LATTICE COMPUTATIONS FOR RANDOM NUMBERS
RAYMOND COUTURE AND PIERRE L’ECUYER

ABSTRACT. We improve on a lattice algorithm of Tezuka for the computation of the $k$-distribution of a class of random number generators based on finite fields. We show how this is applied to the problem of constructing, for such generators, an output mapping yielding optimal $k$-distribution.

1. INTRODUCTION
Extensive classes of random number generators have the following structure. The state space is a finite field $F$ of characteristic 2. We denote by $d$ its degree over $F_2$, and sometimes refer to it as the order of the generator. Any state $y \in F$ evolves into a state $xy$, where the distinguished element, $x \in F$, completely determines the evolution of the generator. Finally, the generator in state $y$ outputs a $w$-bit vector $(y)$ where $y_1, \ldots, y_l$ are suitably chosen non-zero elements of $F$.

The study of the $k$-distribution of the output sequence involves the computation, for all $l \leq w$ and $k \leq d$, of the rank of the mapping $F \rightarrow F_2^{lk}$ defined by

$$y \mapsto \begin{pmatrix} \phi(y_1 y) & \phi(y_1 xy) & \cdots & \phi(y_1 x^{k-1} y) \\ \phi(y_2 y) & \phi(y_2 xy) & \cdots & \phi(y_2 x^{k-1} y) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(y_l y) & \phi(y_l xy) & \cdots & \phi(y_l x^{k-1} y) \end{pmatrix}.$$ (1)

One might naturally use gaussian elimination, as is done in [2, 4] for instance, but there are other methods which are more efficient in terms of both time and space. The efficiency issue becomes critical if the order $d$ of the generator is chosen large. One such method is proposed by Tezuka [7]. He computes the rank of (1), for a given value of $l$ and all $k$, by means of an $l$-dimensional lattice $L_l$ in the space $F_2[X]^l$ of $l$-tuples of polynomials with $F_2$ coefficients. We improve on this method by using instead a “dual” lattice $L'_l \subset F_2[X]^l$ which has the advantage that it has basis vector coordinates which are generally much smaller than those of $L_l$, and that a simple relationship between $L'_l$ and $L'_{l+1}$ allows for recursive computation. We will show how these features are well suited to the problem of constructing, for given $F$ and $x \in F$, an output mapping $\Phi$ with optimal $k$-distribution.

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2. Lattices

We will assume that our distinguished element \( x \) generates \( F \) as a ring so that, as a vector space over \( \mathbf{F}_2 \), \( F \) admits the basis \( 1, x, \ldots, x^{d-1} \). For \( 0 \leq k \leq d \), let \( F_k \subset F \) denote the \( \mathbf{F}_2 \)-subspace generated by the first \( k \) elements in this basis.

Consider the mapping \( \mathbf{F}_2[X]^l \to F^l \) given by

\[
(P_1(X), \ldots, P_l(X)) \mapsto (P_1(x), \ldots, P_l(x)).
\]

The inverse image by this mapping of any \( F \)-linear subspace \( V \) of \( F^l \) is a sublattice \( \Lambda_V \) of \( \mathbf{F}_2[X]^l \). If \( V = 0 \), then \( \Lambda_V \) is the kernel of \( (2) \) and we will denote it by \( K_l \). Clearly, \( K_l = K_1^l \), and \( K_1 \) is an ideal of \( \mathbf{F}_2[X] \). This ideal is generated by a degree \( d \) polynomial, \( P_{ch}(X) \). We define the absolute value of \( P(X) \in \mathbf{F}_2[X] \) to be \( 2^k \) if \( \delta \) is the degree of \( P(X) \), and the length (resp. degree) of \( (P_1(X), \ldots, P_l(X)) \in \mathbf{F}_2[X]^l \) to be the maximum absolute value (resp. degree) of the components.

