THE POSTAGE STAMP PROBLEM:
AN ALGORITHM TO DETERMINE THE h-RANGE
ON THE h-RANGE FORMULA
ON THE EXTREMAL BASIS PROBLEM FOR k = 4

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Abstract. Given an integral “stamp” basis $A_k$ with $1 = a_1 < a_2 < \ldots < a_k$ and a positive integer $h$, we define the $h$-range $n(h, A_k)$ as

$$n(h, A_k) = \max\{N \in \mathbb{N} \mid n \leq N \implies n = \sum_{i=1}^{k} x_i a_i, \sum_{i=1}^{k} x_i \leq h, \ n, x_i \in \mathbb{N}_0\}.$$ 

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For given $h$ and $k$, the extremal basis $A_k^*$ has the largest possible extremal $h$-range

$$n(h, k) = n(h, A_k^*) = \max_{A_k} n(h, A_k).$$

We give an algorithm to determine the $h$-range. We prove some properties of the $h$-range formula, and we conjecture its form for the extremal $h$-range.

1. Background

Given an integral basis $A_k = \{a_1, a_2, \ldots, a_k\}$ with $1 = a_1 < a_2 < \ldots < a_k$ and a positive integer $h$, we define the $h$-range $n(h, A_k)$ as

$$n(h, A_k) = \max\{N \in \mathbb{N} \mid n \leq N \implies n = \sum_{i=1}^{k} x_i a_i, \sum_{i=1}^{k} x_i \leq h, \ n, x_i \in \mathbb{N}_0\}.$$ 

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The integer $n \in \mathbb{N}$ has an $h$-representation by $A_k$ if

$$n = \sum_{i=1}^{k} x_i a_i \mid \sum_{i=1}^{k} x_i \leq h, \ x_i \in \mathbb{N}_0.$$ 

We consider only bases $A_k$ which are $h$-admissible, that is,

$$a_k \leq n(h, A_k).$$

For given $h$ and $k$, the extremal basis $A_k^*$ has the largest possible extremal $h$-range

$$n(h, k) = n(h, A_k^*) = \max_{A_k} n(h, A_k).$$

A popular interpretation arises if we consider the integers $a_i$ as stamp denominations and $h$ as the “size of the envelope.” More information about the “postage
“stamp problem” can be found in E. S. Selmer’s comprehensive research monograph [17]. Here we mainly use Selmer’s notation and presentation.

In the beginning, the main interest was centered around the global aspect, to find an extremal basis \( A_\ast^\kappa \) with extremal \( h \)-range. The “local” aspect is: Determine \( n(h, A_\kappa) \) when \( h, k \) and a particular basis \( A_\kappa \) are given.

In the global case, a convenient approach is to keep \( k \) fixed and let \( h \) increase, asking for asymptotic values of the extremal \( h \)-range \( n(h, k) \). We can also ask for asymptotic values of “local” \( h \)-ranges \( n(h, A_\kappa) = n(h, A_\kappa(h)) \), when the basis elements \( a_i \) are given functions of \( h \). We shall call such bases \( A_\kappa(h) \) parameter bases.

Let \( \varphi \) be the prefactor defined by

\[
n(h, A_\kappa(h)) = \varphi \left( \frac{h}{k} \right)^k (1 + o(1)).
\]

(1)

Both the local and the global problems are trivial for \( k = 2 \), Stöhr [20]. The extremal bases \( A_3^\ast \) were determined by Hofmeister [4], [5]. For \( k \geq 4 \), our knowledge is much more limited. The best known general upper bound is due to Rødseth [15]:

\[
n(h, k) \leq \frac{(k - 1)^{k - 2}}{(k - 2)!} \left( \frac{h}{k} \right)^k + \mathcal{O}(h^{k-1}).
\]

For \( k = 4 \), the prefactor \( \varphi = 4.5 \) is far too large, and Kirfel [7] has the strongest published result:

\[
n(h, 4) \leq 2.35 \left( \frac{h}{4} \right)^4 + \mathcal{O}(h^3).
\]

In [12] the author proved the lower bound

\[
n(h, 4) \geq 2.008 \left( \frac{h}{4} \right)^4 + \mathcal{O}(h^3).
\]

The proof consists in determining a parameter basis \( A_4 = A_4(h) \) whose \( h \)-range equals the bound given. However (May 1991, unpublished), Kirfel and the author have shown that the lower bound 2.008... (more decimals in (32)) is really sharp. Hence, it is natural to investigate the local extremal parameter bases for \( k = 4 \).

For \( k = 5 \), Kolsdorf in [6] has given a parameter basis with asymptotic \( h \)-range 3.06\((h/5)^5\).

