

## SALEM NUMBERS OF NEGATIVE TRACE

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ABSTRACT. We prove that, for all  $d \geq 4$ , there are Salem numbers of degree  $2d$  and trace  $-1$ , and that the number of such Salem numbers is  $\gg d/(\log \log d)^2$ . As a consequence, it follows that the number of totally positive algebraic integers of degree  $d$  and trace  $2d - 1$  is also  $\gg d/(\log \log d)^2$ .

### 1. INTRODUCTION

Recall that a *Salem number* is an algebraic integer  $\tau > 1$ , of degree  $\geq 4$ , all of whose conjugates, apart from  $\tau$  and  $\tau^{-1}$ , have modulus 1. How small can the trace of a Salem number be? It is known that all Salem numbers of degree up to 18 have trace at least  $-1$  (Proposition 6.1).

The aim of this paper is to study the set  $\mathcal{S}_d$  of Salem numbers of degree  $2d$  and trace  $-1$ . This set is tabulated in Table 1 for  $2d \leq 14$ . It is easy to see that  $\mathcal{S}_d$  is finite for all  $d$ . In order to state our main result, we define the subset  $\mathcal{S}'_d$  of  $\mathcal{S}_d$  to be those Salem numbers  $\tau_{d,m}$  with minimal polynomial

$$(1) \quad P_{d,m}(z) = \left( z^{2d} (z^2 - z - 1) + z^{2(d-m)} + z^{2(m+1)} - z^2 - z + 1 \right) / (z - 1)^2.$$

Here  $m$  must be in the range  $1 \leq m \leq \lfloor (d-1)/2 \rfloor$ , and be such that  $P_{d,m}$  is irreducible. Then we have

**Theorem 1.1.** *For every  $d \geq 4$ ,  $\mathcal{S}_d$  is non-empty. Further, for  $d \geq 5$ ,  $\mathcal{S}'_d$  is non-empty, and, for  $d$  sufficiently large,*

$$(2) \quad |\mathcal{S}_d| \geq |\mathcal{S}'_d| > \frac{0.1387d}{(\log \log d)^2},$$

so that certainly  $|\mathcal{S}_d| \rightarrow \infty$  as  $d \rightarrow \infty$ .

In fact, it is likely that  $|\mathcal{S}_d|$  grows at least exponentially with  $d$ .

The Salem number  $\tau_{d,m}$  can in fact be associated with a particular tree, the three-armed star-like tree with  $1, 2m$  and  $2(d-m-1)$  edges on its arms, in a manner described in [MRS].

As a consequence of the theorem, we obtain a similar result for the set  $\mathcal{A}_d$  of totally positive (i.e. all conjugates positive) algebraic integers of degree  $d$  and trace  $2d - 1$ . We define the subset  $\mathcal{A}'_d$  of  $\mathcal{A}_d$  to be those  $\alpha_{d,m}$  in  $\mathcal{A}_d$  with minimal

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polynomial

$$(3) \quad Q_{d,m}(y) = y^d - (2d-1)y^{d-1} + \sum_{k=2}^{d-1} (-1)^k y^{d-k} \left\{ \binom{2d-k}{k} - \sum_{i=\max(0, k-m-1)}^{\min(d-m-2, k-2)} \binom{2d-2m-3-i}{i} \binom{2m-k+1+i}{k-2-i} \right\} + (-1)^d.$$

Again,  $m$  must satisfy  $1 \leq m \leq \lfloor (d-1)/2 \rfloor$  and be such that  $Q_{d,m}$  is irreducible. Then

**Corollary 1.2.** *For every  $d \geq 1$ ,  $\mathcal{A}_d$  is non-empty. Also  $\mathcal{A}'_d$  is non-empty for  $d \geq 5$  and, for  $d$  sufficiently large,*

$$(4) \quad |\mathcal{A}_d| \geq |\mathcal{A}'_d| > \frac{0.1387d}{(\log \log d)^2},$$

so that certainly  $|\mathcal{A}_d| \rightarrow \infty$  as  $d \rightarrow \infty$ .

The proofs of Theorem 1.1 and Corollary 1.2 are based on the following factorization of  $P_{d,m}$ :

**Theorem 1.3.** *For  $d \geq 5$  and  $1 \leq m \leq \lfloor \frac{d-1}{2} \rfloor$ ,  $P_{d,m}(z)$  factors as the product of the minimal polynomial of a Salem number  $\tau_{d,m}$  and a (possibly trivial) cyclotomic polynomial, which is*

$$\begin{cases} C(z)C_{12}(z) & \text{if } d \equiv 3 \pmod 6 \text{ and } m \equiv 1 \pmod 6, \\ C(z)C_{30}(z) & \text{if } d \equiv 4 \pmod{15} \text{ and } m \equiv 1 \text{ or } 2 \pmod{15}, \\ C(z) & \text{otherwise.} \end{cases}$$

Here  $C_{12}(z) = z^4 - z^2 + 1$ ,  $C_{30}(z) = P_{4,1}(z) = z^8 + z^7 - z^5 - z^4 - z^3 + z + 1$  and

$$C(z) = \left( \frac{z^{g_1} - 1}{z - 1} \right) \cdot \left( \frac{z^{g_2} - 1}{z - 1} \right) \cdot \left( \frac{z^{g_3} - 1}{z^{g_4} - 1} \right),$$

where  $g_1 = \gcd(d, 2m + 1)$ ,  $g_2 = \gcd(2d + 1, 2m + 3)$ ,  $g_3 = \gcd(2d + 1, m)$  and  $g_4 = \gcd(g_2, g_3)$  ( $= 1$  or  $3$ ).