If \( \Lambda \) is any sublattice of \( \mathbf{F}_2[X]^l \), its fundamental volume \( |\Lambda| \) is the absolute value of the determinant of any one of its bases, and we have

\[
|\Lambda| = \prod_{i=1}^l |\sigma_i(\Lambda)|,
\]

where \( \sigma_i(\Lambda) \) is the length of the \( i \)th vector of a Minkowski-reduced basis of \( \Lambda \). The fundamental volume \( |\Lambda| \) is also equal to the group theoretical index \( [\mathbf{F}_2[X]^l : \Lambda] \) which, in case \( \Lambda = \Lambda_V \), is simply \( [F^l : V] \). For instance Tezuka’s lattice \( \Lambda_l \) is equal to \( \Lambda_{V(l)} \) with \( V(l) = F \cdot (y_1, \ldots, y_l) \) (see Def. 3 of [7]), and its fundamental volume is thus equal to \( 2^{d(l-1)} \).

We propose to use instead of \( \Lambda_l \), the lattice \( \Lambda'_l \), given by \( \Lambda_{V(l)} \), where we take \( W(l) \) to be the ortho-complement of \( V(l) \) with respect to the standard \( F \)-bilinear scalar product defined for \( v = (x_1, \ldots, x_l) \) and \( v' = (x'_1, \ldots, x'_l) \in F^l \) by

\[
\langle v, v' \rangle = \sum_{i=1}^l x_i x'_i.
\]

The fundamental volume of \( \Lambda'_l \) is equal to \( 2^d \), and is thus much smaller than that of \( \Lambda_l \) unless \( l \) is small. Because of (3), a lattice with a smaller fundamental volume will have, in the mean, smaller successive minima. We will show how to take advantage of this in Section 4. Note that the lattices \( \Lambda_l \) and \( \Lambda'_l \) depend only on the first \( l \) values of the sequence \( y_1, \ldots, y_w \). We will occasionally indicate this dependence by writing \( \Lambda_l(y_1, \ldots, y_l) \) and \( \Lambda'_l(y_1, \ldots, y_l) \), respectively.

We will denote by \( C_k \) the set of all \( (P_1(X), \ldots, P_l(X)) \in \mathbf{F}_2[X]^l \) of length smaller than \( 2^k \). The following lemma establishes further connections between a subspace \( V \subset F^l \) and the lattice \( \Lambda_V \).

**Lemma 1.** (i) The restriction of \( (2) \) to \( C_d \) is one to one, and its image is \( F^l \).
(ii) For \( 0 \leq k \leq d \), \( (2) \) maps \( C_k \) onto \( F_k^l \).
(iii) For any \( F \)-linear subspace \( V \) of \( F^l \), \( (2) \) maps \( \Lambda_V \cap C_d \) onto \( V \).

From this and Theorem 2 of [1] we obtain for any \( F \)-linear subspace \( V \) of \( F^l \)

\[
\dim_{\mathbf{F}_2}(V \cap F_k^l) = \sum_{i=1}^l (k - \lg |\sigma_i(\Lambda_V)|^+) , \quad 0 \leq k \leq d.
\]
3. The kernel of the adjoint

The rank of (11) is equal to \( kl - \dim_{F_2} R_{l,k} \), where \( R_{l,k} \) denotes the vector space over \( F_2 \) of all systems \((\alpha_{i,j})_{i,j} \in F_2^{lk}, 1 \leq i \leq l, 0 \leq j < k \) such that

\[
\sum_{i,j} \alpha_{i,j} \phi(y_i x^j y) = 0, \quad y \in F.
\]

Since the rank of (11) does not depend on the choice of \( \phi \), we will take it to be that \( F_2 \)-linear form over \( F_2 \) which has its kernel equal to \( F_2^d \). The image of \( R_{l,k} \) by the correspondence \( F_2^{lk} \to F^l \) given by

\[
(\alpha_{i,j})_{i,j} \mapsto (\sum_j \alpha_{i,j} x^j)_i
\]

can then be described as follows. We define, in addition to the standard scalar product (14), an \( F_2 \)-bilinear scalar product by

\[
\langle v, v' \rangle_2 = \phi(\langle v, v' \rangle), \quad v, v' \in F^l.
\]

Note that the ortho-complement of an \( F \)-subspace of \( F^l \) is the same for both scalar products (14) and (8). Thus, \( W^{(l)} \) is also the ortho-complement of \( V^{(l)} \) with respect to (8).