It was shown by Kirfel [8] that the limit

\[
c_k = \lim_{h \to \infty} n(h, k)/(h/k)^k
\]

(2)

really exists for all \( k \geq 2 \). It is known that \( c_2 = 1 \), \( c_3 = 4/3 \), and \( c_4 = 2.008... \).

Looking for the extremal bases, we consider parameter bases \( A_k(h) \) for which

\[
n(h, A_k(h)) \text{ has order of magnitude } h^k.
\]

(3)

For the basis elements, this implies that \( a_i(h) \) has order of magnitude \( h^{i-1} \), \( i = 2, 3, \ldots, k \).

**Representations and gain.** The regular representation of \( n \) by \( A_k \),

\[
n = \sum_{i=1}^{k} c_i a_i,
\]

(4)

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where $\gamma$ represents by $A$

For all $k \in A_k$, $i = 2, 3, \ldots, k$, we write

$$a_i = \gamma_{i-1} a_{i-1} - \sum_{j=1}^{i-2} \beta_j a_j,$$

where $\gamma_{i-1} = [a_i/a_{i-1}] \geq 2$, and $\sum_{j=1}^{i-2} \beta_j a_j = \gamma_{i-1} a_{i-1} - a_i$ is the regular representation by $A_{i-2}$. As usual, $[x]$ denotes the smallest integer $\geq x \in \mathbb{R}$. Hofmeister [5] calls (6) the normal form of the basis $A_k$. Let $n \in \mathbb{N}$ have a regular representation (4) by $A_k$, and let $s_i \in \mathbb{Z}$, $i = 2, 3, \ldots, k$. From (6) we get a new representation $n = \sum z_j a_j$ by an $(s_2, s_3, \ldots, s_k)$-transfer:

$$n = \sum_{i=1}^{k} e_i a_i + \sum_{i=2}^{k} s_i \left( \gamma_{i-1} a_{i-1} - a_i - \sum_{j=1}^{i-2} \beta_j a_j \right)$$

(7)

$$= \sum_{j=1}^{k} \left( e_j - s_j + s_{j+1} \gamma_j - \sum_{i=j+2}^{k} s_i \beta_j a_j \right) a_j = \sum_{j=1}^{k} z_j a_j,$$

with $s_1 = s_{k+1} = \gamma_k = 0$. We say that the transfer is possible if $z_j \geq 0$, $j = 1, \ldots, k$.

The sum of the reductions in the coefficients is the gain $G(s_2, s_3, \ldots, s_k)$ in the transfer:

$$G(s_2, s_3, \ldots, s_k) = \sum_{j=1}^{k} (e_j - z_j).$$

The usefulness of such transfers stems from the following result of Hofmeister [5]: Every “legal” representation $n = \sum z_i a_i$ $(z_i \geq 0)$ can be obtained from the regular representation by a suitable $(s_2, s_3, \ldots, s_k)$-transfer with all $s_i \geq 0$. We also cite another result of Hofmeister [5]: If a parameter basis $A_k(h)$ satisfies (3) and is expressed in normal form (6), then the $s_i$ of any possible $(s_2, s_3, \ldots, s_k)$-transfer are bounded as $h \to \infty$. See also Kirfel [7].

In 1963, Hofmeister [5], [3] gave formulas for the regular $h$-range of a basis. If only regular $h$-representations are allowed, we get the regular $h$-range. He also conjectured the formula for the extremal regular $h$-range, later proved by Mrose [14].

Let

$$h_0 = h_0(A_k) = \min\{h \in \mathbb{N} \mid a_k \leq n(h, A_k)\}.$$

For all $k$ and $h \geq h_0$ we trivially have

$$n(h+1, A_k) \geq n(h, A_k) + a_k.$$

Furthermore, Selmer [17] proved that, for arbitrary $k$ and $h \geq h_0$,

$$n(h, A_k) \geq (h+1)a_{k-1} - a_k$$

implies

$$n(h+1, A_k) = n(h, A_k) + a_k.$$

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For all $k$ and $h \geq h_0$ we trivially have

$$n(h+1, A_k) \geq n(h, A_k) + a_k.$$
If $h$ is increased by 1, the right-hand side of (10) increases with $a_{k-1}$, while the left-hand side increases with at least $a_k$. There is consequently an $h_1 \geq h_0$ such that (10) and hence (11) are satisfied for all $h \geq h_1$. This means that for given $h$, $h \geq h_1$, we have

\begin{equation}
(12) \quad n(h, A_k) = n(h_1, A_k) + (h-h_1)a_k.
\end{equation}

We see that for a basis $A_k$ there may be different $h$-range formulas according to the value of $h$, $h_0 \leq h \leq h_1$. From (12), the $h$-range formula is the same for all $h \geq h_1$. In looking for bases with large $h$-range, we often have the same $h$-range formula for all $h \geq h_0$.