From the theorem one can readily read off the trace of  $\tau_{d,m}$ . It is equal to  $-1 + n_1 + n_2 + n_3 + n_4$ , where  $n_1 = 1$  if  $g_1 > 1$ , and 0 otherwise,  $n_2 = 1$  if  $g_2 > 1$ , and 0 otherwise,  $n_3 = 1$  if  $g_3 > g_4$ , and 0 otherwise, and  $n_4 = 1$  if  $d \equiv 4 \pmod{15}$  and  $m \equiv 1$  or  $2 \pmod{15}$ , and 0 otherwise. In particular,  $\tau_{d,m}$  has trace  $-1$  iff it has degree  $2d$ , i.e. iff  $P_{d,m}$  is irreducible.

Of course, we are particularly interested in the pairs  $d, m$  for which  $P_{d,m}$  is irreducible:

**Corollary 1.4.** *For  $d \geq 5$ ,  $1 \leq m \leq \lfloor \frac{d-1}{2} \rfloor$ ,  $P_{d,m}$  has the  $n$ th cyclotomic polynomial  $C_n$  as a factor iff*

- (i)  $n$  odd  $\geq 3$ ,  $d \equiv 0 \pmod n$ ,  $m \equiv \frac{n-1}{2} \pmod n$
- or
- (ii)  $n$  odd  $\geq 3$ ,  $d \equiv \frac{n-1}{2} \pmod n$ ,  $m \equiv 0$  or  $\frac{n-3}{2} \pmod n$
- or

(iii)  $n = 12, d \equiv 3 \pmod{6}, m \equiv 1 \pmod{6}$   
 or  
 (iv)  $n = 30, d \equiv 4 \pmod{15}, m \equiv 1 \text{ or } 2 \pmod{15}$ ,  
 and in no other case. In particular, putting

$$\mathcal{M}_d = \left\{ m : 1 \leq m \leq \lfloor (d-1)/2 \rfloor, m \not\equiv \frac{p-1}{2} \pmod{p} \text{ for all odd primes } p|d, \right. \\ \left. m \not\equiv 0 \text{ or } \frac{q-3}{2} \pmod{q} \text{ for all odd primes } q|2d+1 \right\},$$

$P_{d,m}$  is irreducible iff

$$\begin{cases} m \in \mathcal{M}_d \text{ if } d \not\equiv 4 \pmod{15}, \\ m \in \mathcal{M}_d \cap \{m \not\equiv 1 \text{ or } 2 \pmod{15}\} \text{ if } d \equiv 4 \pmod{15}. \end{cases}$$

The polynomial  $Q_{d,m}$  is defined by  $Q_{d,m}(z + 1/z + 2) := z^{-d}P_{d,m}(z)$ . Its factorization can thus be written down from the factorization of  $P_{d,m}$ . In particular,  $Q_{d,m}$  is irreducible iff  $P_{d,m}$  is irreducible.

The polynomial  $P_{d,m}(z)$  can also be written

$$\begin{aligned} & z^{2d} + z^{2d-1} - z^{2d-3} - 2z^{2d-4} - \dots - (2m-2)z^{2d-2m} \\ & - (2m-1) \left( z^{2d-(2m+1)} + z^{2d-(2m+2)} + \dots + z^{2m+2} + z^{2m+1} \right) \\ & - (2m-2)z^{2m} - \dots - 2z^4 - z^3 + z + 1. \end{aligned}$$

One way in which  $P_{d,m}$  (or, equivalently,  $\tau_{d,m}$ ) arises naturally is the following: the smallest limit point in the set of Pisot numbers is  $\rho = \frac{1}{2}(1 + \sqrt{5})$ , which is a limit of Pisot numbers  $\vartheta_m < \rho$  with minimal polynomial

$$(z^{2m}(z^2 - z - 1) + 1) / (z - 1) \quad (m \geq 1).$$

Then the standard construction ([Sa], [BDGPS]) proving that every Pisot number is a limit from below of Salem numbers shows that  $\vartheta_m$  is a limit from below of the  $\tau_{d,m}$ , as  $d \rightarrow \infty$ .

The factorization of  $P_{d,m}$  described here was first conjectured on the basis of computational evidence obtained for  $d \leq 40$  using Maple.