Lemma 2. For \( k \leq d \), the restriction of (17) to \( R_{l,k} \) is one to one and onto \( W^{(l)} \cap F^l_k \).

Proof. First, the image of \( F_2^{lk} \) by (17) is \( F^l_k \). From (3) a system \((\alpha_{i,j})_{i,j} \in F_2^{lk} \) belongs to \( R^{(l)}_k \) if and only if \((\sum_j \alpha_{i,j} x^j)_i \) is orthogonal to \( V^{(l)} \) with respect to (8); that is, if and only if \((\sum_j \alpha_{i,j} x^j)_i \) belongs to \( W^{(l)} \). The lemma follows.

The main result shows how the computation of the rank of (11) is reduced to the computation of the quantities \( \sigma_l(\Lambda_l^i) \).

Theorem 1. The rank of (11) is equal to

\[
lk - \sum_{i=1}^l (k - \lg \sigma_l(\Lambda_l^i))^+, \quad 0 \leq k \leq d.
\]

Proof. This follows from (5) and Lemma 2.

The quantities \( \sigma_l(\Lambda_l^i) \) can be computed by applying the Lenstra reduction algorithm [5] to a suitably chosen basis of \( \Lambda_l^i \). We digress briefly to establish a remarkable connection between the quantities \( \sigma_l(\Lambda_l) \) and \( \sigma_l(\Lambda_l^i) \). This is closely connected to a result of Mahler (see §10 of [6]). We first establish the following relation.

Proposition 1.

\[
\dim_{F_2}(V^{(l)} \cap F^l_{d-k}) - \dim_{F_2}(W^{(l)} \cap F^l_k) = d - lk, \quad 1 \leq k \leq d.
\]

Proof. We have

\[
\dim_{F_2}(V^{(l)} + F^l_{d-k}) + \dim_{F_2}(V^{(l)} \cap F^l_{d-k}) = \dim_{F_2} V^{(l)} + \dim_{F_2} F^l_{d-k} = d + (d-k)l.
\]
Denote again by \(W_{d-k}(X)\) the ortho-complement of \(F_{d-k}(X)\) with respect to \((X)\), we also have
\[
\dim_F(W_{d-k}(X) \cap F_{d-k}(X)) + \dim_F(W_{d-k}(X) + F_{d-k}(X)) = dl.
\]
The proposition follows by combining these two equations. 

**Corollary 1.** We have, for \(1 \leq i \leq l\),
\[
\lg \sigma_i(\Lambda_i) + \lg \sigma_{l-i+1}(\Lambda_l) = d, \quad 1 \leq i \leq l.
\]

**Proof.** We abbreviate \(\lg \sigma_i(\Lambda_i)\) to \(s_i\), and \(\lg \sigma_{l-i+1}(\Lambda_l)\) to \(s'_i\). Using (5), we can then write (10) as
\[
\sum_{i=1}^{l} (d - k - s_i)^+ - \sum_{i=1}^{l} (k - s'_i)^+ = d - lk.
\]
Combining this with
\[
\sum_{i=1}^{l} (k - s'_i)^+ - \sum_{i=1}^{l} (s'_i - k)^+ = \sum_{i=1}^{l} (k - s'_i) = lk - d,
\]
we obtain
\[
\sum_{i=1}^{l} ((d - s_i) - k)^+ - \sum_{i=1}^{l} (s'_i - k)^+ = 0.
\]
Since \(0 \leq s_i, s'_i \leq d\), this implies that, for \(0 \leq k \leq d\), the sets \(\{i \mid s'_i = k\}\) and \(\{i \mid d - s_i = k\}\) have the same cardinality. The statement of the corollary follows.