**Lemma 1.** Let the basis $A_k$ and the possible transfers $T^{(i)} = (s_2^{(i)}, s_3^{(i)}, \ldots, s_k^{(i)})$, $i = 1, 2, \ldots, \eta$, be given. Let

\begin{equation}
(13) \quad h_2 = \min\{h | n(h, A_k) \geq \max_i s_k^{(i)} a_k\}.
\end{equation}

Then for $h \geq h_2$

\begin{equation}
(14) \quad n(h, A_k) = n(h_2, A_k) + (h-h_2)a_k.
\end{equation}

**Proof.** The minimal representation of a positive integer is independent of the value of $h$. For $h \geq h_2$ we can use all the transfers. From above we know that for $h \geq h_1$ the $h$-range is determined by (12) and we have $h_1 \leq h_2$. Note that only the transfers actually used determine $h_2$.

2. The $h$-Range Algorithm

In the literature we find more or less general $h$-range algorithms by Lunnon [9], Riddell and Chan [16], Mossige [10], and Challis [2].

Let the basis $A_4$ and the possible transfers be given. For each integer $n \in [1, n(h, A_4)]$ given in a regular representation $\sum e_j a_j$, we use the possible transfer with the largest gain to give the minimal representation of $n, \sum z_j a_j$. It satisfies the inequality $\sum e_j - \text{gain} = \sum z_j \leq h$. The algorithm gives sufficient such inequalities that express the conditions that all the integers $n$ have an $h$-representation. The least integer $n$ with $n+1$ not having an $h$-representation is the $h$-range. For a given basis, the algorithm determines $h_0$ and from which $h \geq h_0$ the $h$-range formula is the same. The result is valid for all $h \geq h_0$.

We give the algorithm for $k = 4$, but it may be generalized to $k > 4$.

Now, let the possible transfers $T^{(i)} = (s_2^{(i)}, s_3^{(i)}, s_4^{(i)}), i = 1, \ldots, \eta$, for the basis $A_4$ be given. Then the minimal representation of an integer $n > 0$ is independent of $h$.

The upper bounds for the $e_j$'s are given such that the representation (4) is regular. The conditions for the transfers to be possible give lower bounds for the $e_j$'s. The coefficients $z_j$ of (7) must be $\geq 0$, giving lower bounds on the $e_j$'s. The gain (reduction of coefficient sum) must be positive.

We get the following values of the gain and the lower bounds for $e_j$'s:

\begin{equation}
(15) \quad G_i = s_2^{(i)}(-\gamma_1 + 1) + s_3^{(i)}(\beta_1^{(3)} - \gamma_2 + 1) + s_4^{(i)}(\beta_1^{(4)} + \beta_2^{(4)} - \gamma_3 + 1) \geq 1,
\end{equation}
The corresponding gain $G_i$ is $G_{pqrs}$, is then the largest one which can be used in the case (16), (17). We must always have
\[ e_1 + e_2 + e_3 + e_4 - G(i) \leq h. \]
In the “worst” case $e_1 = L_{p+1} - 1$, $e_2 = M_{q+1} - 1$, $e_3 = N_{r+1} - 1$, and the corresponding integer $n$ has the regular representation
\[ n = L_{p+1} - 1 + (M_{q+1} - 1)a_2 + (N_{r+1} - 1)a_3 + e'_4a_4, \]
where
\[ (15) \quad 0 \leq p \leq r_1, \quad 0 \leq q \leq r_2, \quad 0 \leq r \leq r_3, \quad 0 \leq s \leq r_4. \]
Let $e_1, e_2, e_3$ and $e_4$ be given such that
\[ (16) \quad L_p \leq e_1 \leq L_{p+1} - 1, \quad M_q \leq e_2 \leq M_{q+1} - 1, \]
\[ (17) \quad N_r \leq e_3 \leq N_{r+1} - 1, \quad Q_s \leq e_4 \leq Q_{s+1} - 1. \]
We then scan the quintuples $(G(i), L(i), M(i), N(i), Q(i)), i = 1, 2, \ldots, \eta + 1,$ and register the first time (largest gain) such that
\[ L(i) < L_{p+1}, \quad M(i) < M_{q+1}, \quad N(i) < N_{r+1}, \quad Q(i) < Q_{s+1}. \]
The corresponding gain $G(i) = G_{pqrs}$ is then the largest one which can be used in the case (16), (17). We must always have
\[ e_1 + e_2 + e_3 + e_4 - G(i) \leq h. \]
In the “worst” case $e_1 = L_{p+1} - 1$, $e_2 = M_{q+1} - 1$, $e_3 = N_{r+1} - 1$, and the corresponding integer $n$ has the regular representation
\[ (18) \quad n = L_{p+1} - 1 + (M_{q+1} - 1)a_2 + (N_{r+1} - 1)a_3 + e'_4a_4, \]
with
\[ (19) \quad L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 + e'_4 - G_{pqrs} \leq h. \]
If \( Q_{n+1} < U_4 + 1 \), then \( e_4' = Q_{n+1} - 1 \), and we must have
\[
(20) \quad L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 + Q_{s+1} - 1 - G_{pqrs} \leq h.
\]
The inequality defines a lower bound for \( h \). If \( Q_{n+1} = U_4 + 1 \), then
\[
(21) \quad e_4' = h - (L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 - G_{pqrs})
\]
gives an upper bound for \( e_4' \).