## 2. STANDARD LEMMAS

Let  $\omega_n = e^{2\pi i/n}$ . Then we need

**Lemma 2.1.** For all natural numbers  $n$ ,

- (a)  $-\omega_n$  is a conjugate of  $\omega_n$  iff  $n$  is a multiple of 4;
- (b)  $-\omega_n^2$  is a conjugate of  $\omega_n$  iff  $n$  is divisible by 2 but not by 4;
- (c)  $\omega_n^2$  is a conjugate of  $\omega_n$  iff  $n$  is odd.

The proof is an easy exercise. We also need the standard estimates

**Lemma 2.2.** For  $n \geq 3$

$$\prod_{\substack{p|n \\ p \text{ prime}}} \left( 1 - \frac{1}{p} \right) > \frac{1}{e^\gamma \log \log n + 2.50637 / \log \log n} =: f(n),$$

say, and for  $n > 26$

$$\omega(n) < \frac{\log n}{\log \log n - 1.1714} =: h(n),$$

say. Here  $\omega(n)$  is the number of distinct prime factors of  $n$ , and  $\gamma$  is Euler's constant  $0.577\dots$

For the proofs, see [RS], p.72, and [Robin], respectively, or [MSC].  
 We also need a (presumably well-known) crude sieving estimate:

**Lemma 2.3.** *Let  $\mathcal{D}$  be a finite set of pairwise relatively prime integers, all at least 2, and for each  $p$  in  $\mathcal{D}$  let  $\mathcal{R}_p$  be a set of  $r_p < p$  residue classes mod  $p$ . Then the number  $N$  of positive integers  $m \leq M$  which are  $\not\equiv x_p \pmod p$  for any  $x_p$  in  $\mathcal{R}_p$  and any  $p \in \mathcal{D}$  satisfies*

$$\left| N - M \prod_{p \in \mathcal{D}} \left( 1 - \frac{r_p}{p} \right) \right| \leq \prod_{p \in \mathcal{D}} (1 + r_p).$$

The proof is an easy application of the Principle of Inclusion and Exclusion and the Chinese Remainder Theorem. Alternatively, it is slight extension of the results of [HR], pp. 30-31.

### 3. PROOF OF THEOREM 1.3

We first need

**Lemma 3.1.** *For  $d \geq 5$  and  $1 \leq m \leq \lfloor (d-1)/2 \rfloor$  the polynomial  $P_{d,m}$  has a real root  $\tau_{d,m} > 1$ . All other roots are on  $|z| = 1$  except for  $\tau_{d,m}^{-1}$ . For fixed  $d \geq 5$  the  $\tau_{d,m}$  ( $1 \leq m \leq \lfloor (d-1)/2 \rfloor$ ) are all distinct. For  $d, m$  in this range,  $P_{d,m}(1) \neq 0$ .*

*Proof.* Consider

$$\begin{aligned} R_{d,m}(z) &:= (z-1)^2 P_{d,m}(z) \\ &= z^{2d} (z^2 - z - 1) + z^{2(d-m)} + z^{2(m+1)} - z^2 - z + 1. \end{aligned}$$

Then by a standard Rouché's Theorem argument to be found in [Sa],  $R_{d,m}$  has at most one zero in  $|z| > 1$ . Further, if  $R''_{d,m}(1) < 0$  then  $R_{d,m}$  will have exactly one zero in  $|z| > 1$ . Now

$$R''_{d,m}(1) = 2(4m(m+1) + 1 - 2(2m-1)d) < 0$$

if

$$d \geq \left\lceil \frac{4m(m+1) + 1}{2(2m-1)} \right\rceil = \begin{cases} 5 & \text{for } m = 1, 2, 3 \\ m+2 & \text{for } m \geq 4 \end{cases}$$

This shows that  $R_{d,m}$  has one root in  $|z| > 1$  for  $1 \leq m \leq d-2$  ( $d \geq 5$ ).

Now  $P_{d,d-m-1} = P_{d,m}$ , so that the  $\tau_{d,m}$  can, for fixed  $d$ , be distinct only for  $m \leq d-m-1$ , i.e.  $m \leq \lfloor (d-1)/2 \rfloor$ . Indeed, for  $1 \leq m' < m \leq \lfloor (d-1)/2 \rfloor$  and  $\tau := \tau_{d,m}$ ,

$$\begin{aligned} R_{d,m'}(\tau) - R_{d,m}(\tau) &= \tau^{2(d-m')} + \tau^{2(m'+1)} - \tau^{2(d-m)} - \tau^{2(m+1)} \\ &= (\tau^{2(m-m')} - 1)(-\tau^{2(m'+1)} + \tau^{2(d-m)}) \\ &> 0. \end{aligned}$$

Thus the  $\tau_{d,m}$  are distinct for  $d$  fixed and  $1 \leq m \leq \lfloor (d-1)/2 \rfloor$ .

We now prove the theorem, or rather, Corollary 1.4, which is really an alternative formulation of Theorem 1.3.