4. **Recursivity**

From Theorem 1 and its corollary, the rank of \((X)\) can be obtained, simultaneously for all \(k\), by computation of the quantities \(\sigma_i(\Lambda_i)\) or \(\sigma_{l-i+1}(\Lambda_l)\). This is achieved by use of Lenstra’s reduction algorithm \([5]\) applied to a suitable basis of \(\Lambda_i\) or \(\Lambda_l\) and, as we shall now show, it is advantageous for this to use the latter lattice rather than the former. Assume \(1 < l \leq w\). The \(F\)-linear mappings \(\iota : F^{l-1} \to F^l\) and \(\rho : F^l \to F^{l-1}\), defined by addition of an \(l\)th coordinate taken equal to zero, and deletion of the \(l\)th coordinate respectively, are mutually adjoint; that is,
\[
\langle \iota(w), v \rangle = \langle w, \rho(v) \rangle, \quad w \in F^{l-1}, v \in F^l.
\]

**Lemma 3.** For \(1 < l \leq w\), we have
(i) \(\rho(V^{(l)}) = V^{(l-1)}\),
(ii) \(W^{(l)} = \iota(W^{(l-1)}) \oplus F(y_l, 0, \ldots, 0, y_1)\).

**Proof.** Statement (i) is immediate from the definition of \(V^{(l)}\). To prove (ii), notice that \((X)\) implies that \(\iota(W^{(l-1)})\) is an \(F\)-linear subspace of \(W^{(l)}\). In fact, it is of codimension 1 in \(W^{(l)}\), since it has dimension \(l-1\) while \(W^{(l)}\) has dimension \(l\). The statement now follows since \((y_l, 0, \ldots, 0, y_1)\) belongs to \(W^{(l)} \setminus \iota(W^{(l-1)})\).

We deduce from Lemma 3 the recursivity properties of the lattices \(\Lambda_i\) and \(\Lambda_l\). Denote again by \(\iota\) and \(\rho\) the similarly defined \(F_2[X]\)-linear mappings \(\iota : F_2[X]^{l-1} \to F_2[X]^l\), and \(\rho : F_2[X]^l \to F_2[X]^{l-1}\). Take, \(Q_i(X) \in F_2[X]\) of degree less than \(d\), and such that \(y_iQ_i(x) = y_i\), \(2 \leq i \leq l\).

**Proposition 2.** For \(1 < l \leq w\), we have
(i) \( \rho(\lambda_l) = \lambda_{l-1} \);
(ii) \( \Lambda'_l = \iota(\Lambda'_{l-1}) \oplus F_2[X](Q_l(X), 0, \ldots, 0, 1) \).

Proof. Note that \( \iota \) and \( \rho \) commute with \( \mathfrak{d} \). Therefore, statement (i) of Lemma 3 implies our first statement. Also, since the vector \((Q_l(X), 0, \ldots, 0, 1)\) is mapped by \( \mathfrak{d} \) to the vector \((y_{l}/y_l, 0, \ldots, 0, 1)\), statement (ii) of Lemma 3 implies that

\[ \Lambda'_l = \iota(\Lambda'_{l-1}) + F_2[X](Q_l(X), 0, \ldots, 0, 1) + K_l. \]

But \( K_l = \iota(K_{l-1}) + F_2[X](0, \ldots, 0, P_{ch}(X)) \) so that our second statement follows from the previous equation.

The starting point for the Lenstra reduction algorithm is a lattice basis \( B \) for an \( l \)-dimensional sublattice \( \Lambda \) of \( F_2[X]^l \). The algorithm transforms this basis into another basis of \( \Lambda \), which is Lenstra-reduced and, in particular, Minkowski-reduced. We associate with the basis \( B \) the quantities \( d_s(B) \) and \( d_m(B) \), which are defined as the sum and the maximum of the basis vector degrees, respectively. The storage requirement for the algorithm is then measured by \( ld_s(B) \), and an upper bound for the execution time (the required number of bit operations) is given by

\[ C l^3 d_m(B)(d_s(B) - \lg |\Lambda| + 1), \]

for some absolute constant \( C \) (see Prop. 1.14 in [5]).