Each subset with \( Q_{n+1} = U_4 + 1 \) determines a value \( e_4' \) such that all values
\[ n = \sum c_j a_j \] satisfying (16), (17) have \( h \)-representations, and the value
\[
(22) \quad n' = L_{p+1} - 1 + (M_{q+1} - 1)a_2 + (N_{r+1} - 1)a_3 + (e_4' + 1)a_4
\]
does not, but all other values
\[
m' = e_1 + e_2 a_2 + e_3 a_3 + (e_4' + 1)a_4,
\]
where \( e_1, e_2, e_3 \) satisfy (16), (17), do. Let
\[
(23) \quad m = \min_{pq} \{ n' \};
\]
then \( m \) has no \( h \)-representation, but all values less than \( m \) do, so the \( h \)-range
\( n(h, A_k) = m - 1 \).

If \( Q_{n+1} < U_4 + 1 \), then \( e_4' = Q_{n+1} - 1 \geq 0 \), and the inequality (20) defines a lower bound for \( h \). Then \( h_0 \) is the minimal value of \( h \) that satisfies the inequalities (20) in all the cases with \( e_4' = Q_{n+1} - 1 = 0 \). Let \( h_1 \) be the minimal value of \( h \) such that all the inequalities (20) are satisfied. Then for \( h \geq h_3 \) the \( h \)-range
\( n(h, A_4) \geq \max_i (\sigma(i_4)) a_4 \), with the index running over all used transfers. From
Lemma 1, the basis has the same \( h \)-range formula for all \( h \geq h_3 \). If \( h_3 > h_0 \) and
\( h_0 \leq h \leq h_3 \), then the \( h \)-range is \( m - 1 \), (23).

For given \( h \geq h_3 \), the upper bound on \( e_4' \) is
\[
(24) \quad h - \max (L_{p+1} - 1 + M_{q+1} - 1 + N_{r+1} - 1 - G_{pqrs}),
\]
where the maximum is taken over all the cases with \( Q_{n+1} = U_4 + 1 \), see (21). One
may also use \( e_4' \) to determine the prefactor of the basis; see [12] and Selmer [19].

Let \( h \geq h_3 \). The integers \( n \in [0, h a_4] \) given in regular representation with an
\( h \)-representation may be split into disjoint sets. For each set of integers we perform
the procedure above. Let \( N \) be the smallest one of the integers \( m - 1 \), (23) with
\( Q_{n+1} = U_4 + 1 \). Since we have used the possible transfer with the largest gain for
each integer, \( N \) is the \( h \)-range of the basis.

The algorithm may be easily modified for a parameter basis \( A_4(h) \) where \( \gamma \),
\( \beta \), \( G \), \( L \), \( M \), \( N \) and \( U \) are either linear expressions in \( h \) of the form \( ch + d \) or
constants. This means that the comparisons may have to be done in two steps. Let
\( L_1 = c_1 h + d_1 \) and \( L_2 = c_2 h + d_2 \). Then if \( c_1 \neq c_2 \) we are finished with one
comparison. If \( c_1 = c_2 \) we have to compare \( d_1 \) with \( d_2 \) also.

We have described a constructive procedure to determine the \( h \)-range of a given
explicit basis \( A_4 \) or a parameter basis \( A_4(h) \) with a given set of transfers.

In [12] the author used the algorithm for \( k = 4 \) to determine the \( h \)-range formulas
of the parameter basis that by optimization gave the asymptotic prefactor \( c_4 \). Also
it contributes to the characterization of the \( h \)-range formulas.