We first write  $R_{d,m}(z)/z = 0$  in the form

$$(5) \quad -z^{2d} = \frac{u - z - 1 + \frac{1}{z}}{\frac{1}{u} - \frac{1}{z} - 1 + z},$$

where  $u = z^{2m+1}$ . We assume that  $z = \omega_n$  is a zero of  $P_{d,m}$  and so of (5), and, in order to use Lemma 2.1, separate three cases:

(a) The case  $4|n$ . Here  $z = -\omega_n$  is also a root of (5), so that

$$(6) \quad -z^{2d} = \frac{u - z - 1 + \frac{1}{z}}{\frac{1}{u} - \frac{1}{z} - 1 + z} = \frac{-u + z - 1 - \frac{1}{z}}{-\frac{1}{u} + \frac{1}{z} - 1 - z}$$

which gives

$$(7) \quad 2 \left( z - \frac{1}{z} \right) = u - \frac{1}{u}.$$

To solve (7), put  $z = e^{2\pi i/4k}$  say, with conjugates  $z^r = e^{2\pi ir/4k}$ , where  $(r, 4k) = 1$ . Hence, applying the Galois element  $z \mapsto z^r$ , we get

$$2(z^r - z^{-r}) = (u^r - u^{-r}),$$

so that

$$(8) \quad 2 \left| \sin \frac{\pi r}{2k} \right| = \left| \sin \frac{\pi r(2m+1)}{2k} \right| \leq 1.$$

Thus there can be no  $r$  with  $(r, 2k) = 1$  and  $\frac{k}{3} < r \leq k$ . However, the examples  $(r, k) = (1, 1), (2t - 1, 2t)$  and  $(2t - 1, 2t + 1)$  for  $t \geq 2$  show that every value of  $k$  except  $k = 3$  is impossible. For  $k = 3$ ,  $z = e^{2\pi i/12}$  and  $2 \left( z - \frac{1}{z} \right) = 2i$ , (7) has the unique solution  $u = i = e^{2\pi i(2m+1)/12}$ , giving  $2m + 1 \equiv 3 \pmod{12}$ ,  $m \equiv 1 \pmod{6}$ . Then (5) gives  $-z^{2d} \equiv 1$ ,  $2d \equiv 6 \pmod{12}$ ,  $d \equiv 3 \pmod{6}$ .

(b) The case  $2|n, 4 \nmid n$ . Starting with (5), use Lemma 2.1(b) to replace  $z$  by  $-z^2$ ,  $u$  by  $-u^2$  and eliminate  $z^{2d}$  to obtain

$$(9) \quad (-z^{2d})^2 = \left( \frac{u - z - 1 + \frac{1}{z}}{\frac{1}{u} - \frac{1}{z} - 1 + z} \right)^2 = (-z^2)^{2d} = - \left( \frac{-u^2 - (-z^2) - 1 + \frac{1}{-z^2}}{\frac{1}{-u^2} - \frac{1}{-z^2} - 1 - z^2} \right).$$

Clearing the denominators gives a plane algebraic curve  $f(u, z) = 0$ , independent of  $d$ . Since then also  $f(-u^2, -z^2) = 0$ , the pairs  $(u, z)$  of interest lie on both curves. To find all possible  $(u, z)$  pairs, we use a Maple program [Sm3] which uses a version of the Euclidean algorithm to find all such intersection points, with multiplicities. The program tells us that the only such intersection points with  $z$  and  $u$   $n$ th roots of unity with  $2|n, 4 \nmid n$  are the pairs  $(u, z) = (\alpha^3, \alpha)$  and  $(\alpha^5, \alpha)$ , where  $\alpha$  is a primitive 30th root of unity. Both points have multiplicity one. Hence  $2m + 1 = 3$  or  $5$ ,  $m = 1$  or  $2$ . [Alternatively, one can of course use the classical resultant method to find  $z$ , say, and then back-substitute to find the corresponding values of  $u$ . Doing this, one finds that the cyclotomic factors of this resultant are  $C_{30}^2, (z - 1)^8, (z + 1)^8$

and  $(z^2 + 1)^8$ , from which the pairs  $(z, u)$  can again be found.] Then, using (5), we find that, when  $m = 1, u = z^3$ ,

$$-z^{2d} = \frac{z^3 - z - 1 + 1/z}{z^{-3} - z^{-1} - 1 + z} = -z^8$$

on routine simplification, using  $C_{30}(z) = 0$ . Again, for  $m = 2, u = z^5$ , (4) gives  $-z^{2d} = -z^8$  again. Hence  $2d = 8 \pmod{30}, d = 4 \pmod{15}$ , for  $m = 1$  or  $2$ .

(c) The case  $n$  odd. In a way similar to the previous case, apply Lemma 2.1(c) to (5), and also replace  $z$  by  $z^2$ , to obtain

$$-(-z^{2d})^2 = -\left(\frac{u - z - 1 + \frac{1}{z}}{\frac{1}{u} - \frac{1}{z} - 1 + z}\right)^2 = -(z^2)^{2d} = \frac{u^2 - z^2 - 1 + \frac{1}{z^2}}{\frac{1}{u^2} - \frac{1}{z^2} - 1 + z^2}.$$

Clearing denominators this time gives

$$(u - 1)^2 (uz^2 - 1) (z - u) (z + 1) (z - 1) = 0.$$

Since neither  $\pm 1$  is a zero of  $P_{d,m}$ , we need consider only the subcases where one of the first three factors is 0:

(i)  $u = 1$ . Here  $u = z^{2m+1} = 1, m \equiv \frac{n-1}{2} \pmod{n}$ . Then, from (5),  $z^{2d} = 1, z^d = 1$ , i.e.  $d \equiv 0 \pmod{n}$ .