In case of \( \Lambda_l \), one uses the basis \( B_l \) composed of the vector \((1, Q_2(X), \ldots, Q_l(X))\), and \( P_{ch}(X)\delta_{j}^{(l)}, 2 \leq j \leq l \), where \( \delta_{j}^{(l)} \in F_2[X]^l \) has all its components equal to 0, except for the \( j \)-th which is equal to 1. In case of \( \Lambda'_l \) we may, by (ii) of Proposition 2, take a basis \( B'_l \) composed of the images by \( \mathfrak{d} \) of the vectors belonging to a Lenstra-reduced basis of \( \Lambda'_{l-1} \) and of the vector \((Q_l(X), 0, \ldots, 0, 1)\). The required space to reduce the basis \( B'_l \) is significantly less than for \( B_l \), as we see from Lemma 4.

Lemma 4. We have

(i) \((l - 1)d \leq d_s(B_l) \leq ld - 1;\)
(ii) \( d \leq d_s(B'_l) \leq 2d - 1.\)

Proof. Statement (i) of Lemma 4 follows from the fact that \( P_{ch}(X) \) has degree \( d \), while all \( Q_l(X) \) have it less than \( d \). Using (3) we obtain that the sum of the degrees of the first \( l - 1 \) vectors of \( B'_l \) is equal to \( d \), and this proves statement (ii).

We say that an \( l \)-dimensional lattice \( \Lambda \subset F_2[X]^l \) is regular if

\[ \sigma_1(\Lambda)/\sigma_1(\Lambda) \leq 2. \]

Clearly the rank of \( \Lambda \) is bounded by \( \min(d, lk) \).

Proposition 3. For a given \( l \), the rank of \( \Lambda \) is equal to \( \min(d, lk) \) for all \( k \) if and only if \( \Lambda'_l \) is regular.

Proof. By Theorem 1 when \( lk \leq d \) (resp. \( lk > d \)), the rank of \( \Lambda \) is equal to \( lk \) (resp. \( d \)) if and only if, for all \( i \), \( \lg \sigma_i(\Lambda'_l) \geq k \) (resp. \( \lg \sigma_i(\Lambda'_l) \leq k \)). Thus, the rank of \( \Lambda \) is equal to \( \min(d, lk) \) for all \( k \) if and only if

\[ [d/l] \leq \lg \sigma_i(\Lambda'_l) \leq [d/l] + 1, \quad 1 \leq i \leq l. \]

But, this is equivalent to \( \lg \sigma_i(\Lambda'_l) - \lg \sigma_1(\Lambda'_l) \leq 1 \) since we have, from (3), that \( \sum_{i=1}^{l} \lg \sigma_1(\Lambda'_l) = d. \)
Note, by Corollary 11 the equivalence of the regularity of the lattices \( \Lambda_l \) and \( \Lambda'_l \).

**Theorem 2.** If the lattice \( \Lambda'_{l-1} \) is regular, then the Lenstra basis reduction algorithm applied to the basis \( B'_l \) has running time not exceeding

\[
C'_l l(d+l-1)^2, \quad l \geq 2,
\]

where \( C'_l = (l/(l-1))^2 C + 1/l \), and \( C \) is the constant appearing in [12].