The algorithm requires that all the subsets (16), (17) must be considered in turn.
A slightly different approach might reduce the number of subsets which need to be
considered.
First, choose \( Q_s \leq c_4 \leq Q_{s+1} - 1 \). Now extract from the set of quintuples \((G^{(i)}, L^{(i)}, M^{(i)}, N^{(i)}, Q^{(i)})\) just those which satisfy \( Q^{(i)} < Q_{s+1} \). We do not need to consider other transfers, because they are not possible for these values of \( c_4 \). This set of quintuples defines new subdivisions for \( e_4 \), and there will in general be fewer subdivisions than before. Next, we choose one of these subdivisions \( N_r \leq e_4 \leq N_r + 1 - 1 \), and repeat the process. Finally, when we have chosen subdivisions for \( e_4 \), \( e_3 \), \( e_2 \) and \( e_1 \) we will have a set of quintuples that describes precisely those transfers which are possible for the subset, and so we have only to choose the one with highest gain.

**Properties of the h-range formula.** Since Hofmeister [5] gave explicit formulas for the regular h-range of a basis, we assume that at least one transfer must be applied.

**Theorem 1.** Let \( h, k \geq 3 \), and let the admissible basis \( A_k \) be given in normal form (6). Let \( \sum_{i=1}^{k} \epsilon_i a_i = n(h, A_k) \), \( \epsilon_i \in \mathbb{N} \cup \{0\} \), be the regular representation of the h-range. Let us assume \( \epsilon_1 < a_2 - 2 \). Then

\[
\epsilon_1 = \sum_{i=3}^{k} s_i \beta_1^{(i)} - s_2 \gamma_1 - 2,
\]

where \((s_2, \ldots, s_k)\) is one of the transfers used for \( A_k \). For this transfer to be possible for an integer with regular representation \( \sum_{j=1}^{N} e_j a_j \), it is at least necessary that

\[
\epsilon_1 \geq \sum_{i=3}^{k} s_i \beta_1^{(i)} - s_2 \gamma_1.
\]

**Proof.** Let \( n(h, A_k) = N \). The integer \( N + 1 \) has no h-representation. Consider the integer \( N + 2 = \sum_{i=1}^{k} \epsilon_i a_i + 2 \). Since the basis is admissible, we have one coefficient \( \epsilon_j \geq 1 \), \( j \in [2, k] \). Then the integer

\[
M = N + 2 - a_j = \epsilon_1 + 2 + \sum_{i=2}^{j-1} \epsilon_i a_i + (\epsilon_j - 1)a_j + \sum_{j+1}^{k} \epsilon_i a_i = \sum_{i=1}^{k} z_i a_i,
\]

and the representation is regular with \( \sum_{i=1}^{k} z_i > h \). \( M \) has an h-representation

\[
M = \sum_{i=1}^{k} z_i a_i \text{ with } \sum_{i=1}^{k} z_i \leq h, \text{ since } M \leq N.
\]

If \((s_2, \ldots, s_k)\) is the transfer between the two representations for \( M \), we have at least one \( s_j > 0 \), \( j \in [2, k] \). The h-representation of \( M \) can not be used for \( M - 1 \), since \( N + 1 = M - 1 + a_j \) would then have an h-representation. Hence the representation of \( M \) must have \( z_1 = 0 \), and thus from (7) (with \( \epsilon_j \) replaced by \( z_j' \))

\[
0 = z'_1 - 0 + s_2 \gamma_1 - \sum_{i=3}^{k} s_i \beta_1^{(i)} = \epsilon_1 + 2 + s_2 \gamma_1 - \sum_{i=3}^{k} s_i \beta_1^{(i)}.
\]

From the h-range algorithm and Theorem 1 we have

**Theorem 2.** Let \( k = 4 \), \( h \geq 3 \), and let the admissible basis \( A_4 \) be given in normal form (6) with \( \gamma_2 \geq 3 \), \( \beta_2^{(4)} \geq 1 \) and \( 2a_2 > \beta_1^{(3)} + \beta_1^{(4)} \). Let the used transfers
of the basis be $T^{(j)} = (s_i^{(j)}, s_k^{(j)}), j = 1, \ldots, \eta$. Let the regular representation of the $h$-range of the basis be
\begin{equation}
 n = \epsilon_1 + \epsilon_2 a_2 + \ldots + \epsilon_4 a_4.
\end{equation}
Then
\begin{align*}
 \epsilon_1 &= \sum_{i=3}^{\eta} s_i^{(j)} \beta^{(j)} - s_2^{(j)} \gamma_1 - 2, \\
 \epsilon_2 &= \gamma_2 - \beta^{(j)} - 2 - \delta, \\
 \epsilon_3 &= \gamma_3 - 1 - \delta, \\
 \epsilon_4 &= h - \sum_{i=1}^{3} \epsilon_i + g,
\end{align*}
where $g$ is the gain of the possible transfer of $n$ with the largest gain. For at most one value of $l \in \{2, 3\}$ we have $\epsilon_l = \gamma_l - 1$. Here $\delta = 0$ or $\delta = 1, j_1, j_2, j_3 \in \{1, 2, \ldots, \eta\}$.