(ii)  $u = z^{-2}, z^{2m+3} = 1, m \equiv \frac{n-3}{2} \pmod{n}$ , and, from (5),  $z^{2d+1} = 1, d \equiv \frac{n-1}{2} \pmod{n}$ .

(iii)  $u = z, z^{2m} = 1, z^m = 1, m \equiv 0 \pmod{n}$ , and, from (5),  $z^{2d+1} = 1, d \equiv \frac{n-1}{2} \pmod{n}$ .

This completes the proof of Corollary 1.4. Theorem 1.3 now follows readily by collecting together all the cyclotomic factors  $C_n(z)$  of  $P_{d,m}$  for  $n$  odd, and noting that  $\gcd(g_1, g_2) = \gcd(g_1, g_3) = 1$ , and  $g_4 = \gcd(g_2, g_3) = 1$  or  $3$ .

#### 4. PROOF OF THEOREM 1.1

For the proof, we need to find a positive lower bound for  $|\mathcal{S}'_d|$ . First we show that

**Lemma 4.1.** *The set  $\mathcal{S}'_d$  is non-empty for  $5 \leq d \leq B := 7.98 \times 10^{12}$ .*

*Proof.* First, direct Maple computation of the set  $\mathcal{M}_d$  shows that  $\mathcal{M}_d$ , and hence  $\mathcal{S}'_d$  is non-empty for  $5 \leq d \leq 2998$ . The set  $\mathcal{M}_d$  is shown for  $d \leq 60$  in Table 2 (at the end of this paper). Next, we find, again using Maple, that the primes  $m' \in \{5, 29, 53, 89, 113, 173, 509, 659, 743, 809, 1013, 1499\}$  have the property that, for each of these primes  $m'$ , the numbers  $2m' + 1$  and  $2m' + 3$  are also both prime. Further, there is no repeated prime in the multiset of all such  $m', 2m' + 1, 2m' + 3$  for  $m'$  in the above set of primes.

Now suppose that  $d \geq 2999$ . Then, by Lemma 3.1, the polynomials  $P_{d,m}$  for fixed  $d$  and  $1 \leq m \leq 1499 = (2999 - 1)/2 \leq \lfloor (d - 1)/2 \rfloor$  all are divisible by the minimal polynomials of distinct Salem numbers. I claim that for  $m$  equal to at least one  $m'$  on the above list,  $m' \in \mathcal{M}_d$ , so that  $\mathcal{M}_d$  and hence  $\mathcal{S}'_d$  is non-empty. For, if not, then, from the definition of  $\mathcal{M}_d$ , either  $m'|2d + 1$  or  $(2m' + 3)|2d + 1$  or

$(2m' + 1)|d$ , implying that  $m'' := m'$  or  $2m' + 1$  or  $2m' + 3$  divides  $d(2d + 1)$ . But now

$$\prod m'' \geq \prod m' = 4.08 \times 10^{27} > 1.27 \times 10^{26} = B(2B + 1) \geq d(2d + 1)$$

gives a contradiction.

We next find a lower bound for  $|\mathcal{S}'_d|$  for large  $d$ , i.e. for  $d > B$ . To do this, we apply Lemma 2.3, using the description of the integers  $m$  in  $\mathcal{S}'_d$  given by Corollary 1.4.

First consider the case  $d \not\equiv 4 \pmod{15}$ . Take  $\mathcal{D}$  to be the set of odd primes dividing  $d(2d + 1)$ , and  $\mathcal{R}_p = \{\frac{1}{2}(p - 1)\}$  if  $p$  is an odd prime dividing  $d$ , and  $\mathcal{R}_q = \{0, \frac{1}{2}(q - 3)\}$  if  $q$  is a prime dividing  $2d + 1$ . Put  $r_p = |\mathcal{R}_p|$ . Then  $r_p = 1$  for  $p|d$ ,  $r_3 = 1$  if  $3|2d + 1$ ; otherwise  $r_q = 2$  if  $q|2d + 1$ ,  $q \neq 3$ . Hence, applying Lemma 2.3 with  $M = \lfloor (d - 1)/2 \rfloor$ , we obtain

$$(10) \quad |\mathcal{S}'_d| \geq M \prod_{p|d_3} \left(1 - \frac{1}{p}\right) \prod_{\substack{q|2d+1 \\ q \neq 3}} \left(1 - \frac{2}{q}\right) - 2^{\omega(d)} 3^{\omega(2d+1)}.$$

Here  $\omega(r)$  is the number of prime factors of  $r$ , and  $d_3 = 3d$  if  $3|2d + 1$ , while  $d_3 = d$ , otherwise.