**Proof.** Since \( \Lambda'_{l-1} \) is assumed regular, the first \( l-1 \) vectors of \( B'_l \) have their degree bounded by \( d/(l-1) + 1 \). In a first phase, the algorithm will reduce (in length) the \( l \)th vector by the repeated operation of adding to it one of the first \( l-1 \) vectors, premultiplied by a suitable power of \( X \). Each such operation requires at most \( d/(l-1) + 2 \) bit operations. We thus need at most \( d + 2l - 2 \) bit operations to diminish by 1 the degree of the \( l \)th vector, and at most

\[
\left( \frac{l-2}{l-1} \right) d(d+2l-2)
\]

(13)

to diminish its degree to a value bounded by \( d/(l-1) \). After termination of this first phase, the algorithm terminates, according to [12], using at most

\[
C'l^3 \left( \frac{d}{l-1} + 1 \right)^2
\]

(14)

further bit operations. The sum of (13) and (14) is bounded by \( C'_l l(d+l-1)^2 \), and the theorem follows. \( \square \)

For given \( F, x \in F \), and a subset \( E \subset F^w \), it is a problem of interest to determine \( (y_1, \ldots, y_w) \in E \), such that the rank of \( \Pi \) is equal to \( \min (d, lk) \) for all \( l \leq w \), and all \( k \leq d \); that is, such that the lattices \( \Lambda_l (y_1, \ldots, y_l) \) (or, equivalently, \( \Lambda'_l (y_1, \ldots, y_l) \)) are regular for all \( l \leq w \). This type of question arises when one wants to construct an optimally equidistributed output mapping \( \Phi(x) = (\phi(y_1x), \ldots, \phi(y_wx)) \) for a generator based on the field \( F \). Consider the rooted tree \( T = T(E) \) whose vertices of depth \( l \) (or \( l \)-vertices for short) are those \( l \)-tuples \( (y_1, \ldots, y_l) \in F^l \) for which there exists \( y_{l+1}, \ldots, y_w \) such that \( (y_1, \ldots, y_w) \in E \), and whose edges link an \( (l-1) \)-vertex to an \( l \)-vertex if and only if they have the same first \( l-1 \) components. We associate with an \( l \)-vertex the lattices \( \Lambda_l = \Lambda_l (y_1, \ldots, y_l) \) and \( \Lambda'_l = \Lambda'_l (y_1, \ldots, y_l) \). We will say that an \( l \)-vertex \( (y_1, \ldots, y_l) \) of \( T \) is **regular** if its associated lattice \( \Lambda_l \) (or, equivalently, \( \Lambda'_l \)) is regular. A **regular path** in \( T \) is a path visiting only regular vertices. One may then reformulate our problem as the determination of a regular path in \( T \) joining the root to a \( w \)-vertex.

For any \( l \)-vertex \( (y_1, \ldots, y_l) \) of \( T \) we may, as above, construct lattice bases \( B_l \) and \( B'_l \) for the associated lattices \( \Lambda_l \) and \( \Lambda'_l \). We denote them by \( B_l (y_1, \ldots, y_l) \) and \( B'_l (y_1, \ldots, y_l) \), respectively. The regularity of an \( l \)-vertex \( (y_1, \ldots, y_l) \) can be determined by application of Lenstra’s basis reduction algorithm, either to \( B_l (y_1, \ldots, y_l) \) or \( B'_l (y_1, \ldots, y_l) \). If we use \( B_l (y_1, \ldots, y_l) \), then, according to (12), the execution time does not exceed \( Cl^3 d^2 \). It does not exceed \( C'_l l(d+l-1)^2 \) for \( l \) and \( d/l \) large, according to Theorem 2 if we use \( B'_l (y_1, \ldots, y_l) \) instead, and if the \( (l-1) \)-vertex \( (y_1, \ldots, y_{l-1}) \) is regular. Obviously, in the latter case, one needs a Lenstra-reduced basis of the lattice \( \Lambda'_{l-1} \) associated with the \( (l-1) \)-vertex \( (y_1, \ldots, y_{l-1}) \), but such a basis is already available when constructing a regular path, visiting a regular \( (l-1) \)-vertex before any adjacent \( l \)-vertex. Memorizing a reduced basis of \( \Lambda'_{l-1} \) for a regular \( (l-1) \)-vertex also permits one to verify the regularity of several
l-vertices adjacent to it, without recomputing the reduced basis. We finally note that, given a regular path of length \( l - 1 \) and an adjacent \( l \)-vertex \((y_1, \ldots, y_l)\), the regularity of the latter can be obtained by successively constructing and reducing (by Lenstra’s algorithm) the bases \( B'_l(y_1, y_2), \ldots, B'_l(y_1, \ldots, y_l) \), in a time which, by Theorem 2, does not exceed \( C''_l(l^2/2)(d + l - 1)^2 \sim C(l^2/2)d^2 \), for \( l \) and \( d/l \) large. Here the constants \( C''_l \) are given by