**Proof.** From the $h$-range algorithm we have that the values of $\epsilon_l$ are given by either the conditions for the transfers to be possible or the conditions for $n$ to be in regular representation, [12]. In the algorithm we may have $p = r_1$, giving $L_{r_1+1} = \gamma_1$ and, from (23), $\epsilon_1 = \gamma_1 - 2$. If $p < r_1$ we find $\epsilon_1$ from the algorithm or Theorem 1. The possible transfer of $n$ with the largest gain and with the conditions on the $\epsilon_j$ such that we can have $\epsilon_j \leq \epsilon_j, j = 1, 2, 3$, gives the gain $g \geq 0$. If no possible transfer for $n$ exists, then $g = 0$.

**Conjecture 1.** For $k \geq 3$, there exist an $h_s \in N$, a set of transfers $T^{(j)} = (s_i^{(j)}, s_k^{(j)}), j = 1, \ldots, \eta$, and a $\sigma \in [1, \eta]$ such that for $h > h_s$ we have the extremal parameter basis $A_k^{(h)}$ given in normal form (6) uses the transfers $T^{(j)}, j = 1, \ldots, \eta$. If the regular representation of the $h$-range of the basis is
\begin{equation}
 n = \epsilon_1 + \epsilon_2 a_2 + \ldots + \epsilon_k a_k,
\end{equation}
then
\begin{align*}
 \epsilon_1 &= \sum_{i=3}^{k} s_i^{(\sigma)} \beta_1^{(\sigma)} - s_2^{(\sigma)} \gamma_1 - 2, \\
 \epsilon_l &= \gamma_l - 2, \\
 \epsilon_k &= h - \sum_{i=1}^{k-1} \epsilon_i + g,
\end{align*}
for $l = 2, 3, \ldots, k - 1$, and

where $g$ is the gain of the possible transfer of $n$ with the largest gain.

Also from a numerical point of view the conjecture is quite interesting, to find a upper bound for a given basis.

For $k = 3, h > 22$, the $(0,1)$-transfer with the condition $\epsilon_1 \geq \beta^{(3)}_1 = \beta$, see (7), is the only transfer used for the $A_k^*$ basis, Hofmeister [4]. But with $\epsilon_1 = \beta = 2$, we cannot apply it on $n(h, 3)$. Hence, the extremal $h$-range $n(h, 3)$ is a minimal regular $h$-representation. The extremal bases for $k = 3$ and $h \geq 6$ have $\epsilon_2 = \gamma_2 - 2$. 

All the known extremal bases for $k = 4$ are determined numerically and have for $h \geq 43$ and 23 other values, $6 \leq h < 42$, $\epsilon_2 = \gamma_2 - 2$ and $\epsilon_3 = \gamma_3 - 2$. See Challis [2], Mossige [10] and [11].

3. The conjecture in the case $k = 4$

Let $k = 4$, $h = 12\alpha t + i$, $i \in [0, 11]$, $\alpha \geq 1$, $\alpha \in \mathbb{Q}$, $t \in \mathbb{N}$. The parameter basis $A_M = A_M(h, b, p)$ we are going to use in normal form is $(a_1 = 1)$

\[ a_2 = 9\alpha t + b_1t + p_1, \]
\[ a_3 = (3\alpha t + b_3t + p_3)a_2 - (5\alpha t + b_2t + p_2), \]
\[ a_4 = (2\alpha t + b_6t + p_6)a_3 - (\alpha t + b_5t + p_5)a_2 - (6\alpha t + b_4t + p_4), \]

where $b_l, p_l \in \mathbb{Z}$ (to be chosen suitably) and where we put $b = (b_1, b_2, \ldots, b_6)$ and $p = (p_1, p_2, \ldots, p_6)$. We shall also consider the basis $A_S = A_S(h, h, p)$, given by replacing the coefficient 5 in (26) by 7 and the coefficient 6 by 4. Let $A_M = A_M(h, b)$ be the basis (26) with $p = (0, \ldots, 0)$, and similarly, $A_S = A_S(h, b)$.

Since 1971, the “record” prefactor $\varphi = 2$ was held by the parameter basis $A_M(h, b)$ discovered by Hofmeister and Schell [5] with

\[ b = (0, 0, \ldots, 0), \quad \alpha = 1 \]

and the transfers that give a positive gain

\[ T_1 = (0, 1, 0), \quad T_2 = (0, 0, 1), \quad T_3 = (1, 1, 2), \quad T_4 = (1, 0, 2). \]

In 1988 Braunschädel [1] gave the basis $A_S$ with (27), using the transfers

\[ T_1, T_2, T_3, T'_4 = (0, 0, 2). \]

He examined (on a computer) all bases $A_4(h)$ of the form,

\[ h = Ht; \quad a_2 = c_1t, \quad a_3 = c_2ta_2 - c_3t, \quad a_4 = c_4ta_3 - c_5ta_2 - c_6t, \]

with $c_i \in \mathbb{N}$, allowing only $(s_2, s_3, s_4)$-transfers with $s_2, s_3, s_4 \leq 2$. He then always found $\varphi \leq 2$, and $\varphi = 2$ only for the bases $A_M(h, b)$ and $A_S(h, b)$ with (27) (see also Selmer [19]).