Similarly, for the case  $d \equiv 4 \pmod{15}$  we have  $2d + 1 \equiv 9 \pmod{15}$ , so  $3|2d + 1$ , but  $3 \nmid d$ ,  $5 \nmid d$ ,  $5 \nmid 2d + 1$ . Thus there are seven excluded residue classes mod 15:  $m \not\equiv 0, 1, 2, 3, 6, 9, 12 \pmod{15}$ , and the lemma gives

$$(11) \quad |\mathcal{S}'_d| \geq M \prod_{p|d} \left(1 - \frac{1}{p}\right) \prod_{\substack{q|2d+1 \\ q \neq 3}} \left(1 - \frac{2}{q}\right) \left(1 - \frac{7}{15}\right) - 2^{\omega(d)} 3^{\omega(2d+1)-1} (1 + 7).$$

We now apply Lemma 2.2 to (10) and (11). Thus for  $d \not\equiv 4 \pmod{15}$ , and  $3 \nmid 2d + 1$  we get

$$\begin{aligned} |\mathcal{S}'_d| &\geq M \prod_{p|d} \left(1 - \frac{1}{p}\right) \prod_{q|2d+1} \left(1 - \frac{2}{q}\right) - 2^{\omega(d)} 3^{\omega(2d+1)} \\ &> M \prod_{p|d} \left(1 - \frac{1}{p}\right) \prod_{q|2d+1} \left(1 - \frac{1}{q}\right)^2 \prod_{\substack{q \geq 5 \\ q \text{ prime}}} \left(1 - \frac{1}{(q-1)^2}\right) - 2^{h(d)} 3^{h(2d+1)} \\ (12) \quad &> M f(d(2d+1)) f(2d+1) \left(1 - \frac{1}{2^2}\right)^{-1} \times 0.66 - 2^{h(d)} 3^{h(2d+1)} \end{aligned}$$

as  $\prod_{\substack{q \geq 3 \\ q \text{ prime}}} \left(1 - \frac{1}{(q-1)^2}\right) > 0.66$ . Now if  $3|2d + 1$  we obtain similarly

$$\begin{aligned} |\mathcal{S}'_d| &\geq M \left(1 - \frac{1}{3}\right) \left(1 - \frac{2}{3}\right)^{-1} \prod_{p|d} \left(1 - \frac{1}{p}\right) \prod_{q|2d+1} \left(1 - \frac{2}{q}\right) - 2^{h(d)} 3^{h(2d+1)} \\ &= 2M f(d(2d+1)) f(2d+1) \times 0.66 - 2^{h(d)} 3^{h(2d+1)}, \end{aligned}$$

which is stronger than (12). Hence (12) certainly holds for  $d \not\equiv 4 \pmod{15}$ .

For  $d \equiv 4 \pmod{15}$ , we obtain, from (11), using  $3|2d + 1$  and  $5 \nmid 2d + 1$ , that

$$\begin{aligned}
 |\mathcal{S}'_d| &\geq M \cdot \frac{8}{15} \left(1 - \frac{2}{3}\right)^{-1} \prod_{p|d} \left(1 - \frac{1}{p}\right) \prod_{q|2d+1} \left(1 - \frac{1}{q}\right)^2 \\
 &\quad \times \prod_{\substack{q \geq 3 \\ q \text{ prime}}} \left(1 - \frac{1}{(q-1)^2}\right) \left(1 - \frac{1}{4^2}\right)^{-1} - \frac{8}{3} 2^{h(d)} 3^{h(2d+1)} \\
 (13) \quad &> \frac{384}{225} \times 0.66 M f(d(2d+1)) f(2d+1) - \frac{8}{3} 2^{h(d)} 3^{h(2d+1)}.
 \end{aligned}$$

Hence, from (12) and (13), we have

$$(14) \quad |\mathcal{S}'_d| > c_1 M f(d(2d+1)) f(2d+1) \left(1 - \frac{2^{h(d)} 3^{h(2d+1)}}{c_2 M f(d(2d+1)) f(2d+1)}\right),$$

where  $c_1 = 1.1264, c_2 = 0.4224$  for  $d \equiv 4 \pmod{15}$ , and  $c_1 = c_2 = 0.88$  otherwise. Thus we see that for

$$\frac{2^{h(d)} 3^{h(2d+1)}}{[(d-1)/2] f(d(2d+1)) f(2d+1)} < 0.4224$$

we have  $|\mathcal{S}'_d| > 0$ . A straightforward Maple calculation shows that this happens for  $d \geq B = 7.98 \times 10^{12}$ .

Finally, from (13) and the definition of  $f(d)$  we see that, for large  $d$ ,

$$\begin{aligned}
 |\mathcal{S}'_d| &> \left(0.88 \times \frac{1}{2} \times e^{-2\gamma} - o(1)\right) d / (\log \log d)^2 \\
 &> 0.1387 d / (\log \log d)^2.
 \end{aligned}$$

### 5. PROOF OF COROLLARY 1.2

First, note that, from my tables [Sm1],  $|\mathcal{A}_d| > 0$  for  $1 \leq d \leq 7$ . For larger values of  $d$ , we use the correspondence  $\tau + \tau^{-1} + 2 = \alpha$ . This shows that  $\mathcal{A}'_d > 0$  for all  $d$ , and gives the asymptotic lower bound (4).

