zum Reichweitenproblem, Mainzer Seminarberichte in Additiven Zahlentheorie, 1 (1983), 30-52.

[6] G. Hofmeister, C. Kirfel and H. Kolsdorf, Extremale Reichweiten, Inst. Rep. No 60, Dept. of pure Math., Univ. of Bergen, 1991.

[7] C. Kirfel, On Extremal Bases for the $h$-range Problem, I, II, Inst. Rep. Nos. 53, 55, Dept. of pure Math., Univ. of Bergen, 1989, 1990.

[8] C. Kirfel, Extremale asymptotische Reichweitenbasen, Acta Arith. (1992) 279-288. MR **92m:**11012

[9] W.F. Lunnon, A Postage Stamp Problem, Compt. J. 12 (1969), 377-380. MR **40:**6745

[10] S. Mossige, Algorithms for computing the $h$-range of the Postage Stamp Problem, Math. Comp., 36 (1981), 575-582. MR **82e:**11095

[11] S. Mossige, On the extremal $h$-range of the Postage Stamp Problem with four Stamp denominations, Dissertation, Inst. Rep. No. 41, Dept. of pure Math., Univ. of Bergen, 1986.

[12] S. Mossige, On extremal $h$-bases $A_4$, Math. Scand. 61 (1987), 5-16. MR **89e:**11008

[13] S. Mossige, The Postage Stamp Problem. An algorithm to determine the $h$-range. On the $h$-range formula. On the extremal basis problem for $k = 4$, Inst. Rep. No. xx, Dept. of pure Math., Univ. of Bergen, 1995, 1-74.

[14] A. Mrose, Die Bestimmung der extremalen regulären Abschnittbasen mit Hilfe einer Klasse von Kettenbruchdeterminanten, Dissertation, Freie Universität Berlin, 1969.

[15] Ø. Rødseth, An upper bound for the $h$-range of the Postage Stamp Problem, Acta Arith. 54 (1990), 301-306. MR **91h:**11013

[16] J. Riddell and C. Chan, Some extremal 2-bases, Math. Comp. 32(1978), 630-634. MR **57:**16244

[17] E.S. Selmer, The Local Postage Stamp Problem, I-III, Inst. Rep. Nos. 42, 44, 47, Dept. of Pure Math., Univ. of Bergen, 1986, 1990.

[18] E.S. Selmer, Associate Bases in the Postage Stamp Problem, J. Number Theory, 42, 3,(1992), 320-336. MR **94b:**11012

[19] E.S. Selmer, Asymptotic $h$-ranges and dual bases, Inst. Rep. No. 56, Dept. of Pure Math., Univ. of Bergen, 1990.

[20] A. Stöhr, Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe, I, J. Reine Angew. Math. 194 (1955), 40-65. MR **17:**713a

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