The author’s idea was to make small variations of the leading coefficients of the elements of the basis (26), by varying $b$ around the six-tuple $(0, \ldots, 0)$, to see whether an increase of the prefactor is possible. Let

\[ b_M = (15, 1, -15, 6, -13, -20). \]

In 1985 he found a basis $A_M(h, b_M)$ with $\varphi > 2$ (see [12]):

To get the prefactor $\varphi$ of this basis we consider the polynomial

\[ g(\gamma) = -32\gamma - 168\gamma^2 - 22\gamma^3 + 3\gamma^4 \]

and determine the solution $\gamma_1$ of $g'(\gamma_1) = 0$, where

\[ \gamma_1 = \frac{11}{6} + \frac{1}{3}\sqrt{457} \cos \frac{\xi + 4\pi}{3}, \quad \cos \xi = \frac{7163}{\sqrt{457}}, \quad 0 < \xi < \pi/2, \]

giving $\gamma_1 = -0.09712372$...

With this $\gamma_1$ we put $\alpha_1 = -20/\gamma_1$ and

\[ \sigma = 2 + 3^{-1}2^{-6}g(\gamma_1) = 2.0080397... \]
For given $\varepsilon > 0$ we can choose $t$ so large that for $h = 12[\alpha t]$ the basis $A_M(h, b_M)$ has the prefactor

$$\varphi > \sigma - \varepsilon.$$  

In fact, here $\sigma$ is a cubic irrationality, and can only be approximated by “rational” bases (26). We obtain a very good approximation if we put $\alpha = 206$, that is, $h = 2472t$, giving $\sigma$ of (32) with all seven decimals correct. As usual, $[x]$ denotes the largest integer $\leq x \in \mathbb{R}$.

In [11] we developed formulas for the possible $h$-ranges of the parameter bases $A_M(h, b, p)$, and based the optimization on the determination of the local $h$-range $n(h, A_M)$. In addition to the transfers (28), we discovered that it was possible to use

$$T_5 = (1, 2, 1), \quad T_6 = (1, 0, 3).$$

In spite of the very small improvement on $\varphi = 2$, this result gave quite a new situation. Let

$$b_S = (15, -1, -15, 2, -13, -20).$$

In 1988 Selmer [19] showed that also the basis $A_S(h, b_S)$ has the prefactor (32). In [13] we show that my cited result (31), (32) for the basis $A_M(h, b_M)$ is valid also for the basis $A_S(h, b_S)$ with $b_M$ replaced by $b_S$. Selmer [18] calls the two bases an associate pair of bases.

**Computational results.** When we apply our $h$-range algorithm to the parameter bases $A_M$, (26) and $A_S$, it gives for each basis the sufficient inequalities that express the conditions that all the integers $n \in [1, n(h, A)]$ have an $h$-representation and it gives all the $h$-range formula candidates. By extensive computations for $h \leq 620000$ we came to two constructions of two bases. For details see [13].

**Construction 1.** Given $h = 12\alpha t + i \geq 1236$, where $\alpha \in \mathbb{Q}$, $t \in \mathbb{N}$ and $0 \leq i \leq 11$. Let $b_M = (15, 1, -15, 6, -13, -20)$. Let $P_t = (p_1, p_2, \ldots, p_6)$ and $r$ be given by Table 1. Let $\beta = at = [h/12]$, $i = h - [h/12]12$ and $q = r + i$. The basis $A(t)$ has the elements $(a_1 = 1)$

$$a_2 = 9\beta + 15t + p_1,$$

$$a_3 = (3\beta - 15t + p_3)a_2 - (5\beta + t + p_2),$$

$$a_4 = (2\beta - 20t + p_6)a_3 - (\beta - 13t + p_5)a_2, -(6\beta + 6t + p_4),$$

and $h$-range formula

$$n(t) = (3\beta + 45t + q + 1)a_4 + (2\beta - 20t + p_6 - 2)a_3$$

$$+(3\beta - 15t + p_3 - 2)a_2 + 5\beta + t + p_2 - 2.$$  

Let $j = \lfloor \beta/\alpha_1 \rfloor$, where $\alpha_1$ is given in the cited result (31), (32). If $i = 0$ then put $j = \lfloor \beta/\alpha_1 \rfloor + 1$. If $n(j + 1) > n(j)$ then $t = j + 1$, else $t = j$.