It remains only to show that if  $\tau$  has minimal polynomial  $P_{d,m}(z)$ , then  $\alpha = \tau + \tau^{-1} + 2$  has minimal polynomial  $Q_{d,m}(y)$  given by (3). Now, using (1), we can write

$$R_{d,m}(z) = P_{d,m}(z)(z-1)^2 = (z^{2d+1} - 1)(z-1) - z^2(z^{2(d-m-1)} - 1)(z^{2m} - 1),$$

so that

$$\begin{aligned}
 \frac{P_{d,m}(z)}{z^d} &= \frac{z^{d+1/2} - z^{-(d+1/2)}}{z^{1/2} - z^{-1/2}} - \frac{z^{d-m-1} - z^{-(d-m-1)}}{z^{1/2} - z^{-1/2}} \cdot \frac{z^m - z^{-m}}{z^{1/2} - z^{-1/2}} \\
 &= U_{2d}(x) - U_{2(d-m)-3}(x) \cdot U_{2m-1}(x),
 \end{aligned}$$

where  $x = \sqrt{z} + 1/\sqrt{z}$  and [Robins]

$$(15) \quad U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}$$

is the  $n$ th Chebyshev polynomial of the second kind, with defining property

$$U_n(t+1/t) = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}}.$$

Now, for  $\alpha = \tau + \tau^{-1} + 2$  we have  $\sqrt{\alpha} = \sqrt{\tau} + 1/\sqrt{\tau}$ , so that  $y = \alpha$  is a root of

$$Q_{d,m}(y) = U_{2d}(\sqrt{y}) - U_{2(d-m)-3}(\sqrt{y}) \cdot U_{2m-1}(\sqrt{y})$$

which, using (15), gives (3).

## 6. TABLES

Table 1 shows that, for  $2d = 8, 10, 12, 14$ , there are respectively 1, 3, 9, 39 elements of  $\mathcal{S}_d$ . It was obtained from the tables in [Sm1], using the transformation  $\tau + \tau^{-1} + 2 = \alpha$ , where  $\alpha$  is totally positive of degree  $d$  and trace  $2d - 1$ . Several examples of Salem numbers of trace  $-1$ , including the unique degree 8 example, had been found earlier by Boyd (personal communication).

It is interesting to note [Sm1] that there are in fact 40 totally positive algebraic integers of degree 7 and trace 13. All but one of them has exactly one conjugate  $> 4$ , giving the 39 elements of  $\mathcal{S}_7$  mentioned above. The exception is the number  $\alpha$  having minimal polynomial  $z^7 - 13z^6 + 62z^5 - 135z^4 + 140z^3 - 67z^2 + 14z - 1$ , which has two such conjugates. For this  $\alpha$ , the  $\tau$  defined by  $\tau + \tau^{-1} + 2 = \alpha$  has, of course, two conjugates in  $(1, \infty)$ , so is not a Salem number.

The results of [Sm1], combined with further computation using the same method as in that paper, also show that

**Proposition 6.1.** *For  $2d \leq 18$ , all Salem numbers of degree  $2d$  have trace at least  $-1$ .*

This further computation consisted of an unsuccessful search for totally positive algebraic integers of degree  $d = 8$  or  $9$  and trace  $\leq 2d - 2$ . There are, however, examples of totally positive algebraic integers of large degree  $d$  and trace  $< 2d - 1$  ([Sm3]). Thus there may well be Salem numbers of large degree and trace  $< -1$ .

Table 2 shows, for  $d \leq 60$ , the set  $\mathcal{M}_d$  of those  $m$  for which  $P_{d,m}$  is irreducible.

TABLE 1. Minimal polynomials of all Salem numbers of trace  $-1$  and degree  $2d$  up to 14.