\[
C''_l = \left(1 + \frac{5}{l} + \frac{6\ln 4 + 4}{l^2}\right) C + \frac{2(l - 1)}{l^2}.
\]

5. Computation of a random regular path

The advantage of using the lattices \( \Lambda'_l \) instead of \( \Lambda_l \) is confirmed by extensive computer experiments. We give a typical illustration. We take \( F \) to be the field of degree 19937 over \( F_2 \), and \( x \in F \) to be a root of

\[
P_{ch}(X) = X^{19937} + X^{9842} + 1.
\]

This trinomial is primitive (see the table in [3]). Let \( w = 32 \) and \( E = (F\setminus\{0\})^w \). We seek to determine a regular path in \( T(E) \) recursively. Having found a regular \((l - 1)\)-vertex \((y_1, \ldots, y_{l-1})\), a regular \( l \)-vertex \((y_1, \ldots, y_l)\) is determined by randomly choosing \( y \in F\setminus\{0\} \), each outcome being equally likely, and taking for \( y \) the first value of \( y \) for which the vertex \((y_1, \ldots, y_{l-1}, y)\) is regular. The regularity is determined by using either of the lattices \( \Lambda_l(y_1, \ldots, y_{l-1}, y) \) and \( \Lambda'_l(y_1, \ldots, y_{l-1}, y) \). In the first case, Lenstra’s reduction algorithm is applied to the basis \( B_l(y_1, \ldots, y_{l-1}, y) \), while in the second case it is applied to the basis \( B'_l(y_1, \ldots, y_{l-1}, y) \) constructed with the help of the previously reduced basis for the lattice \( \Lambda'(y_1, \ldots, y_{l-1}) \).

For each value of \( l \), from 2 to 32, the CPU time (in seconds) for the reduction required at the \( l \)-vertex and the total cumulative CPU time to determine the first \( l \) vertices, are recorded in Table 1. In most cases, the first \( y \) that was tried already gave a regular vertex. When more than one value of \( y \) was needed, their number is indicated in parentheses, and the reduction time given is the mean reduction time for all these values of \( y \). Since in both computations the same values of \( y \) are used, the same regular path is determined. It appears from Table 1 that the reduction itself takes almost all of the CPU time, and that it is always much quicker to determine the regularity of a vertex using the lattice \( \Lambda'_l \) rather than \( \Lambda_l \). In this instance, there is as much as a 10-fold time reduction for dimension \( l = 18 \), and this increases with \( l \) up to a 16-fold time reduction for \( l = 32 \).

Here, we have taken \( E = (F\setminus\{0\})^w \). When dealing with the problem of constructing an output mapping

\[
\Phi(y) = (\phi(y_1 y), \ldots, \phi(y_w y))
\]

for some generator based on the field \( F \), one must choose \( E \) such that each of its members \((y_1, \ldots, y_w)\) defines an efficient mapping \( \Phi \), when viewed as depending on a computer memory image of the state of the generator (i.e., an output mapping for which a fast computer implementation is available). A description of a specific case, with a new class of random number generators, will be the subject of a forthcoming paper.
Table 1. Efficiency comparison for a random regular path. The first column under \( \lambda_l \) (resp. \( \lambda'_l \)) gives the (mean) reduction time, and the second one, the total cumulative execution time.

<table>
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<th>( \lambda_l )</th>
<th>( \lambda'_l )</th>
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<td>.84</td>
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References


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