Then the basis $A^*_M = A_M(h, b_M, P_t) = A(t)$ has $h$-range

$$n(h, A^*_M) = n(t).$$

For $i$ even, $h - h_0 = 1$. For $i$ odd, $h - h_0 = 0$.

**Construction 2.** Given $h = 12\alpha t + i \geq 1236$, where $\alpha \in \mathbb{Q}$, $t \in \mathbb{N}$ and $0 \leq i \leq 11$. Let $b_S = (15, -1, -15, 2, -13, -20)$. Let $P_t = (p_1, p_2, \ldots, p_6)$ and $r$ be given by
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<th>p₄</th>
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Table 2. Let $\beta = \alpha t = \lfloor h/12 \rfloor$, $i = h - \lfloor h/12 \rfloor 12$ and $q = r + i$. The basis $A(t)$ has the elements $(a_{1} = 1)$

\[
\begin{align*}
a_2 &= 9\beta + 15t + p_1, \\
a_3 &= (3\beta - 15t + p_3)a_2 - (7\beta - t + p_2), \\
a_4 &= (2\beta - 20t + p_6)a_3 - (\beta - 13t + p_5)a_2, -(4\beta + 2t + p_4),
\end{align*}
\]

and h-range formula

\[
n(t) = (3\beta + 45t + q + 1)a_4 + (2\beta - 20t + p_6 - 2)a_3 + (3\beta - 15t + p_3 - 2)a_2 + 4\beta + 2t + p_4 - 2.
\]

Let $j = \lfloor \beta/\alpha_1 \rfloor$, where $\alpha_1$ is given in the cited result (31), (32). If $i \leq 4$, then put $j = \lfloor \beta/\alpha_1 \rfloor + 1$. If $n(j + 1) > n(j)$, then $t = j + 1$, else $t = j$.

Then the basis $A_S^* = A_S(h, b_S, P_i) = A(t)$ has h-range

\[
n(h, A_S^*) = n(t).
\]

For $i$ even, $h - h_0 = 1$. For $i$ odd, $h - h_0 = 0$.

Two parameter bases $A^{(1)}(h)$ and $A^{(2)}(h)$ is said to be asymptotically equal if $a_j^{(1)} / a_j^{(2)} \to 1$ when $h \to \infty$ for $2 \leq j \leq 4$.

In an unpublished work from 1991, Kirfel and the author have shown the following result. For $h \to \infty$, all the bases $A = A(h)$ with prefactor $\varphi > 2.008$ are either asymptotically equal to $A_M(h, b_M)$ or equal to $A_S(h, b_S)$. The sets $b_M$ and $b_S$ are given in (31) and (34) respectively. In [12] it is shown that such bases exist.

Let $A_M(h, b_S)$ denote the class of all bases $A = A(h)$ that are asymptotically equal to $A_M(h, b_M)$, and let $A_S(h, b_S)$ be the similar class for $A_S(h, b_S)$.

We now ask for the best choice of the basis $A = A(h)$ in the class $A_M(h, b_M)$ and the best choice of the basis in $A_S(h, b_S)$. 

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Extensive computations for $h \leq 620000$ give the following results.

**Result A.** For $9793 \leq h \leq 620000$, $A^*_M = A_M(h)$ in the class $A_M(h, b_M)$ is the basis with the largest $h$-range.

**Result B.** For $10653 \leq h \leq 620000$, $A^*_S = A_S(h)$ in the class $A_S(h, b_S)$ is the basis with the largest $h$-range.

**Result C.** For $11385 \leq h \leq 620000$, $A^*_M$ has larger $h$-range than $A^*_S$.

For both the bases $A^*_M$ and $A^*_S$ the $h$-range formulas are of the type stated in Conjecture 1.

**Conjecture 2.** For $h \geq 11385$, $A^*_M = A_M(h)$ is the extremal basis.

For $h < 11385$ we may get better bases when we replace the set $b_M$ or $b_S$ by other sets $b$ and suitable sets $p$.

Using a result of Selmer [18], we prove in [13] that if $b = (b_1, \ldots, b_6)$ and

$$b' = (b_1, b_1 - b_2 + b_3, b_3, b_1 - b_4 - b_5 + b_6, b_5, b_6),$$

then for each basis $A_M(h, b)$ with prefactor $\varphi$ there is a basis $A_S(h, b')$ with the same prefactor, and vice versa. If $b = b'$, then

$$2b_2 = b_1 + b_3,
2b_4 = b_1 - b_5 + b_6.$$  

For further details see [13].

**References**


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