#	$2d$	Coefficients of $z^{2d}, \dots, z^d$									
1	8	1	1	-1		-4	-5				
2	10	1	1	-1		-5	-9	-11			
3	10	1	1	0		-1	-1	-1			
4	10	1	1	0		-2	-4	-5			
5	12	1	1	-2		-6	-6	-3	-1		
6	12	1	1	-2		-7	-11	-14	-15		
7	12	1	1	-2		-7	-10	-11	-11		
8	12	1	1	-1		-3	-3	-3	-3		
9	12	1	1	-1		-3	-2	0	1		
10	12	1	1	-1		-4	-6	-8	-9		
11	12	1	1	-1		-5	-10	-14	-15		
12	12	1	1	0		-1	-2	-3	-3		
13	12	1	1	0		-2	-4	-5	-5		
14	14	1	1	-4		-15	-26	-31	-29	-27	
15	14	1	1	-4		-16	-32	-48	-59	-63	
16	14	1	1	-4		-17	-36	-56	-70	-75	
17	14	1	1	-3		-10	-15	-17	-17	-17	
18	14	1	1	-3		-10	-13	-8	2	7	
19	14	1	1	-3		-11	-19	-25	-28	-29	
20	14	1	1	-3		-11	-18	-20	-17	-15	
21	14	1	1	-3		-11	-17	-16	-9	-5	
22	14	1	1	-3		-12	-24	-37	-47	-51	
23	14	1	1	-3		-12	-23	-33	-39	-41	
24	14	1	1	-3		-12	-22	-29	-31	-31	
25	14	1	1	-3		-13	-28	-45	-58	-63	
26	14	1	1	-3		-13	-27	-41	-50	-53	
27	14	1	1	-2		-6	-7	-6	-5	-5	
28	14	1	1	-2		-6	-6	-2	3	5	
29	14	1	1	-2		-7	-11	-13	-12	-11	
30	14	1	1	-2		-7	-11	-14	-16	-17	
31	14	1	1	-2		-7	-10	-9	-5	-3	
32	14	1	1	-2		-7	-10	-10	-8	-7	
33	14	1	1	-2		-7	-9	-5	3	7	
34	14	1	1	-2		-7	-9	-6	0	3	
35	14	1	1	-2		-8	-16	-25	-31	-33	
36	14	1	1	-2		-8	-15	-22	-27	-29	
37	14	1	1	-2		-8	-14	-18	-19	-19	
38	14	1	1	-2		-8	-13	-14	-11	-9	
39	14	1	1	-2		-9	-19	-30	-38	-41	
40	14	1	1	-2		-9	-18	-27	-33	-35	
41	14	1	1	-1		-3	-3	-3	-4	-5	
42	14	1	1	-1		-4	-7	-10	-11	-11	
43	14	1	1	-1		-4	-6	-6	-4	-3	
44	14	1	1	-1		-4	-6	-7	-7	-7	
45	14	1	1	-1		-4	-5	-3	1	3	
46	14	1	1	-1		-4	-5	-4	-2	-1	
47	14	1	1	-1		-5	-11	-18	-23	-25	
48	14	1	1	-1		-5	-10	-15	-18	-19	
49	14	1	1	-1		-5	-9	-12	-13	-13	
50	14	1	1	-1		-6	-13	-21	-27	-29	
51	14	1	1	0		-1	-2	-3	-3	-3	
52	14	1	1	0		-2	-4	-6	-7	-7	

TABLE 2. Values of  $m$  for which the polynomial  $P_{d,m}$  is irreducible, for  $d \leq 60$ .

2d	Values of $m$
10	1
12	2
14	2
16	1 2 3
18	2 3
20	1 4
22	1 2 3 4
24	2 3
26	1 2 4 5
28	1 2 4 5 6
30	3 5 6
32	1 2 5 7
34	3 4
36	2 3 5 6 8
38	4 7 8
40	1 3 4 5 6 8 9
42	2 5 6 8 9
44	2 4 7 8
46	1 2 3 4 5 6 7 8 9 10
48	3 5 6 8 11
50	1 4 5 8 10 11
52	1 2 3 4 5 7 8 9 10 11 12
54	2 3 8 9 12
56	1 2 4 5 7 11 13
58	1 2 3 4 5 6 7 8 9 10 11 12 13
60	3 5 6 8 9 11 14
62	1 4 5 8 10 11 13
64	2 3 4 7 8 9 12 14
66	2 3 6 8 9 11 12 14 15
68	4 5 7 11 13 14
70	1 4 5 6 8 9 11 13 14 15 16
72	2 3 5 6 8 9 11 12 14 15 17
74	2 4 7 8 13 14 17
76	1 3 5 6 8 10 12 13 17 18
78	2 3 5 8 9 11 12 14 15 17 18
80	1 4 5 8 10 11 13 14 16 19
82	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
84	2 8 9 12 14 18
86	1 2 4 5 7 8 10 11 14 16 17 19 20
88	1 2 3 4 6 7 8 9 10 11 12 13 14 15 17 18 19 20 21
90	3 6 8 11 15 20
92	1 2 4 5 7 8 10 13 16 17 19 20 22
94	2 3 4 7 9 12 13 14 17 18 22
96	2 3 5 6 8 9 11 12 14 15 17 18 20 21 23
98	5 7 8 13 14 19 20 23
100	1 3 4 5 6 8 9 10 11 13 14 15 16 18 19 20 21 23 24
102	2 3 5 6 9 11 12 14 15 17 18 20 21 23 24
104	4 8 13 17 22
106	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
108	2 3 5 6 8 9 11 12 14 15 17 18 20 21 23 24 26
110	1 4 8 10 11 13 14 19 20 23 25 26
112	1 2 4 5 6 7 8 9 11 12 13 14 15 16 18 19 20 21 22 23 25 26 27
114	2 3 8 12 14 17 18 24 27
116	1 2 4 7 8 10 11 16 17 19 20 22 23 25 28
118	1 3 4 5 6 8 10 11 12 13 15 18 19 20 22 25 26 27
120	3 5 6 8 9 14 18 20 21 23 24 29

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